Signal-Noise Ratio Maximization Using the Pontryagin Maximum Principle

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The applicability of the Pontryagin maximum principle to signal-noise ratio maximization is explored. Attention is focused on the reformulation of the problem so that the maximum principle may be used. The basic aspect of the reformulation is to cast the problem into the form of differential equations instead of integral equations.

Two problems are solved. The first, a variation of the matched filter problem, could have been solved by other methods. However, the maximum principle provided a very neat and systematic approach. The second problem, signal design with both an energy and an amplitude constraint imposed on the signal, is solved numerically. It appears to be intractable by other methods. One of the advantages of the maximum principle formulation is that, by working with differential equations rather than with integral equations, numerical techniques may be more easily used.

I. INTRODUCTION

The Pontryagin maximum principle may be considered to be a generalization of methods of calculus of variations that permits solution of optimization problems with inequality contraints. During the last few years, it has been extensively used to attack control theory problems. The use of the principle to solve signal optimization problems is introduced in Ref. 1. The maximum principle is briefly discussed in connection with wave-form optimization in Ref. 2.

The purpose of this present paper is to develop techniques for the application of the maximum principle, in particular to problems of signal-noise ratio maximization. We shall show how the maximum principle may be used to solve some problems with inequality constraints (e.g., the amplitude of a signal may be constrained to be less than or equal to some maximum value) which were heretofore considered intractable. We shall also show how a problem, solvable by other methods, may be very conveniently attacked with the formalism of the maximum principle.

It is interesting to note that the maximum principle is, in a sense, more applicable to communication theory than to control theory for which it was originally developed (this is also pointed out in Ref. 1). The maximum principle yields a function of time to maximize a functional subject to constraints and for prescribed initial conditions. The answer to most communication theory problems is a function of time. On the other hand, in control problems, the function of time for specific initial conditions is called an "open loop" solution. What is actually needed is the "closed loop" or "feedback" solution which is a function of the present state. This is only indirectly determined using the maximum principle.

After introducing the maximum principle, we shall first solve a constrained matched filter problem and then a signal design problem. Attention will be focused on the reformulation of the problems so that the maximum principle is applicable.

II. THE MAXIMUM PRINCIPLE

We shall briefly discuss the maximum principle. Our discussion is an abstraction of some material in Ref. 3. Another excellent introduction to the maximum principle, which is presently available, is the Introduction and Chapter I of Ref. 4. Consider a system whose state is described by a vector $x = (x_1, x_2, \cdots x_n)$ which satisfies the differential equation

$$\dot{x} = f(x,u,t) \qquad t \in [t_0, t_1] \tag{1}$$

where $u = (u_1, \cdots, u_r)$ is an r-dimensional control vector and $f(x,u,t) = (f_1(x,u,t), f_2(x,u,t), \cdots, f_n(x,u,t))$ is a given n-dimensional vector valued function of x, u, and t. We require that

$$u(t) \in \Omega \qquad t \in [t_0, t_1] \tag{2}$$

where the set Ω is the set of admissible control vectors. Let F be the class of all piecewise continuous functions* from $[t_0, t_1]$ into Ω. If $\mathscr{u}$ is a control function in the class F we denote the trajectory corresponding to $\mathscr{u}$ by $x(t;\mathscr{u})$ which satisfies the following relations

$$\dot{x}(t;\mathscr{u}) = f(x(t;\mathscr{u}), u(t), t) \quad \text{a.e. } t \in [t_0, t_1] \tag{3}$$

$$x(t_0; \mathscr{u}) = 0. \tag{4}$$

* Corinthian script (e.g., $\mathscr{u}, \mathscr{v}$) is used to denote control functions. Small English letters (e.g., $u(t)$, $v(t)$) denote values of functions at specific times. The function $\mathscr{u}$ is the function whose value at time t is $u(t)$.

(There is no loss of generality in assuming zero initial conditions; a transformation of coordinates may be used for nonzero initial conditions.)

The optimization problem is as follows. Let $\{s_1, s_2, \cdots, s_m\}$ be a given set of real numbers where $0 \leqq m \leqq n - 1$. We prescribe the final values (at $t = t_1$) of the first m coordinates of the state vector x to be

$$\{s_1, s_2, \cdots, s_m\}$$

and we require the final value of x_n to be maximum. The optimization problem is formally stated as follows: we are given the set

$$S = \{x: x_i = s_i \quad \text{for} \quad i = 1, \cdots, m\} \tag{5}$$

and we want to find a control function v in the class F such that

(i) $x(t_1; v) \in S$

(ii) for all $u \in F$ such that (6)

$$x(t_1; u) \in S \tag{7}$$

the following relation holds:

$$x_n(t_1; u) \leqq x_n(t_1; v). \tag{8}$$

The control function v is called the optimal control function and $x(t;v)$ is the optimal trajectory.

The Pontryagin maximum principle is a necessary condition that an optimal control function must satisfy. To state the principle, we first define the Hamiltonian, $H(x,u,t,p)$,

$$H(x,u,t,p) = \langle f(x,u,t) \mid p \rangle \tag{9}$$

where $p = (p_1, \cdots, p_n)$ is an n-dimensional vector and $\langle a \mid b \rangle$ denotes the scalar product of a and b.

2.1 *Maximum Principle*

If v is an optimal control function then there exists a nonidentically zero continuous vector valued function $p(t)$ such that

(i) $H(x(t;v), v(t), t, p(t)) \geqq H(x(t;v), u, t, p(t))$

for a.e $t \in [t_0, t_1]$ and all $u \in \Omega$, (10)

$$(ii)\ \dot{p}(t) = -\left[\frac{\partial f(x,v(t),t)}{\partial x}\right]^T_{x=x(t;\upsilon)} \cdot p(t) \tag{11}$$

for a.e. $t \in [t_0, t_1]$
(superscript T denotes transpose),

$$(iii)\ p_i(t_1) = 0 \qquad i = m+1, \cdots, n-1, \tag{12}$$

$$(iv)\ p_n(t_1) \geqq 0. \tag{13}$$

Relation (i) states that the Hamiltonian, evaluated along the optimal trajectory, takes on its maximum value with $v(t)$. Note that the maximization is over u, with $x(t;\upsilon)$ and $p(t)$ held fixed.

Relation (ii) may alternatively be expressed as follows:

$$\dot{p}_i(t) = -\sum_{j=1}^{n}\left[\frac{\partial^j f(x,v(t))}{\partial x_i}\right]_{x=x(t;\upsilon)} \cdot p_j(t), \qquad i = 1, 2, \cdots, n. \tag{14}$$

Relation (iii) states that the final value of an element of the vector $p(t)$ is zero if it corresponds to an element of the vector $x(t)$ which is left free at $t = t_1$.

Relation (iv) states that the nth element of $p(t)$, which corresponds to the element of $x(t)$ which is being maximized at $t = t_1$, is nonnegative at $t = t_1$.

Verification that the maximum principle is satisfied is seen to be equivalent to verification that a set of differential equations with mixed boundary values is satisfied. Conditions on $x(t;\upsilon)$ must be satisfied at both t_0 and t_1 and $p(t)$ must satisfy conditions at t_1. This boundary value problem is not always solvable analytically but much progress has been made in the numerical solution of such problems.

Communication theory problems are not usually stated in the form just described with differential equations. So the first order of business in applying the maximum principle to a communication theory problem is to convert it into the appropriate form with differential equations.

III. A MATCHED FILTER PROBLEM

To illustrate the formalism involved, we first solve a variation of the matched filter problem. The use of the maximum principle, in this case, is actually equivalent to using the classical calculus of variations. The basic matched filter problem is as follows (Ref. 5, p. 244). We have signal, $s_1(t)$, and noise, $n(t)$, entering a linear filter and we wish to design the linear filter so that the output signal-noise ratio is maximized at a specific time, t_1. This problem can be trivially solved using the maximum principle and by other methods. To make the problem a little more interesting, suppose that we also specify that the output to a second signal input, $s_2(t)$, is to be equal to some real number, α. For example, if we chose $\alpha = 0$, we could be interested in detecting the presence of $s_1(t)$ while discriminating against $s_2(t)$.

We assume white noise with correlation function

$$R_n(t - u) = N\delta(t - u). \tag{15}$$

Then the mean square noise at t_1, $\sigma^2(t_1)$, is (if we start the problem at $t = 0$ and if we employ the usual formal operations with white noise)

$$\sigma^2(t_1) = N \int_0^{t_1} h^2(\tau)d\tau \tag{16}$$

where $h(t)$ is the impulse response of the linear filter. The outputs due to $s_1(t)$ and $s_2(t)$ at $t = t_1$ are, respectively,

$$y_1(t_1) = \int_0^{t_1} h(\tau)\ s_1(t_1 - \tau)\ d\tau \tag{17}$$

$$y_2(t_1) = \int_0^{t_1} h(\tau)\ s_2(t_1 - \tau)\ d\tau. \tag{18}$$

The problem is then to choose $h(t)$ to maximize

$$\frac{[y_1(t_1)]^2}{\sigma^2(t_1)} \tag{19}$$

while satisfying the relationship

$$y_2(t_1) = \alpha. \tag{20}$$

If we let

$$\dot{x}_1(t) = u(t)s_1(t_1 - t), \tag{21}$$

$$\dot{x}_2(t) = u(t)s_2(t_1 - t), \tag{22}$$

$$\dot{x}_3(t) = -Nu^2(t), \tag{23}$$

$$x_1(0) = x_2(0) = x_3(0) = 0, \tag{24}$$

and if we identify $u(t)$ with $h(t)$, then

$$x_1(t_1) = y_1(t_1), \tag{25}^*$$

$$x_2(t_1) = y_2(t_1), \tag{26}$$

$$x_3(t_1) = -\sigma^2(t_1). \tag{27}$$

* Note that for $t \neq t_1$, $x_1(t)$ does not necessarily equal $y_1(t)$.

An equivalent problem is to choose $u(t)$ so as to maximize $x_3(t_1)$ subject to

$$x_1(t_1) = 1 \tag{28}$$

$$x_2(t_1) = \alpha. \tag{29}$$

That is, we can minimize $\sigma^2(t_1)$ with $y_1(t_1)$ constrained since the signal-noise ratio is not changed if $h(t)$ is multiplied by a constant factor.

Now that we have recast the problem into differential equation form, we can solve it using the maximum principle. Using (14) we see that, since f is independent of x,

$$p_1(t) = \text{constant} \equiv p_1 \tag{30}$$

$$p_2(t) = \text{constant} \equiv p_2 \tag{31}$$

$$p_3(t) = \text{constant} \equiv 1 \tag{32}$$

(we let $p_3(t) = 1$ for convenience).*
The Hamiltonian, H, is

$$H = p_1 u(t) s_1(t_1 - t) + p_2 u(t) s_2(t_1 - t) - N u^2(t). \tag{33}$$

Since there are no constraints on $u(t)$, we maximize H by differentiating and get

$$u(t) = \frac{1}{2N} [p_1 s_1(t_1 - t) + p_2 s_2(t_1 - t)]. \tag{34}$$

To satisfy (28) and (29), we must have

$$\frac{1}{2N} \int_0^{t_1} [p_1 s_1(t_1 - \tau) + p_2 s_2(t_1 - \tau)]\, s_1(t_1 - \tau)\, d\tau = 1 \tag{35}$$

$$\frac{1}{2N} \int_0^{t_1} [p_1 s_1(t_1 - \tau) + p_2 s_2(t_1 - \tau)]\, s_2(t_1 - \tau)\, d\tau = \alpha. \tag{36}$$

We can then solve for p_1 and p_2 and get

$$p_1 = \frac{2N(S_2 - \alpha S_{12})}{S_1 S_2 - S_{12}^2} \tag{37}$$

$$p_2 = \frac{2N(\alpha S_1 - S_{12})}{S_1 S_2 - S_{12}^2}, \tag{38}$$

* This involves an assumption of normality (in the sense of the classical calculus of variations).

where

$$S_1 = \int_0^{t_1} s_1^2(t_1 - \tau)\, d\tau \tag{39}$$

$$S_2 = \int_0^{t_1} s_2^2(t_1 - \tau)\, d\tau \tag{40}$$

$$S_{12} = \int_0^{t_1} s_1(t_1 - \tau)\, s_2(t_1 - \tau)\, d\tau. \tag{41}$$

As a simple example, let

$$s_1(t) = 1 \quad t \in [0,t_1] \tag{42}$$

$$\begin{aligned} s_2(t) &= 1 \quad t \in [0,t_1/2] \\ &\, 0 \quad t \in (t_1/2,t_1]. \end{aligned} \tag{43}$$

We would then get

$$u(t) = h(t) = \frac{2(1-\alpha)}{t_1}\, s_1(t_1 - t) + \frac{2}{t_1}\,(2\alpha - 1)\, s_2(t_1 - t). \tag{44}$$

IV. A SIGNAL DESIGN PROBLEM

The following problem is taken from the thesis by M. I. Schwartz (Ref. 6). The system is depicted in Fig. 1. We have a signal passing through a linear time-invariant filter, represented by impulse response function $h(t)$, after which the signal is corrupted by noise. The resultant signal plus noise is processed by a correlation-type receiver. The object is to maximize the signal-noise ratio at the output of the receiver at $t = t_1$ by choosing forms for both $s(t)$ and receiver function $q(t)$. M. I.

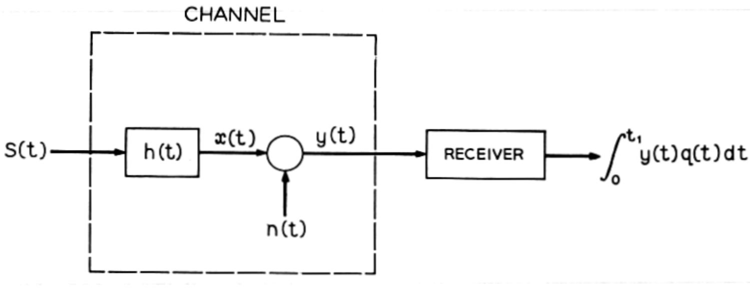

Fig. 1 — Signal design problem.

Schwartz solves this problem with an energy constraint on $s(t)$. We will show how to simultaneously handle an energy constraint and an amplitude constraint on $s(t)$. That is, we require that

$$\int_0^{t_1} s^2(\tau)\, d\tau = \mathcal{E} \tag{45}$$

and

$$| s(t) | \leqq S_{\max} . \tag{46}$$

To make the problem meaningful, we require that

$$(S_{\max})^2 t_1 > \mathcal{E}.$$

It may be easily shown that the problem is equivalent to the problem with the equality of (45) replaced by $\leqq$.

Again, we assume that the noise is white, i.e.,

$$R_n(t - u) = N\delta(t - u), \tag{47}$$

and we further specify that $h(t)$ has a rational Fourier transform with just poles* for simplicity. The assumption of rational Fourier transform facilitates recasting the problem into the differential equation form. Thus,

$$\mathcal{F}[h(t)] = \frac{\alpha}{D(i\omega)} \tag{48}$$

where α is a real number and $D(i\omega)$ is a polynomial in $i\omega$. Letting

$$D(i\omega) = (i\omega)^l + a_1(i\omega)^{l-1} + a_2(i\omega)^{l-2} + \cdots + a_{l-1}(i\omega) + a_l , \tag{49}$$

(i.e., we have an lth order differential equation in $h(t)$) and

$$x_1 = x \tag{50}$$

$$x_2 = \dot{x} \tag{51}$$

$$\vdots$$

$$x_l = x^{(l-1)} \tag{52}$$

we can represent the effect of $h(t)$ by the following set of first-order differential equations

* If we also assumed zeros in the Fourier transform, we would solve for a derivative of $h(t)$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \\ \dot{x}_l \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & & & 1 \\ -a_l & -a_{l-1} & -a_{l-2} & & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_l \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \alpha s \end{bmatrix} \tag{53}$$

$$x_1(0) = x_2(0) = \cdots = x_l(0) = 0.$$

We have converted an lth order differential equation into l first-order differential equations.

Let

$$x^* = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{bmatrix} \tag{54}$$

and let B be the $l \times l$ matrix in (53). Then (53) may be more concisely written as

$$\dot{x}^* = B\, x^* + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha s \end{bmatrix} \tag{55}$$

$$x^*(0) = 0.$$

Now that we have put the effect of $h(t)$ into differential equation form, it remains to cast the signal-noise considerations into differential equations. Recall that the object is to maximize $[s_0(t_1)]^2/\sigma^2(t_1)$ subject to (45) and (46) where

$$s_0(t_1) = \int_0^{t_1} dt\, q(t) \int_0^t du\, s(u)\, h(t-u) \tag{56}$$

$$\sigma^2(t_1) = N \int_0^{t_1} q^2(t)\, dt. \tag{57}$$

If we let

$$\dot{z}_1(t) = s^2(t) \tag{58}$$

$$\dot{z}_2(t) = Nq^2(t) \tag{59}$$

$$\dot{z}_3(t) = x_1(t)q(t) \qquad (z_3(t_1) = s_0(t_1)) \tag{60}$$

$$\dot{z}_4(t) = 2z_3(t)x_1(t)q(t) \qquad (z_4(t_1) = s_0^2(t_1)) \tag{61*}$$

$$z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0 \tag{62}$$

$$\text{control vector} = u(t) = (s(t), q(t)),$$

then the optimization problem is to choose $s(t)$, subject to relation (46), and $q(t)$ to maximize $z_4(t_1)$ (which equals $s_0^2(t_1)$) subject to $z_1(t_1) = \mathcal{E}$ and $z_2(t_1) = \sigma^2$. That is, we fix $\sigma^2(t_1)$ at some arbitrary real number and maximize $s_0^2(t_1)$. Just as in the matched filter problem, multiplying $q(t)$ by a constant does not affect the signal-noise ratio.

Now our state vector is the $(l + 4)$-vector, $(x^*, z_1, z_2, z_3, z_4)$. Equation (14) will take the following form ($p(t)$ is an $(l + 4)$-vector):

$$\dot{p}(t) = -\begin{bmatrix} \begin{bmatrix} B^T \\ (l \times l) \\ \\ \\ \end{bmatrix} & \begin{matrix} 0 & 0 & q & 2z_3q \\ 0 & 0 & 0 & 0 \\ & & & \vdots \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2x_1q \\ 0 & 0 & 0 & 0 \end{matrix} \end{bmatrix} p(t) \tag{63}$$

(B is the $l \times l$ matrix in (53)).
The final conditions on $p(t)$ are

$$p_1(t_1) = 0 \tag{64}$$

$$p_2(t_1) = 0 \tag{65}$$

$$\vdots$$

$$p_l(t_1) = 0, \tag{66}$$

$p_{l+1}(t_1)$, $p_{l+2}(t_l)$ unspecified (they correspond to $z_1(t)$ and $z_2(t)$ which are fixed at $t = t_1$)

* This differential equation is derived as follows:

$$dt\ (z_3^2(t)) = 2z_3(t)\dot{z}_3(t) = 2z_3(t)x_1(t)q(t)$$

$$z_4(t_1) = z_3^2(t_1) = s_0^2(t_1).$$

$$p_{l+3}(t_1) = 0 \tag{67}$$

$$p_{l+4}(t_1) \geqq 0. \tag{68}$$

An example of the use of the maximum principle for this signal design problem is given in the next section.

V. EXAMPLE OF SIGNAL DESIGN

We shall consider the case of

$$\mathfrak{F}[h(t)] = \frac{\alpha}{i\omega + \alpha}. \tag{69}$$

Following the method described in Section IV, the problem is first recast into the following differential equation form

$$\dot{x} = -\alpha x + \alpha s \tag{70}$$

$$\dot{z}_1 = s^2 \qquad (z_1(t_1) = \mathcal{E}) \tag{71}$$

$$\dot{z}_2 = Nq^2 \qquad (z_2(t_1) = \sigma^2) \tag{72}$$

$$\dot{z}_3 = xq \qquad (z_3(t_1) = s_0(t_1)) \tag{73}$$

$$\dot{z}_4 = 2z_3xq \qquad (z_4(t_1) = s_0^2(t_1)) \tag{74}$$

$$x(0) = z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0. \tag{75}$$

For this problem the necessary condition (maximum principle) for an optimal solution is that there exists a nonzero vector function $p(t) = (p_1(t), p_2(t), \cdots, p_5(t))$ such that

$$H = p_1(-\alpha x + \alpha s) + p_2 s^2 + p_3 N q^2 + p_4 x q + p_5 2 z_3 x q \tag{76}$$

is maximized over the allowable s and q and such that

$$\dot{p}_1 = \alpha p_1 - q p_4 - 2z_3 q \tag{77}$$

$$\dot{p}_2 = 0 \tag{78}$$

$$\dot{p}_3 = 0 \tag{79}$$

$$\dot{p}_4 = -2xq \tag{80}$$

$$\dot{p}_5 = 0 \tag{81}$$

$$p_1(t_1) = 0 \tag{82}$$

$$p_4(t_1) = 0 \tag{83}$$

$$p_5(t_1) \geqq 0 \tag{84}$$

(we can let $p_5(t_1) = p_5(t) = 1$ under a normality assumption).
The maximization of H leads to

$$q(t) = -\left[\frac{p_4(t)\,x(t) + 2z_3(t)\,x(t)}{2p_3N}\right] \qquad (85)^*$$

$$s(t) = s^*(t) \qquad \text{if} \quad |s^*(t)| \leqq S_{\max} \qquad (86)$$

$$= \frac{s^*(t)}{|s^*(t)|} S_{\max} \quad \text{if} \quad |s^*(t)| > S_{\max}, \qquad (87)$$

where

$$s^*(t) = \frac{-p_1(t)\alpha}{2p_2}.$$

To verify satisfaction of the maximum principle, it is necessary to solve the differential equations (70) to (74) and (77) to (81) with satisfaction of the above mentioned initial and final conditions and in such a way that maximization of the Hamiltonian is satisfied.

The numerical method of satisfying the maximum principle is based on iteration of the initial values of p-vector to successively improve the final conditions. That is, we know the initial conditions of x, z_1, z_2, z_3, z_4 (see (75)) and we wish to constrain the final values of z_1, z_2, p_1, and p_4:

$$z_1(t_1) = \mathcal{E} \qquad (88)$$

$$z_2(t_1) = \sigma^2 \qquad (89)$$

$$p_1(t_1) = 0 \qquad (90)$$

$$p_4(t_1) = 0. \qquad (91)$$

Suppose we guess at $p(0) = (p_1(0), p_2(0), p_3(0), p_4(0), 1)$ and integrate the differential equations (70) to (74) and (77) and (81) and evaluate the following error in final conditions

$$E = |z_1(t_1) - \mathcal{E}| + |z_2(t_1) - \sigma^2| + |p_1(t_1)| + |p_4(t_1)|. \qquad (91)\dagger$$

We wish to decrease E. To this end, let

$$(p_1(0))_{\text{new}} = (p_1(0))_{\text{old}} + \delta p_1(0) \qquad (92)$$

and re-integrate the differential equations. If E decreases, change

* Since $\dot{p}_4(t) + 2\dot{z}_3(t) = 0$, $q(t)$ is proportional to $x(t)$. Thus, we are equivalently maximizing the signal energy into the receiver (see (73)) and then correlating with $x(t)$. This is consistent with (and, in fact, rederives) well-known properties of matched filters.

† Actually, the second term is not required as σ^2 can be adjusted by changing $q(t)$ by a multiplicative factor (without changing the signal-noise ratio).

$p_2(0)$. If E does not increase, try

$$(p_1(0)_{\text{new}} = (p_1(0))_{\text{old}} - \delta p_1(0).$$

Again see whether E has decreased. If it has, change $p_1(0)$ to $(p_1(0))_{\text{new}}$ and change $p_2(0)$. If E has not decreased, retain $(p_1(0))_{\text{old}}$ and try changing $p_2(0)$.

Thus, the method is to successively change $p_1(0)$, $p_2(0)$, $p_3(0)$, $p_4(0)$ to decrease E. When E becomes sufficiently small, the maximum principle may be said to be satisfied. After we present some results, we will discuss the method further.

5.1 *Results*

Two cases were run. They were for the following parameters:

$$t_1 = 1$$

$$\mathcal{E} = 1$$

$$N = 1$$

$$\alpha = 1.$$

(As mentioned previously, σ^2 is determined by the scaling of $q(t)$.) The difference between the two cases is that in the first, the amplitude constraint was not imposed and in the second, $S_{\max}$ was set at 1.1. The first case was already treated by other methods in Ref. 6. Our results for that case were in agreement with those of Ref. 6. Fig. 2 shows $q(t)$ for both cases and Fig. 3 shows $s(t)$. For the case of no amplitude constraint the signal-noise ratio $(\sqrt{s_o^2(t_1)/\sigma^2})$ was 0.44 and the signal-noise ratio for the amplitude constrained case was 0.43. One could (nonoptimally) impose the amplitude constraint by scaling down the results for the amplitude unconstrained case (and not use all the signal energy available). That is, the peak amplitude of the signal in the first case is 1.27. The entire signal ($s(t)$ for $t \in [0,t_1]$) could be reduced by a factor of 1.1/1.27. The signal-noise ratio would also be reduced by that factor (0.865). Whether or not this signal-noise ratio reduction is significant is not actually germane to this investigation. What is of consequence is the fact that the optimum can be determined and any sub-optimum scheme can be compared with it.

5.2 *Comments on the Numerical Method*

The basic method is similar to that of Ref. 7. In Ref. 7, the gradient (relating changes in the error to changes in $p(t_0)$) is evaluated and used

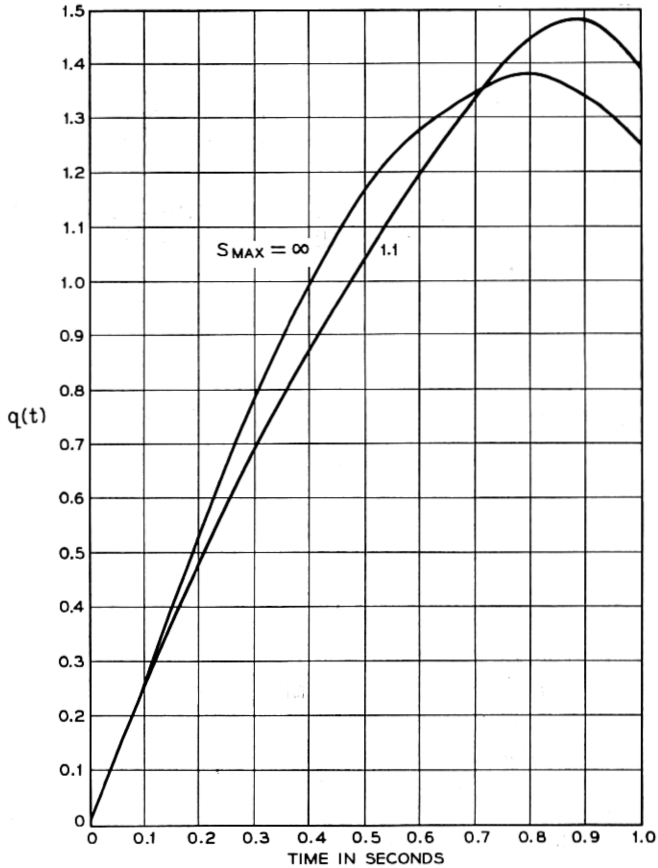

Fig. 2 — Optimum correlation waveforms.

to determine a steepest descent change in $p(t_0)$. This was not done in the present problem for two reasons. First of all, the evaluation of the gradient (as done in Ref. 7) is not valid for the problem with inequality constraints. Secondly, even if the gradient can be conveniently evaluated, it still requires extra integrations and the problem of step size is left unresolved. (This is not intended as criticism of the method of Ref. 7 which may be quite useful in many applications.) We decided to frankly treat the problem as a systematic trial and error. Our method of seeing how the error changes as $p_i(t_0)$ is changed to $p_i(t_0) + \delta p_i(t_0)$ may be loosely interpreted as evaluating $\partial E/\partial p_i(t_0) \cdot \delta p_i(t_0)$.

The numerical method may be considered to be semiautomatic. There is little *a priori* information available as to the initial choices of the

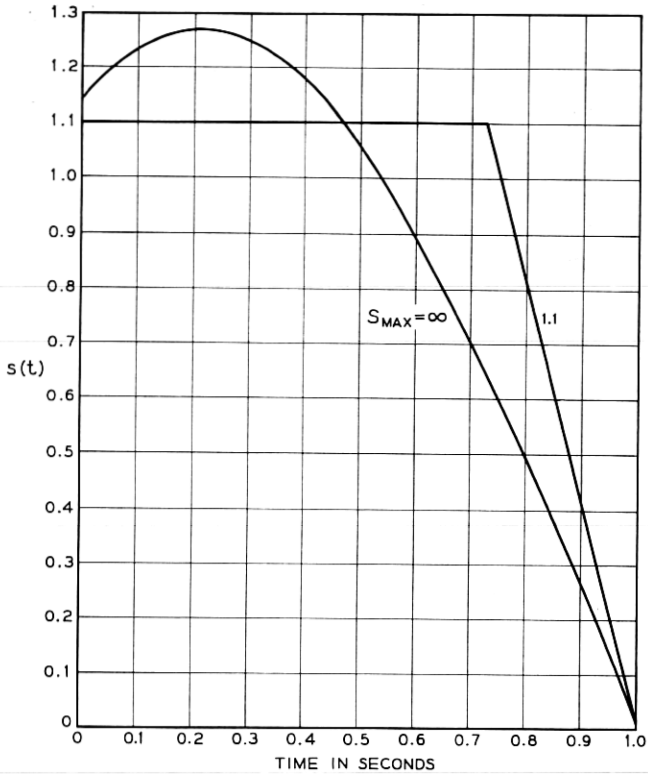

Fig. 3 — Optimum signal waveforms.

$p_i(t_0)$ and the $\delta p_i(t_0)$. A few runs on the computer offer the optimizer some insight as to appropriate choices. About 30 iterations were found to be needed for convergence (approximately 0.01 hours of computer time). No convergence proof is offered for this method. In fact, even though it did not happen in the problem considered in this paper, it is conceivable that $\partial E/\partial p_i(t_0)$ (assuming the derivative exists) can be so large that the smallest $\delta p_i(t_0)$ that can be used by the computer would result in a much too large change in E. There is also the possibility of local minima of E (with $E > 0$). These problems (which may not even occur; we are trying to anticipate the worst) could be presumably resolved by changing the metric defining E and by trying a wide range of $p_i(t_0)$.* It may be noted that convergence proofs do not seem to be avail-

* More efficient methods of adjusting the $p_i(t_0)$ may be possible. See, for example, Wilde, D. J., *Optimum Seeking Methods,* Prentice-Hall, 1962.

able for competitive algorithms (e.g., steepest descent) for these optimization problems.

VI. CONCLUSION

The maximum principle has been used here to attack two signal-noise ratio maximization problems. The first one (matched filter problem) could have been solved without the maximum principle. However, the maximum principle provided a very neat and systematic approach. The second problem (with the amplitude constraint included) appears to be unsolvable except by the maximum principle.* In this paper, it was assumed that the noise is white. The handling of non-white noise merits further attention, in particular, the conversion to differential equations and the presence of impulses (see Ref. 5, Appendix 2). It should also be noted that the maximum principle is not conceptually limited to time-invariant problems.

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REFERENCES

1. Schweppe, F. C., Optimization of Signals, Lincoln Laboratories Group Report 1964–4, January 16, 1964.
2. Tufts, D. W. and Shnidman, D. A., Optimum Waveforms Subject to Both Energy and Peak-Value Constraints, Proc. of the IEEE, September, 1964, pp. 1002–1007.
3. Halkin, H., Mathematical Foundations of System Optimization, to appear as a chapter in *Optimization Techniques*, Vol. 2, ed. by G. Leitmann.
4. Pontryagin, L. S., et al, *The Mathematical Theory of Optimal Processes*, (translation) John Wiley and Sons, Inc., New York, 1962.
5. Davenport, W. B., Jr. and Root, W. L., *An Introduction to the Theory of Random Signals and Noise*, McGraw-Hill Book Company, Inc., New York, 1958.
6. Schwartz, M. I., Binary Signal and Receiver Design for Linear Time Invariant Channels, Sc.D. Thesis, New York University, September, 1964.
7. Knapp, C. H. and Frost, P. A., Determination of Optimum Control and Trajectories Using the Maximum Principle in Association with a Gradient Technique, IEEE Trans. on Automatic Control, *AC-10*, No. 2, April, 1965, pp. 189–193.

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* Another possible approach to the problem would be to assume that $s(t) = S_{max} \sin [\theta(t)]$ and use classical methods to solve for $\theta(t)$ (since the problem of inequality constraint is thereby avoided). However, the maximum principle attacks the problem directly.

RECENT RELATED REFERENCES*

1. Schweppe, F. C. and Gray, D., Radar Amplitude Modulation Using Pontryagin's Maximum Principle, Lincoln Laboratory preprint, 1965.
2. Athans, M. and Schweppe, F. C., On the Design of Optimal Modulation Schemes Via Control Theoretic Concepts. I: Formulation, Lincoln Laboratory preprint, Aug. 31, 1965.

* I wish to thank Prof. M. Athans of MIT for sending these references.