The Statistical Effects of Random Variations in the Components of a Beam Waveguide

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The effects of variations in the components of a beam waveguide are considered. These variations statistically cause the Gaussian beam spot size of the light propagating down the waveguide to grow and cause the beam center to oscillate about the waveguide axis with ever-increasing amplitude. Random variations in lens focal length and spacing and random lateral lens displacements are considered. It is also shown how random variations in focal length and spacing can be included in the published analyses for short random bends in the waveguide axis.

When the number of lenses is large, it is shown that the beam displacement and beam spot size both grow exponentially with distance.

As an example, a confocal waveguide with lenses spaced one meter apart and built to somewhat optimistic tolerances will require a beam redirector every 2.5 kilometers to prevent the beam oscillations from exceeding an rms value of 2 millimeters.

I. INTRODUCTION

A long sequence of spaced lenses is of considerable interest for optical communications. It is known that the diffraction losses in such an optical beam waveguide can be kept very small for moderate size lenses.[1,2] This means that if a transmission line is made of identical low-loss lenses, spaced identically along a straight line with each lens centered on this straight line, there is a mode of propagation which is low loss. However, if there are imperfections in the transmission line, the light beam will begin to wander from the axis or the beam size will grow and the beam will eventually strike the edge of the lens and be lost. Since the diffraction loss of the beam in a perfect line can be kept very small, it is the line imperfections, the line axis curvature, and the scattering and absorption at each lens which will primarily determine the optical loss. Gas lenses have been considered for reducing the scattering and absorption losses.[3]

Rowe[4] and Hirano and Fukatsu[5] have shown how the beam position is affected by random lateral lens displacements. Berreman,[6] Marcuse,[7] and Unger[8] have considered correlated lateral lens displacements in the form of bends. All of these analyses have assumed perfect lenses and perfect spacing and have shown the growth of the beam displacement due to lateral lens displacements only. It is the purpose of this paper to show how the previously obtained results are altered when the lens focal lengths and lens spacings have random variations.

In this paper, we shall consider the statistical effect of random variations in lens focal length and lens spacing and random lateral lens displacements. We shall also consider random bends whose correlation length is much smaller than the total line length. It is shown how the various line imperfections couple to one another and cause the beam deviation from the axis to grow. The growth of the spot size of a Gaussian beam is also considered.

It is shown here that the random variations in f and L cause the rms expected value of beam displacement and the rms expected value of the beam spot size to grow exponentially with distance when the number of lenses is large. For bends, the variations in f and L cause an exponential increase in the average allowed bending radius of the guide when the number of lenses is large. In contrast, when f and L are perfect these effects grow more slowly with distance, and increase only as the square root of the number of lenses.

We shall use geometric optics since it is known that in the paraxial approximation the center of a Gaussian beam in a beam waveguide behaves as a ray.[9] The geometric optics analysis is extended to find the behavior of the beam spot size by replacing the Gaussian beam by its equivalent ray packet.[10]

II. GENERAL PROBLEM FORMULATION

We shall consider the problem in two dimensions only for simplicity. It has been shown that the three-dimensional problem can be split into two independent two-dimensional problems.[5] For aberration-free lenses, the motion of the beam in one transverse dimension is dependent only on the initial conditions and lens displacements in the same transverse direction.

Consider the transmission line shown in Fig. 1. We define r_n and r_n' as the position and slope of the ray just to the right of the nth lens. The

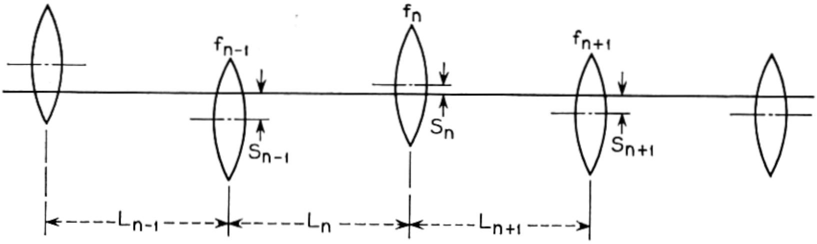

Fig. 1 — Beam waveguide notation.

ray position and slope are measured with respect to a straight line which is the nominal transmission line axis. The spacing between lenses is labeled L_n and the convergence of the lenses as C_n where

$$C_n = 1/f_n$$

and f_n is the focal length of the nth lens. The lateral distance between the center of the nth lens and the reference line is s_n. The displacement s_n is positive if the lens center is above the reference line.

Using this notation we can write

$$r_n = r_{n-1} + L_n r_{n-1}' \tag{1}$$

$$r_n' = -C_n r_{n-1} + (1 - C_n L_n) r_{n-1}' + C_n s_n . \tag{2}$$

If we define

$$R_n = \begin{bmatrix} r_n \\ r_n' \end{bmatrix},$$

$$M_n = \begin{bmatrix} 1 & L_n \\ -C_n & 1 - L_n C_n \end{bmatrix},$$

and

$$V_n = \begin{bmatrix} 0 \\ C_n s_n \end{bmatrix}$$

we can write (1) and (2) in matrix form as

$$R_n = M_n R_{n-1} + V_n . \tag{3}$$

This relates the ray position and slope at the n plane to the ray position and slope at the $n - 1$ plane. We shall be interested in the rms expected value of the output beam displacement and hence in the square of the output beam slope and position. We shall, therefore, square the matrix (3). To do this, we take the Kronecker product[11] of each side of (3) with itself.

$$R_n \times R_n = (M_n \times M_n)(R_{n-1} \times R_{n-1}) + V_n \times V_n + (M_n R_{n-1}) \times V_n + V_n \times (M_n R_{n-1}). \tag{4}$$

Now take the expected value of (4)

$$\langle R_n \times R_n \rangle = \langle (M_n \times M_n)(R_{n-1} \times R_{n-1}) \rangle + \langle V_n \times V_n \rangle + \langle (M_n R_{n-1}) \times V_n \rangle + \langle V_n \times (M_n R_{n-1}) \rangle. \tag{5}$$

The expected value of a matrix is the matrix of the expected values.

We will now state our statistical assumptions. We assume small variations about L and C

$$L_n = L(1 + l_n)$$

$$C_n = C(1 + c_n).$$

$$\langle l_n \rangle = \langle c_n \rangle = \langle s_n \rangle = 0$$

$$\langle l_n l_k \rangle = \langle c_n c_k \rangle = 0 \qquad n \neq k$$

$$\langle l_n c_k \rangle = \langle l_n s_k \rangle = \langle c_n s_k \rangle = 0 \quad \text{all} \quad n$$

$$\langle c_n^2 \rangle = \sigma_c^2$$

$$\langle l_n^2 \rangle = \sigma_L^2$$

$$\langle s_n^2 \rangle = y^2.$$

Hence, l_n, c_n, and s_n are mutually independent random variables of zero mean. There is no correlation between the variations in spacing, the variations in focal length, and the variations in lateral displacement. The variations in L and C are completely random with no correlation between adjacent L's and adjacent C's. For the first sections of the paper we shall consider the lateral lens displacements to also be completely random with no correlation between adjacent displacements. In a later section it will be shown how correlation of the lens displacements in the form of random bends in the waveguide axis can be included.

III. A TRANSMISSION LINE WITH RANDOM LATERAL LENS DISPLACEMENTS

In this section we shall consider only random lateral lens displacements. Hence, we impose the additional statistical restriction that

$$\langle s_k s_n \rangle = 0 \qquad n \neq k.$$

The lateral displacement of any lens is unaffected by the lateral displacements of any other lens.

If we repeatedly substitute (3) into the last two matrices of (5), we see that they contain elements of the general form

$$\langle G(L_k\,,\ L_{k+1}\,,\ \cdots L_n\,,\ C_k\,,\ C_{k+1}\,,\ \cdots C_n) s_{k-1} s_n \rangle, \qquad k \leqq n$$

and

$$\langle F(L_1\,, L_2\,, \cdots L_n\,, C_1\,, C_2\,, \cdots C_n) R_0 s_n \rangle$$

where $G(\ )$ and $F(\ )$ are some functions. In view of the statistical assumptions, these can be written as

$$\langle G(L_k\,,\ L_{k+1} \cdots L_n\,,\ C_k\,,\ C_{k+1}\,,\ \cdots C_n) \rangle \langle s_{k-1} s_n \rangle, \qquad k \leqq n$$

and

$$\langle F(L_1\,, L_2\,, \cdots L_n\,, C_1\,, C_2\,, \cdots C_n) \rangle R_0 \langle s_n \rangle$$

which are zero since $\langle s_{k-1} s_n \rangle = 0$ for $k \leqq n$ and $\langle s_n \rangle = 0$. Hence, the last two matrices of (5) are both zero.

When there is some correlation between lateral lens displacements, i.e., a wavy transmission line axis, these two matrices are not zero. It is through these matrices that the correlation will enter.

Also, by using the above statistical assumptions M_n is independent of R_{n-1} and $\langle M_n \times M_n \rangle$ and $\langle V_n \times V_n \rangle$ are not functions of n. Equation (5) can, therefore, be written as

$$\langle R_n \times R_n \rangle = \langle M_n \times M_n \rangle^n R_0 \times R_0 + \sum_{k=0}^{n-1} \langle M_n \times M_n \rangle^k \langle V_n \times V_n \rangle \quad (6)$$

where R_0 is the matrix of the initial ray slope and position.

The Kronecker products in (6) are either 4×4 or 4×1 matrices. These can be reduced to 3×3 and 3×1 matrices by combining the two redundant terms.[11] For clarity and ease of computation we shall reduce the matrices and write them out explicitly below. We have assumed here that $\sigma_L^2 \ll 1$ and $\sigma_C^2 \ll 1$ hence, we have neglected $\sigma_L^2 \sigma_C^2$ compared to σ_L^2 or σ_C^2.

$$\langle R_n \times R_n \rangle = \begin{bmatrix} \langle r_n^2 \rangle \\ \langle r_n r_n' \rangle \\ \langle r_n'^2 \rangle \end{bmatrix}$$

$$R_0 \times R_0 = \begin{bmatrix} r_0^2 \\ r_0 r_0' \\ r_0'^2 \end{bmatrix}$$

$$\langle V_n \times V_n \rangle = \begin{bmatrix} 0 \\ 0 \\ C^2(1 + \sigma_C^2) y^2 \end{bmatrix}$$

$$\langle M_n \times M_n \rangle = \begin{bmatrix} 1 & 2L & L^2(1 + \sigma_L^2) \\ -C & 1 - 2LC & L - L^2C(1 + \sigma_L^2) \\ C^2(1 + \sigma_C^2) & \begin{matrix} -2C + 2LC^2 \\ (1 + \sigma_C^2) \end{matrix} & \begin{matrix} 1 - 2LC + L^2C^2 \\ (1 + \sigma_L^2 + \sigma_C^2) \end{matrix} \end{bmatrix}.$$

3.1 *The Characteristic Roots of* $\langle M_n \times M_n \rangle$

To evaluate (6) will require the raising of $\langle M_n \times M_n \rangle$ to the kth power. To do this it will be necessary to find the characteristic roots of $\langle M_n \times M_n \rangle$. The characteristic roots of $\langle M_n \times M_n \rangle$ can be found from the equation

$$| \langle M_n \times M_n \rangle - I\lambda | = 0$$

where λ is the characteristic root and I is the unity matrix. This leads to the following cubic equation for λ:

$$\begin{aligned} \lambda^3 - \lambda^2[3 - 4LC + L^2C^2 + L^2C^2(\sigma_L^2 + \sigma_C^2)] \\ - \lambda[-3 + 4LC - L^2C^2 + L^2C^2(\sigma_L^2 + \sigma_C^2)] - 1 = 0. \end{aligned} \tag{7}$$

We assume in (7) that σ_L^2 and σ_C^2 are very small quantities, hence terms of higher power than 2 in σ_L and σ_C are neglected. Since σ_L^2 and σ_C^2 are assumed very small it is reasonable to assume that the roots of (7) are very near the roots for the perfect transmission when $\sigma_L^2 = \sigma_C^2 = 0$. For the perfect transmission line, the roots are

$$\lambda = 1, \quad e^{2i\theta}, \quad e^{-2i\theta},$$

where $\theta = \cos^{-1}(1 - LC/2)$. We therefore write the three roots of (7) as

$$\lambda_1 = 1 + q_1$$
$$\lambda_2 = e^{2i\theta}(1 + q_2)$$
$$\lambda_3 = e^{-2i\theta}(1 + q_3)$$

where

$$| q_1 |, | q_2 |, | q_3 | \ll 1.$$

For the case of $LC \neq 2$, i.e., a nonconfocal system, (7) gives

$$q_1 = \frac{2LC}{4 - LC}(\sigma_L^2 + \sigma_C^2)$$

$$q_2 - q_3 = \frac{-LC}{4 - LC}(\sigma_L^2 + \sigma_C^2).$$

If we define

$$a = \frac{2LC}{4 - LC}(\sigma_L^2 + \sigma_C^2)$$

then

$$\lambda_1 = 1 + a$$
$$\lambda_2 = e^{2i\theta}\left(1 - \frac{a}{2}\right)$$
$$\lambda_3 = e^{-2i\theta}\left(1 - \frac{a}{2}\right).$$

For the confocal case, $LC = 2$, the roots of (7) are

$$\lambda_1 = 1 + a$$
$$\lambda_2 = -1 + a$$
$$\lambda_3 = -1.$$

These two sets of solutions of (7) are valid so long as $a \ll 1$. This means $\sigma_C^2 \ll 1$, $\sigma_L^2 \ll 1$, and LC is not near 4.

3.2 *Sylvesters Theorem*

For raising the matrix $\langle M_n \times M_n \rangle$ to the kth power, it is helpful to use Sylvesters Theorem.[12] If λ_1, λ_2, and λ_3 are the characteristic roots of the matrix A then

$$A^k = \frac{(A - \lambda_2 I)(A - \lambda_3 I)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \lambda_1^k + \frac{(A - \lambda_1 I)(A - \lambda_3 I)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \lambda_2^k + \frac{(A - \lambda_1 I)(A - \lambda_2 I)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \lambda_3^k$$

where I is the unity matrix.

In (6) we shall be interested in the case where n is a very large number, i.e., many lenses in the transmission line. The difference between the matrix $\langle M_n \times M_n \rangle$ and $M \times M$ where M is the perfect transmission line matrix is very small (terms of the order of σ_C^2 and σ_L^2). For the nonconfocal case, it will therefore only be necessary to consider the deviations from a perfect transmission line in λ_1^k, λ_2^k, and λ_3^k. In the coefficients of λ_1^k, λ_2^k, and λ_3^k we can assume $\sigma_L^2 = \sigma_C^2 = 0$.

Hence, for the nonconfocal case, we can write

$$\langle M_n \times M_n \rangle^k = \frac{(M \times M - e^{2i\theta} I)(M \times M - e^{-2i\theta} I)}{(1 - e^{2i\theta})(1 - e^{-2i\theta})} e^{ka} + \frac{(M \times M - I)(M \times M - e^{-2i\theta} I)}{(e^{2i\theta} - 1)(e^{2i\theta} - e^{-2i\theta})} e^{2ik\theta} e^{-ka/2} + \frac{(M \times M - I)(M \times M - e^{2i\theta} I)}{(e^{2i\theta} - 1)(e^{-2i\theta} - e^{2i\theta})} e^{-2ik\theta} e^{-ka/2}. \quad (8)$$

Here we have used 1, $e^{2i\theta}$, and $e^{-2i\theta}$ for the roots of $M \times M$ and

$$\lambda_1^k = (1 + a)^k \approx e^{ka}$$

$$\lambda_2^k = e^{2ik\theta} \left(1 - \frac{a}{2}\right)^k \approx e^{2ik\theta} e^{-ka/2}$$

$$\lambda_3^k = e^{-2ik\theta} \left(1 - \frac{a}{2}\right)^k \approx e^{-2ik\theta} e^{-ka/2}.$$

In (6) we shall be interested in only the first element of $\langle R_n \times R_n \rangle$ which is $\langle r_n^2 \rangle$. To calculate this we will only need the first row of elements of $\langle M_n \times M_n \rangle^k$. These can be found from (8). The 1,1 element for the nonconfocal case is

$$b_{11} = e^{ka}\left[\frac{4\sin^2\theta - 2LC\cos\theta}{4\sin^2\theta}\right]$$
$$+ e^{2ik\theta}e^{-ka/2}\left[\frac{-2LC\cos\theta}{(e^{2i\theta}-1)2j\sin 2\theta}\right]$$
$$+ e^{-2ik\theta}e^{-ka/2}\left[\frac{-2LC\cos\theta}{(e^{-2i\theta}-1)(-2j\sin 2\theta)}\right].$$

After some simplification

$$b_{11} = \frac{2}{4-LC}\,[e^{ka} + e^{-ka/2}\cos(2k-1)\theta].$$

Similarly, we can write b_{12} and b_{13} for the nonconfocal case as

$$b_{12} = \frac{2L}{4-LC}\left[e^{ka} + e^{-ka/2}\left(\sqrt{\frac{4-LC}{LC}}\sin 2k\theta - \cos 2k\theta\right)\right]$$

$$b_{13} = \frac{2L^2}{LC(4-LC)}\,[e^{ka} - e^{-ka/2}\cos 2k\theta].$$

For the confocal case, $LC = 2$, we are close to a degenerate case where two characteristic roots are close to being equal. If we retain terms to no higher power than 2 in σ_L^2 and σ_C^2 in $\langle M_n \times M_n\rangle^k$ we can write

$$\langle M_n \times M_n\rangle^k = \frac{(M\times M + I)(M\times M + I)}{4}\,e^{ka}$$
$$- \frac{(\langle M_n\times M_n\rangle - (1+a)I)(\langle M_n\times M_n\rangle + I)}{2a}\,(-1)^k e^{-ka} \quad (9)$$
$$+ \frac{(\langle M_n\times M_n\rangle - (1+a)I)(\langle M_n\times M_n\rangle + (1-a)I)}{2a}\,(-1)^k.$$

We have retained the σ_L^2 and σ_C^2 in the coefficients of the last two terms since σ_L^2 and σ_C^2 appear in the denominators.

We again are interested only in the first row of $\langle M_n \times M_n\rangle^k$. For the confocal case, the elements of the first row are

$$b_{11} = e^{ka}$$

$$b_{12} = L\left[e^{ka} - \frac{2(-1)^k}{a}(\sigma_C^2 e^{-ka} + \sigma_L^2)\right]$$

$$b_{13} = \frac{L^2}{2}\left[e^{ka} - \frac{2(-1)^k}{a}(\sigma_C^2 e^{-ka} + \sigma_L^2)\right].$$

3.3 *Calculation of* $\langle r_n^2 \rangle$

For the nonconfocal case using (6) and the values for the elements in the first row of $\langle M_n \times M_n \rangle^k$, we can calculate $\langle r_n^2 \rangle$.

$$\begin{aligned}\langle r_n^2 \rangle = \frac{2}{4 - LC} \Big\{ & r_0^2 [e^{na} + e^{-(na/2)} \cos (2n - 1)\theta] \\ & + r_0 L r_0' \left[e^{na} + e^{-(na/2)} \left(\sqrt{\frac{4 - LC}{LC}} \sin 2n\theta - \cos 2n\theta \right) \right] \\ & + \frac{L^2 r_0'^2}{LC} [e^{na} - e^{-(na/2)} \cos 2n\theta] \\ & + LCy^2 \sum_{k=0}^{n-1} (e^{ka} - e^{-(ka/2)} \cos 2k\theta) \Big\}.\end{aligned}$$

In the 3,1 position of $\langle V_n \times V_n \rangle$ we have neglected the σ_C^2 as compared to 1.

The summations can be evaluated as

$$\sum_{k=0}^{n-1} e^{ka} = \frac{e^{na} - 1}{e^a - 1} \approx \frac{e^{na} - 1}{a}$$

$$\begin{aligned}&\sum_{k=0}^{n-1} e^{-(ka/2)} \cos 2k\theta \\ &= \frac{1 - e^{-(a/2)} \cos 2\theta - e^{-(na/2)} \cos 2n\theta + e^{-[(n+1)a/2]} \cos 2(n - 1)\theta}{1 + e^{-a} - 2e^{-a/2} \cos 2\theta} \\ &\approx \frac{1}{2} + \frac{e^{-(na/2)} \sin (2n - 1)\theta}{2 \sin \theta}.\end{aligned}$$

We have used the facts that $a \ll 1$ and $n \gg 1$.

The expected value of the square of the output beam position for the nonconfocal case is therefore,

$$\begin{aligned}\langle r_n^2 \rangle = \frac{2}{4 - LC} \Big\{ & r_0^2 [e^{na} + e^{-(na/2)} \cos (2n - 1)\theta] \\ & + r_0 L r_0' \left[e^{na} + e^{-(na/2)} \left(\sqrt{\frac{4 - LC}{LC}} \sin 2n\theta - \cos 2n\theta \right) \right] \\ & + \frac{L^2 r_0'^2}{LC} [e^{na} - e^{-(na/2)} \cos 2n\theta] \\ & + LCy^2 \left[\frac{e^{na} - 1}{a} - \frac{1}{2} - \frac{e^{-(na/2)} \sin (2n - 1)\theta}{2 \sin \theta} \right] \Big\}\end{aligned} \tag{10}$$

where

$$a = \frac{2LC}{4 - LC} (\sigma_L^2 + \sigma_C^2).$$

For confocal spacing, $LC = 2$, the square of the expected value of r_n is

$$\begin{aligned} \langle r_n^2 \rangle &= r_0^2 e^{na} + r_0 L r_0' \left[e^{na} - \frac{2(-1)^n}{a} (\sigma_C^2 e^{-na} + \sigma_L^2) \right] \\ &+ \frac{L^2 r_0'^2}{2} \left[e^{na} - \frac{2(-1)^n}{a} (\sigma_C^2 e^{-na} + \sigma_L^2) \right] \\ &+ 2y^2 \left[\frac{e^{na} - 1}{a} - \frac{1}{2} - \frac{\sigma_C^2 e^{-na}}{a} + \frac{(-1)^n \sigma_L^2}{a} \right] \end{aligned} \tag{11}$$

where, for $LC = 2$, $a = 2(\sigma_L^2 + \sigma_C^2)$.

It is of interest to consider how close to $LC = 2$ one must be to have (11) hold rather than (10). If we retain terms to only the first power in a, it can be shown that (11) is valid when

$$| LC - 2 | < \sigma_L^2 + \sigma_C^2.$$

If

$$| LC - 2 | > \sigma_L^2 + \sigma_C^2$$

then (10) holds. Since σ_L^2 and σ_C^2 will be of the order of 10^{-4} we must be very close to $LC = 2$ for (11) to hold.

If the lenses and spacing are perfect so that $a = 0$, the first three terms of (10) and of (11) give the square of the output beam position due to the input beam slope and position. The last term gives the increased displacement due to random lateral lens displacements. Both parts agree with Hirano and Fukatsu[5] when $a = 0$.

The random errors in focal length and spacing cause an exponential increase in the expected value of the square of the output beam displacement. This can be seen more clearly for the case where n is very large. If $na > 2$, then (10) and (11) reduce to the same result. In this case,

$$\langle r_n^2 \rangle \approx \frac{2e^{na}}{4 - LC} \left[r_0^2 + r_0 L r_0' + \frac{L^2 r_0'^2}{LC} \right] + \frac{2LC}{4 - LC} y^2 \left[\frac{e^{na} - 1}{a} \right]. \tag{12}$$

IV. TRANSMISSION LINE WITH RANDOM BENDS

We will now assume the axis of the beam waveguide is bent so that there is some correlation between adjacent lateral lens displacements.

As noted in Section III this correlation will appear in (5) in the last two terms, $\langle (M_n R_{n-1}) \times V_n \rangle$ and $\langle V_n \times (M_n R_{n-1}) \rangle$.

It is shown in the Appendix that these two terms to first order do not contain σ_L^2 or σ_C^2, i.e., they are not affected by the random variations in focal length and spacing. It is also shown that if the axis of the guide is composed of a series of uncorrelated bends whose average bend length is much smaller than the total length of the transmission line, these two terms are not functions of n. This type of bending might typically be the case for a very long transmission line laid to follow the gentle bends of the terrain.

It was shown in Section III that to first order $\langle V_n \times V_n \rangle$ also has these properties, i.e., it is not a function of σ_L^2, σ_C^2, or n. Because of this similarity, the matrix $\langle (M_n R_{n-1}) \times V_n \rangle + \langle V_n \times (M_n R_{n-1}) \rangle$ can be considered as an added part of $\langle V_n \times V_n \rangle$ and can be carried through the analysis in this manner. Hence, the random errors in focal length and spacing affect the beam displacement due to short uncorrelated bends in the same way they affected the beam displacement due to random lens displacements.

From (10) or (11) for "a" small we can show how σ_L^2 and σ_C^2 couple to the random displacements by writing

$$\langle r_n^2 \rangle = \langle r_n^2 \rangle_{a=0} \frac{e^{na} - 1}{na} .$$

In this expression, $\langle r_n^2 \rangle_{a=0}$ is the expected value of the square of the beam displacement due to random lens displacements when $a = 0$. Because of the similarity pointed out above, the beam displacement due to short random bends is also multiplied by $(e^{na} - 1)/na$ to account for the focal length and spacing errors. Let us assume a transmission line axis is specified which fits the conditions, i.e., it is composed of a series of uncorrelated bends whose bend length is much shorter than the total line length. From this we can calculate $\langle r_n^2 \rangle$ assuming L and C are perfect. This has been done for some specific cases by Marcuse,[7] Berreman,[6] and Unger.[8] To include random imperfections of L and C if a is small we multiply this result by

$$\frac{e^{na} - 1}{na} .$$

This analysis does not hold if the correlation extends along the entire line (for example a serpentine bend) or if the correlation extends over a large portion of the line.

V. STATISTICAL GROWTH OF BEAM SPOT SIZE

We have been concerned thus far with the behavior of light rays in an imperfect transmission line. Our primary concern, however, is the behavior of Gaussian light beams rather than light rays. It has previously been shown that in the paraxial approximation the center of a Gaussian light beam does behave like a ray.[9] We can regard r_n and r_n' therefore, as the position and slope of the center of a Gaussian beam at lens n and r_0 and r_0' as the initial conditions of the beam center. We lack information on the effects of the transmission line imperfections on the size of the beam.

Using the complex beam parameter law of Kogelnik[13] it would be possible to find the statistical growth of the spot size due to the line imperfections. It will be simpler, however, to use Steier's ray packet equivalent to the Gaussian beam.[10] This approach conveniently uses the already derived statistical behavior of the light rays to find the beam size behavior.

Just to the right of a lens, the ray packet equivalent of the normal Gaussian mode of the transmission line is

$$\left.\begin{aligned} r_0 &= w_0 \cos\varphi + \frac{L}{kw_0}\sin\varphi \\ r_0' &= \frac{-2}{kw_0}\sin\varphi \end{aligned}\right\} \varphi \text{ has all values from 0 to } 2\pi \tag{13}$$

where

$$w_0 = \text{spot size at the beam waist} = \left[\frac{L(4 - LC)}{k^2C}\right]^{\frac{1}{4}}$$

$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \text{wavelength.}$$

If the path through the transmission line of each ray of the packet is found then the behavior of the equivalent Gaussian mode through the transmission line can be found. At any point in the transmission line, the envelope or the distance between the extreme rays of the ray packet is equal to twice the beam spot size and the curves which are perpendicular to the average ray slope are the beam phase fronts.

To find the effect of transmission line imperfections on the beam spot size, let us launch the ray packet given by (13) into the transmission line. If we substitute the value for r_0 and r_0' from (13) into (10) for the nonconfocal case we find

$$\langle r_n^2 \rangle = \frac{2w_0^2}{4 - LC} e^{na} + \frac{2w_0^2}{4 - LC} e^{-(na/2)} [\cos 2\varphi \cos (2n - 1)\theta - \sin 2\varphi \sin (2n - 1)\theta] \tag{14}$$

where φ ranges from 0 to 2π. We have not included the last term of (10) since the lateral lens displacements have no effect on the growth of the spot size.

The expected value of the square of the spot size at the nth lens, $\langle w_n^2 \rangle$, is given by the envelope of these rays. Taking the maximum value of (14) as φ goes from 0 to 2π.

$$\langle w_n^2 \rangle = \frac{2w_0^2}{4 - LC} (e^{na} + e^{-(na/2)}).$$

The normal mode spot size at a lens, w, is given by

$$w^2 = \frac{4w_0^2}{4 - LC}.$$

Hence, for the nonconfocal case

$$\frac{\langle w_n^2 \rangle}{w^2} = \frac{e^{na} + e^{-(na/2)}}{2} \tag{15}$$

where

$$a = \frac{2LC}{4 - LC} (\sigma_L^2 + \sigma_C^2).$$

For the confocal case, $LC = 2$, we substitute from (13) into (11). Taking the maximum value as φ ranges from 0 to 2π.

$$\frac{\langle w_n^2 \rangle}{w^2} = \frac{1}{2} \left\{ e^{na} + \frac{\sigma_C^2 e^{-na} + \sigma_L^2}{\sigma_C^2 + \sigma_L^2} \right\}. \tag{16}$$

If na is small so that

$$e^{na} \approx 1 + na,$$

then for the nonconfocal case (15) becomes

$$\frac{\langle w_n^2 \rangle}{w^2} = 1 + \frac{LCn}{2(4 - LC)} (\sigma_L^2 + \sigma_C^2), \tag{17}$$

and for the confocal case (16) becomes

$$\frac{\langle w_n^2 \rangle}{w^2} = 1 + n\sigma_L^2. \tag{18}$$

Equations (17) and (18) agree with the results of the perturbation analysis of Hirano and Fukatsu.[5]

Hence, for na small, the random errors in C do not affect the spot size in the confocal case. However, as pointed out in Section 3.3, LC must be almost 2 for this to be true. If $| LC - 2 | > \sigma_L^2 + \sigma_C^2$ the result for a nonconfocal system should be used. Since $\sigma_L^2, \sigma_C^2 \approx 10^{-4}$, this is a very stringent requirement on LC. It is doubtful if LC can be held close enough to 2 to gain this advantage in reduced spot size growth.

If $na > 2$ then the nonconfocal and the confocal results are very nearly the same and for both cases

$$\frac{\langle w_n^2 \rangle}{w^2} \approx \frac{1}{2} e^{na}.$$

VI. SUMMARY

The results derived here show statistically how imperfections in an optical transmission line affect the output beam from the transmission line. The imperfections cause the beam center to wander from the transmission line axis and cause the beam size to grow. We have considered the errors in focal length and spacing to be random and the lateral lens displacements to be random or with short correlation lengths. For this case, statistically the size of the beam and the distance of the beam center from the axis grow exponentially as the number of lenses when there are many lenses.

For $na > 2$, and random lateral lens displacements, the results can be summarized as follows. The beam center launched at r_0, r_0' has an rms expected value of

$$\sqrt{\langle r_n^2 \rangle} = \sqrt{\frac{2}{4 - LC}}\left[e^{na}\left(r_0^2 + r_0 L r_0' + \frac{L^2 r_0'^2}{LC}\right) + LCy^2 \frac{e^{na} - 1}{a}\right]^{\frac{1}{2}}.$$

The beam spot size has an rms expected value of

$$\sqrt{\langle w_n^2 \rangle} = e^{na/2} \frac{w}{\sqrt{2}}.$$

For random transmission line bends of short correlation length the random variations in L and C increase the expected value of the square of the output beam displacement as

$$\langle r_n^2 \rangle = \frac{e^{na} - 1}{na} \langle r_n^2 \rangle_{a=0}$$

where $\langle r_n^2 \rangle_{a=0}$ is the value computed assuming no variations in L and C.

If a beam is launched into a straight line on axis with no slope its position is not affected by random errors in f and L and only the size of the beam is affected. This is obviously true since a ray through the center of a lens does not bend no matter what the lens focal length. If, however, the axis is curved or the lenses have random lateral displacements, the beam begins to wander from the axis and is now affected by the errors in f and L. This coupling is clearly shown in these results.

These calculations are pertinent when n is large. This is the case of a transmission line with relatively closely spaced lenses which would be able to control the light beam around gentle bends in the terrain.

As an example, let us consider a confocal beam waveguide with lenses spaced every one meter and built to the following rms expected value tolerances:

(*i*) focal length variations — 1 per cent
(*ii*) spacing variations — much less than 1 per cent
(*iii*) random lateral lens displacements — 2×10^{-2} mm.

These tolerances give

$$a = 2 \times 10^{-4}$$

$$y^2 = 4 \times 10^{-10} \text{ m}^2.$$

If we assume an rms output beam deviation of 2 millimeters is acceptable, we can go approximately 3.5 kilometers ($n = 3.5 \times 10^3$) with the line described above. If the line is allowed to have gentle circular bends of an average radius of curvature of 5 kilometers and an average bend length[7] of 100 meters then the distance which can be traveled before there is an rms beam deviation of 2 millimeters is reduced to 2.5 kilometers ($n = 2.5 \times 10^3$). This means a beam redirector is required every 2.5 kilometers.

In the above example, the dominant Gaussian mode spot size at each lens for the perfect line is 0.45 mm rad. Because of the line imperfections, this grows to an rms expected value of 0.48 mm rad at $n = 2.5 \times 10^3$. This spot size growth is insignificant compared to the beam deflection. In general, unless LC is very small the spot size growth is not as important as the beam deflection growth for closely spaced lenses.

In calculating these numbers we have assumed the lenses are perfect and have neglected any aberrations. Additional work is required to determine the effects of aberrations on these results.

These numbers were calculated at $LC = 2$. If we make LC smaller the effect of the random lens displacements becomes less but the effect of correlated bends becomes larger.[7] At very small LC, the effect of spot size growth becomes important. If we increase LC the effect of correlated bends is reduced[7] but the effect of random lens displacements is increased. Clearly there is an optimum LC depending on line construction tolerances and line laying tolerances. It appears this optimum is near $LC = 2$ in a typical case.

Fig. 2 shows rms expected beam deviation as a function of n for a confocal line. This clearly shows how the distance between redirectors must be reduced as the errors in f and L become larger.

VII. ACKNOWLEDGMENTS

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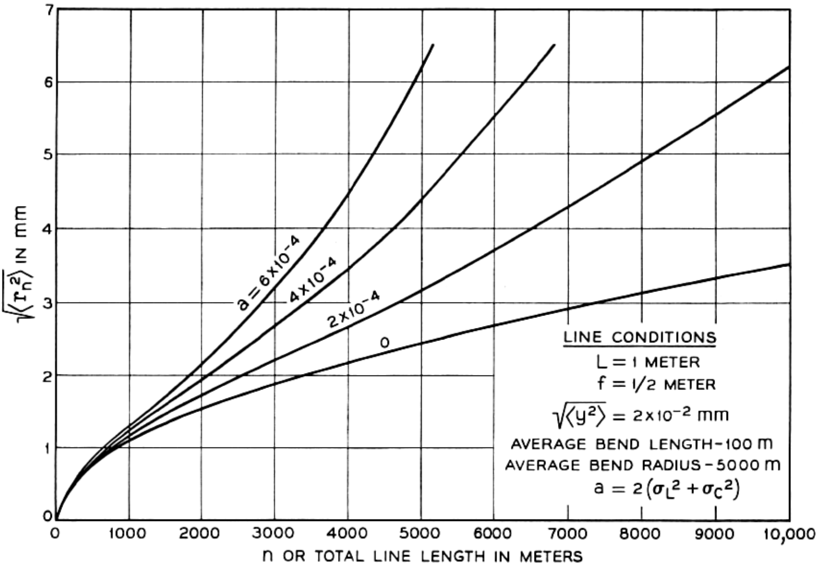

Fig. 2 — Expected value of the beam deflection as a function of transmission line length.

APPENDIX

Analysis of Matrices for Bent Waveguide Axis

We are interested in the last two terms of (5) since they contain the correlation between lateral lens displacements. These terms are

$$\langle (M_nR_{n-1}) \times V_n \rangle + \langle V_n \times (M_nR_{n-1}) \rangle.$$

For simplicity let us consider only $\langle (M_nR_{n-1}) \times V_n \rangle$, since the two terms are very similar. By repeated substitution of

$$R_n = M_nR_{n-1} + V_n$$

into

$$(M_nR_{n-1}) \times V_n ,$$

we find

$$(M_nR_{n-1}) \times V_n = (M_nP_1R_0) \times V_n + \sum_{k=2}^{n} (M_nP_kV_{k-1}) \times V_n \qquad (19)$$

where

$$P_k = M_{n-1}M_{n-2}M_{n-3} \cdots M_{k+1}M_k .$$

If we take the expected value of (19), the first term on the right side is zero since it is the product of independent terms and $\langle V_n \rangle = 0$.

We must look at the individual elements of $(M_nP_kV_{k-1}) \times V_n$. We can write the matrix P_k as

$$P_k = \begin{bmatrix} p_{k(1,1)} & p_{k(1,2)} \\ p_{k(2,1)} & p_{k(2,2)} \end{bmatrix}.$$

The p_k contain $L_{n-1}, L_{n-2}, \cdots, L_k$, and $C_{n-1}, C_{n-2}, \cdots, C_k$ but only to the first power.

$$M_nP_k = \begin{bmatrix} p_{k(1,1)} + L_np_{k(2,1)} & p_{k(1,2)} + L_np_{k(2,2)} \\ -C_np_{k(1,1)} + p_{k(2,1)}(1 - L_nC_n) & -C_np_{k(1,2)} + p_{k(2,2)}(1 - L_nC_n) \end{bmatrix}$$

and

$$(M_nP_kV_{k-1}) \times V_n = \begin{bmatrix} 0 \\ p_{k(1,2)}C_n + p_{k(2,2)}L_nC_n \\ -p_{k(1,2)}C_n^2 + p_{k(2,2)}(C_n + L_nC_n^2) \end{bmatrix} C_{k-1}s_{k-1}s_n .$$

From this result we can write the last two terms in (5) as

$$\langle (M_n R_{n-1}) \times V_n \rangle + \langle V_n \times (M_n R_{n-1}) \rangle$$

$$= \sum_{k=2}^{n} \begin{bmatrix} 0 \\ \langle p_{k(1,2)} \rangle C + LC \langle p_{k(2,2)} \rangle \\ \langle p_{k(1,2)} \rangle C + LC \langle p_{k(2,2)} \rangle \\ 2C^2(1 + \sigma_C^2)(L\langle p_{k(2,2)} \rangle - \langle p_{k(1,2)} \rangle) \\ \qquad + 2\langle p_{k(2,2)} \rangle C \end{bmatrix} C \langle s_{k-1} s_n \rangle. \quad (20)$$

Since the p_k contain the L's and C's only to the first power in each, $\langle p_k \rangle$ will contain only L and C and will not contain σ_L^2 and σ_C^2. If we neglect σ_C^2 as compared to 1 (the same approximation is used in $\langle V_n \times V_n \rangle$), then $\langle (M_n R_{n-1}) \times V_n \rangle + \langle V_n \times (M_n R_{n-1}) \rangle$ does not contain σ_L^2 or σ_C^2.

We will now find under what conditions $\langle (M_n R_{n-1}) \times V_n \rangle + \langle V_n \times (M_n R_{n-1}) \rangle$ is independent of n.

The n dependence of (20) is in the terms

$$\sum_{k=2}^{n} \langle p_{k(1,2)} \rangle \langle s_{k-1}\, s_n \rangle$$

and

$$\sum_{k=2}^{n} \langle p_{k(2,2)} \rangle \langle s_{k-1}\, s_n \rangle.$$

Since

$$\langle P_k \rangle = \langle M \rangle^{n-k}$$

$$\langle p_{k(1,2)} \rangle = \frac{L \sin\,(n-k)\,\theta}{\sin\theta}$$

$$\langle p_{k(2,2)} \rangle = -\frac{L^2 C \sin\,(n-k)\,\theta}{2 \sin\theta} + L \cos\,(n-k)\,\theta.$$

And we can write

$$\frac{\sin\theta}{L} \sum_{k=2}^{n} \langle p_{k(1,2)} \rangle \langle s_{k-1}\, s_n \rangle = \langle s_1 s_n \rangle \sin\,(n-2)\,\theta$$

$$+ \langle s_2 s_n \rangle \sin\,(n-3)\,\theta \cdots + \langle s_{n-2} s_n \rangle \sin\theta + \langle s_{n-1} s_n \rangle\, 0.$$

We will assume, as did Marcuse,[6] that

$$\langle s_k s_n \rangle = f(n-k).$$

The correlation depends on the distance between the lenses. We also assume that

$$f(n - k) = 0 \quad \text{for} \quad n - k > N.$$

That is, the correlation length is finite and extends only N lenses away. This means the waveguide axis is a series of random bends, the "average length" of each bend is NL. Therefore, if $n > N$ we can write

$$\frac{\sin\theta}{L}\sum_{k=2}^{n}\langle p_{k(1,2)}\rangle\langle s_{k-1}\, s_n\rangle = f(2)\sin\theta + f(3)\sin 2\theta$$
$$\cdots f\,(N-1)\sin\,(N-2)\theta + f(N)\sin\,(N-1)\theta,$$

which is not a function of n. We can write a similar equation for

$$\sum_{k=2}^{n}\langle p_{k(2,2)}\rangle\langle s_{k-1}\, s_n\rangle.$$

However, we must consider all n down to 1. For these small n, $\langle (M_nR_{n-1}) \times V_n\rangle + \langle V_n \times (M_nR_{n-1})\rangle$ will be a function of n. This means that all bends contribute the same to the output beam displacement except the initial bend which is within NL of the beginning of the transmission line. However, if we assume that $n \gg N$ (the average bend length is much smaller than the length of the transmission line) the contribution of this initial bend will be very small and can be neglected.

In summary, the conditions imposed on the transmission line axis are that it is composed of a series of random bends whose average length is NL. The average bend length is much smaller than the total length of the line. We have neglected the effect of any bend within NL of the beginning of the line. These are essentially the same conditions used by Marcuse,[7] Berreman,[6] and Unger[8] in their analyses of random bends of the transmission line.

Under these conditions $\langle (M_nR_{n-1}) \times V_n\rangle$ and $\langle V_n \times (M_nR_{n-1})\rangle$ are independent of n.

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