Optics of General Guiding Media

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Weakly focusing transparent media provide possible means for guided transmission of coherent light beams with relatively small loss. The scalar wave equation for the eigenmodes of propagation in such a medium is formally identical with Schroedinger's wave equation. Hence, the methods used in the solution of quantum-mechanical problems, such as the Wentzel-Kramers-Brillouin (WKB) approximation, are immediately applicable to this problem. Solutions for the eigenmodes and eigenvalues in the case of focusing in one dimension are given, and the Pöschl-Teller medium, whose index varies as

$$n = n_o[1 - (\alpha/2) \tan^2 \eta x]$$

is discussed in some detail. In addition, the relationship between the wave solutions and geometrical (ray) optics is examined.

I. INTRODUCTION

Weakly focusing transparent media, exemplified by the gas lens,[1] provide a possible means for guided transmission of coherent light beams with relatively small loss. The optics of a medium whose refractive index decreases quadratically away from some spatial axis (the z axis, say) has been the subject of much discussion in the literature.[2] In this paper, we would like to examine the properties of more general guiding media. Some work on this problem has been carried out recently by S. E. Miller.[3]

Consider a light beam traveling paraxially in the z direction in space, guided in a weakly focusing transparent medium. For simplicity, we shall consider focusing only in one of the two transverse dimensions. Thus, the refractive index of the medium and the electromagnetic field will be assumed to be dependent only on the x dimension, and to be independent of y.

Our concern is with transparent media whose index of refraction has the form

$$n = n_o(1 - \tfrac{1}{2}f(x)).$$

We make the following assumptions about $f(x)$

$$f(x) \ll 1 \quad \text{in the range of interest} \tag{1}$$

$$\frac{\partial^2 f}{\partial x^2} \geqq 0 \tag{2}$$

$$f(0) = 0; \qquad f(x) = f(-x). \tag{3}$$

We will investigate both geometrical and physical optics of such a medium in the realm of validity of the paraxial ray equation. Of the assumptions, (1) implies only gradual changes of index, (2) insures focusing properties, and is used in the approximation procedure of Section VII, while (3) is made only for mathematical convenience and can easily be relaxed.

II. GEOMETRICAL (RAY) OPTICS

The well-known paraxial ray equation

$$\frac{d^2x}{dz^2} = \frac{1}{n}\frac{\partial n}{\partial x} = -\frac{1}{2}\frac{\partial f}{\partial x} \tag{4}$$

has the following general solution.

If we let

$$p = \frac{dx}{dz}, \tag{5}$$

i.e., p is the slope of the ray path; then (4) becomes

$$p\frac{dp}{dx} = -\frac{1}{2}\frac{\partial f}{\partial x}.$$

Hence,

$$p^2 + f = \text{const} \equiv \xi. \tag{6}$$

Inserting (6) in (5), and solving, we get

$$z = \int \frac{dx}{\sqrt{\xi - f(x)}} + \text{const}. \tag{7}$$

Note from (6), (5) and (3) that $\pm\xi^{\frac{1}{2}}$ is the slope of the ray path as it crosses the axis $x = 0$.

III. PHYSICAL OPTICS

In the paraxial ray approximation, the electromagnetic fields are always very nearly perpendicular to the "direction" of propagation, which we take to be the z axis. Hence, we can use the scalar wave equation. For harmonic [exp $(-i\omega t)$] fields that are independent of y, we arrive at the wave equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + k^2 n_o^2(1 - f(x))\psi = 0 \qquad [k \equiv \omega/c] \tag{8}$$

where we have used (1) to expand n^2. We look for solutions to (8) with a propagation constant β in the z direction, i.e.,

$$\psi \propto e^{i\beta z}; \qquad \frac{\partial^2\psi}{\partial z^2} = -\beta^2\psi \tag{9}$$

yielding, for (8)

$$\frac{\partial^2\psi}{\partial x^2} + [\beta_o^2 - \beta^2 - \beta_o^2 f(x)]\psi = 0 \tag{10}$$

where we have substituted $n_o^2 k^2 = \beta_o^2 \equiv (2\pi/\lambda)^2$. With the further substitution

$$1 - \beta^2/\beta_o^2 = \xi \tag{11}$$

(10) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \beta_o^2(\xi - f(x))\psi = 0. \tag{12}$$

Equation (12) is Schroedinger's wave equation for a particle in a one dimensional potential well.[4] It has a sequence of eigensolutions ψ_m and corresponding eigenvales ξ_m. The eigenfunctions ψ_m here represent the transverse distributions of the propagating field modes, while the eigenvalues ξ_m give the propagation constants. In accord with (1), the eigenvalues will be much smaller than unity, and so we can find the propagation constants β_m from (11) as

$$\beta_m = \left(1 - \frac{\xi_m}{2}\right)\beta_o. \tag{13}$$

If we use the notation of the classical ray path equation (6), and set

$$p = p(x) = \sqrt{\xi - f(x)} \tag{14}$$

then (12) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \beta_o^2 p^2 \psi = 0. \tag{15}$$

We note from (2) and (3) that $f(x)$ increases monotonically with $|x|$. Consider the points $x = \pm A$, where

$$f(x) = f(A) = \xi. \tag{16}$$

For $|x| < A$, p^2 is positive, and ψ has oscillatory behavior. For $|x| > A$, p^2 is negative, and ψ has decreasing exponential behavior. The points $|x| = \pm A$ are inflection points of ψ. In the ray solution, the points $|x| = A$, where $p = 0$, are the turning points, where the ray has zero slope and, correspondingly, maximum excursion from the axis.

IV. THE WKB APPROXIMATION

A well-known solution to (15), valid in the range $|x| < A$ if p can be considered approximately constant over a few cycles of the oscillatory behavior of ψ, is

$$\psi \propto p^{-\frac{1}{2}} \cos\left\{\beta_o \int_0^x p\, dx + \alpha\right\}. \tag{17}$$

With the symmetry of f assumed in (3), one can show directly from (12) that ψ must be either symmetrical or antisymmetrical in x. Thus, α must be a multiple of $\pi/2$. The eigenvalues of ξ are then determined by the condition that ψ must be matched through the turning point $x = A$ (where another approximate solution to (15) is necessary) into the *decreasing* exponential solution for $x > A$. Discussions of this problem may be found in most quantum mechanics texts.[5] If $f(x)$ can be assumed linear in x over a suitable region near A, then the asymptotic formulae for connection through $x = A$ are

$$\psi \propto p^{-\frac{1}{2}} \cos\left(\theta - \frac{\pi}{4}\right) \qquad x < A \tag{18a}$$

$$\rightarrow \left(\frac{2\pi\theta}{3p}\right)^{\frac{1}{2}} [J_{\frac{1}{3}}(\theta) + J_{-\frac{1}{3}}(\theta)] \qquad x \lesssim A \tag{18b}$$

$$\rightarrow \left(\frac{2\pi\theta}{3p}\right)^{\frac{1}{2}} [-I_{\frac{1}{3}}(\theta) + I_{-\frac{1}{3}}(\theta)] \qquad x \gtrsim A \tag{18c}$$

$$\rightarrow \tfrac{1}{2} p^{-\frac{1}{2}} \exp(-\theta) \qquad x > A \tag{18d}$$

where

$$\theta = \theta(x) = \left| \beta_o \int_A^x |p(x')|\, dx' \right|. \tag{18e}$$

The phase of ψ as it approaches the turning point must be as in (18a). Otherwise the connection through the turning point would give rise to the increasing exponential in the region $x > A$, and this is unallowable.

Expressions (18a) and (17) for ψ can be equated only when

$$\beta_o \int_0^A p\, dx = (m + \tfrac{1}{2}) \frac{\pi}{2} \tag{19}$$

where m is an integer. Even and odd symmetry solutions correspond respectively to even or odd values of m. Equation (19) is Bohr's quantization rule, and its solutions give the eigenvalues ξ_m. The eigenfunctions are then given by (18).

V. RELATION BETWEEN WKB AND GEOMETRICAL RAY PATHS

Referring to (19), let

$$I(\xi) = \frac{2\beta_o}{\pi} \int_0^A p\, dx. \tag{20}$$

For successive eigensolutions, I changes by unity. Now,

$$\frac{dI}{d\xi} = \frac{\beta_o}{\pi} \int_0^A \frac{dx}{p} \tag{21}$$

where we have used (14); i.e., $p = \sqrt{\xi - f(x)}$. For a change of I by unity, the propagation constant (13) changes approximately by

$$\delta\beta \approx \frac{d\beta}{dI} = \frac{d\beta}{d\xi} \bigg/ \frac{dI}{d\xi} = \frac{\pi/2}{\int_0^A dx/p}. \tag{22}$$

Now from the ray solution (7), the z distance a ray with the same value of ξ takes to go through one complete cycle of its transverse oscillatory motion is

$$z_\sim = 4 \int_0^A dx/p \tag{23}$$

hence, putting this in (22), we find

$$(\delta\beta) z_\sim = 2\pi. \tag{24}$$

Hence, adjacent modes undergo a relative phase shift of approximately 2π in the same distance the corresponding geometrical ray takes to complete one transverse cycle.

We can make the same argument more general by examining the behavior of solutions of (15) which are sums of component solutions (17) with nearby values of ξ. We look for trajectories $x(z)$ such that the phase differences between components remain constant. Then if we make up a wave packet, the packet will follow one of these trajectories.

Putting the z dependence back into (17), and keeping, for example, only the positive imaginary part of the cosine term, we have

$$\psi \propto p^{-\frac{1}{2}} \left\{ \exp i \left[\beta z + \beta_o \int_0^x p \, dx + \alpha \right] \right\}. \tag{25}$$

The phase we have to maintain constant over a range of ξ is thus

$$\beta z + \beta_o \int_0^x p \, dx + \alpha. \tag{26}$$

Differentiating with respect to ξ, setting the result equal to zero, and using (13) and (14), we have

$$z = \int_0^x \frac{dx}{p} + \frac{2}{\beta_o} \frac{d\alpha}{d\xi}. \tag{27}$$

To complete the picture, we note that in order to make a wave packet initially concentrated at some point (x_o, z_o) should take all the phases initially equal at that point, so that

$$\alpha = -\beta z_o - \beta_o \int_0^{x_o} p \, dx + \text{const}$$

$$\frac{2}{\beta_o} \frac{d\alpha}{d\xi} = z_o - \int_0^{x_o} \frac{dx}{p}.$$

Hence, the wave packet trajectories of stationary phase are given by

$$z - z_o = \int_{x_o}^x \frac{dx}{p}. \tag{28}$$

This is identical with the ray path (7). In this last discussion we have neglected the discrete nature of the eigenvalues. Such an approximation should be valid for reasonably large mode numbers m.

VI. THE PÖSCHL-TELLER POTENTIAL

A number of functions $f(x)$ yield analytically integrable equations. The square law medium $f(x) = (x/b)^2$ and the square well medium $f(x) = 0$ for $|x| < A$, $f(x) \to \infty$ for $x > A$, suitably joined, are media with well-known solutions both for the geometrical and physical optics equations. An interesting function, which in a slightly more general form goes by the name of the Pöschl-Teller potential[6] in quantum mechanics, is the function

$$f(x) = \alpha \tan^2 (\eta x); \qquad -\pi/2\eta < x < \pi/2\eta. \tag{29}$$

Near $x = 0$

$$f(x) = \alpha\eta^2x^2 \tag{30}$$

and there are impenetrable boundaries at $x = \pm\pi/2\eta$. For this function the ray equation (7), the wave equation (12), and the WKB approximation (17) can be directly integrated and compared. As the parameters α and η are changed, the function varies smoothly between a square well and a square law type.

Consider first the ray equation,

$$z = \int \frac{dx}{[\xi - \alpha \tan^2 \eta x]^{\frac{1}{2}}} + \text{const.} \tag{31}$$

This integrates to [Burrington #258]

$$z = \frac{1}{\eta[\xi + \alpha]^{\frac{1}{2}}} \sin^{-1} ([1 + \alpha/\xi]^{\frac{1}{2}} \sin \eta x) + \text{const.} \tag{32}$$

If we set the constant equal to zero, then $z = 0$ when $x = 0$. This expression generates a ray path, with x taking on values between the turning points

$$x = \pm A = \pm\eta^{-1} \sin^{-1} [1 + \alpha/\xi]^{-\frac{1}{2}} \tag{33}$$

while the period of oscillation is

$$z_{\sim} = 2\pi/\eta[\xi + \alpha]^{\frac{1}{2}}. \tag{34}$$

Note that at the turning points, $f(x) = \xi$. We see that for $\alpha \gg \xi$, ηx is always small, and (32) reduces to

$$x = (\xi/\eta^2\alpha)^{\frac{1}{2}} \sin (\eta\alpha^{\frac{1}{2}}z)$$

which is appropriate to a square law medium, while for $\alpha \ll \xi$, the ray travels back and forth with constant slope $\xi^{\frac{1}{2}}$ between reflecting walls separated by π/η.

Pöschl and Teller have found exact solutions for the wave equation (12) with this potential. They have shown that the eigenvalues follow the simple law

$$\xi_m = (\eta^2/\beta_o^2)(m^2 + 2ma + a) \tag{35a}$$

where a is the positive root of the equation

$$a(a - 1) = \alpha\beta_o^2/\eta^2. \tag{35b}$$

The corresponding eigenfunctions are

$$\psi_m = \cos^a (\eta x) \sum_{\substack{k=0,\text{even} \\ k=1,\text{odd}}}^{m\,\text{even},\ \text{odd}} c_k \sin^k (\eta x) \tag{36}$$

where

$$\frac{c_{k+2}}{c_k} = \frac{(m - k)(m + k + 2a)}{(k + 1)(k + 2)} .$$

From (35) and (13), we find that the propagation constants of the modes follow the law

$$\beta_m = \beta_o - (\eta^2 a/2\beta_o)\left(1 + 2m + \frac{m^2}{a}\right). \tag{37}$$

To examine the square law limit, we set $\alpha \gg \xi$. Then [see (35)], $a \approx \alpha^{\frac{1}{2}}\beta_o/\eta \gg 1$,

and, from (37)

$$\beta_m \approx \beta_o - \eta\alpha^{\frac{1}{2}}(m + \tfrac{1}{2}) \tag{38}$$

while from (36)

$$\psi_m \approx \exp\,(-\tfrac{1}{2}\eta\alpha^{\frac{1}{2}}\beta_o x^2) H_m(\eta\alpha^{\frac{1}{2}}\beta_o x^2). \tag{39}$$

To examine the square well limit, we set $\alpha \ll \xi$. Then $a \approx 1$, and

$$\beta_m \to \beta_o - \tfrac{1}{2}(\eta^2/\beta_o)(m + 1)^2 \tag{40}$$

$$\begin{aligned} \psi_m &\to \cos\,[(m + 1)\eta x] \quad \text{for} \quad m \text{ even} \\ \psi_m &\to \sin\,[(m + 1)\eta x] \quad \text{for} \quad m \text{ odd.} \end{aligned} \tag{41}$$

Returning now to the general solution for the propagation constants, note from (37) that the average of the propagation constant differences between the mth mode and its two neighbors is given by

$$\delta\beta = \tfrac{1}{2}\,(\beta_{m-1} - \beta_{m+1}) = (\eta^2/\beta_o)(m + a). \tag{42}$$

If we look at the ray equation, and insert the values of α and ξ_m (35) into (34), we find

$$z_\sim = 2\pi/\delta\beta. \tag{43}$$

This is a more precise version of (24).

Using the Pöschl-Teller potential, the WKB result (19) for the eigenvalues ξ_m can also be integrated. Referring to (21), (23), and (34), we have

$$\begin{aligned} \frac{dI}{d\xi} &= (\beta_o/\pi)\int_0^A dx/p = \beta_o z_\sim/4\pi \\ &= \beta_o/2\eta(\xi + \alpha)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$I = \beta_o(\xi + \alpha)^{\frac{1}{2}}/\eta - \beta_o\alpha^{\frac{1}{2}}/\eta. \tag{44}$$

Here the constant of integration is determined by the condition that $I = 0$ at $\xi = 0$. Thus, (19) [see also (20)] is

$$\beta_o(\xi_m + \alpha)^{\frac{1}{2}}/\eta = \beta_o\alpha^{\frac{1}{2}}/\eta + (m + \tfrac{1}{2}) \tag{45}$$

or

$$\begin{aligned} \xi_m &= [(\eta/\beta_o)(m + \tfrac{1}{2}) + \alpha^{\frac{1}{2}}]^2 - \alpha \\ &= (\eta/\beta_0)^2[(m + \tfrac{1}{2})^2 + (2m + 1)\sqrt{a(a - 1)}] \end{aligned} \tag{46}$$

where we have used (35b). We see that the WKB result (46) agrees with the exact result (35a) in the limit of the square law medium, $a \gg 1$. In the square well limit, $a \rightarrow 1$, the WKB method gives eigenvalues proportional to $(m + \frac{1}{2})^2$ rather than $(m + 1)^2$ as does the exact result (35). This last discrepancy can be traced to incorrect matching of the boundary conditions by the WKB method, since here the WKB wave function (17) is the correct one. The factor $\frac{1}{2}$ in $m + \frac{1}{2}$ arises from matching the boundary condition, and in fact, we can get the WKB answer to equal the exact one for all a and m here by the artifice of replacing $m + \frac{1}{2}$ in (46) by

$$m + a - \sqrt{a(a - 1)} = m + a - \alpha^{\frac{1}{2}}\beta_o/\eta. \tag{47}$$

The number added to m varies between the value $\frac{1}{2}$ for large a to one as a approaches unity.

VII. FURTHER APPLICATION OF THE WKB METHOD

The results of Section VI give us reasonable faith in the WKB method of obtaining eigenvalues even at small values of m. The number we add to m may however, take on values between $\frac{1}{2}$ and 1 depending in general on m as well as on the form of the potential.

Let us consider the WKB result further. Equation (19), which we repeat here for convenience, may be written

$$\beta_o \int_0^{A_m} \sqrt{\xi_m - f(x)}\, dx = (m + \tfrac{1}{2})\frac{\pi}{2} \tag{48}$$

where

$$f(A_m) = \xi_m .$$

We can see two simple bounds. For functions that satisfy (2) and (3), we have that in the range $0 \leqq x \leqq A_m$,

$$\xi_m \geqq \xi_m - f(x) \geqq \xi_m \left(1 - \frac{x}{A_m}\right) \tag{49}$$

hence, performing the integral for the two bounds, (48) gives

$$\beta_o \sqrt{\xi_m} A_m \geqq (m + \tfrac{1}{2}) \frac{\pi}{2} \geqq \tfrac{2}{3} \beta_o \sqrt{\xi_m} A_m .$$

We can write this as

$$\sqrt{\xi_m} A_m = r(m + \tfrac{1}{2})\pi/2\beta_o \tag{50}$$

where

$$1 \leqq r \leqq 1.5.$$

Note that

$$r^{-1} = \frac{1}{\xi^{\frac{1}{2}} A} \int_0^A \sqrt{\xi - f(x)}\, dx. \tag{51}$$

If we further remember that we should replace the $\frac{1}{2}$ in $m + \frac{1}{2}$ by some number between $\frac{1}{2}$ and 1 which we can denote by s, we have

$$\sqrt{\xi_m} A_m = r(m + s)\pi/2\beta_o \tag{52}$$

with

$$1 \leqq r \leqq 1.5 \quad \text{and} \quad 0.5 \leqq s \leqq 1.$$

Let us examine the implications of this formula in a definite example. We take the potential to be a pure power law[3]

$$f(x) = \left(\frac{x}{b}\right)^{2n}. \tag{53}$$

We need the value of A_m;

$$f(A_m) = \xi_m \quad \text{therefore,} \quad A_m = b\xi_m^{1/2n}.$$

Then (52) is easily solved, yielding

$$\xi_m = \left\{\frac{\pi r(m + s)}{2\beta_o b}\right\}^{2n/n+1}. \tag{54}$$

In this case, examination of (51) reveals that r is given by

$$r^{-1} = \int_0^1 \sqrt{1 - y^{2n}}\, dy \tag{55}$$

so that r is independent of m, and varies between

$$\frac{4}{\pi} \quad \text{for} \quad n = 1 \qquad \text{and} \qquad 1 \quad \text{for} \quad n \to \infty.$$

Further, we know that $s = \frac{1}{2}$ for $n = 1$ (square law), and $s = 1$ for $n \to \infty$ (square well). As a guess, we would be tempted to try

$$s = \frac{n}{n + 1}$$

as a suitable interpolation. Then we have as our final result

$$\xi_m = \left\{\left(m + \frac{n}{n + 1}\right) \frac{\pi r}{2\beta_o b}\right\}^{2n/n+1} \tag{56}$$

hence for the phase constant [repeating (13)]

$$\beta_m = \beta_o(1 - \xi_m/2). \tag{57}$$

VIII. ACKNOWLEDGMENT

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$$\frac{a}{\sin^2 \alpha x} + \frac{b}{\cos^2 \alpha x}$$

and

$$\frac{a}{\sinh^2 \alpha x} - \frac{b}{\cosh^2 \alpha x}.$$

The parameters a, b, and α are not the same as those used in the text. In addition, treatment of the function

$$-(b/\cosh^2 \alpha x)$$

can be found in Landau, E. M. and Lifshitz, E. M., *Quantum Mechanics, Non-Relativistic Theory,* Pergamon Press Ltd., 1958, pp. 69–70.