# Theory of Cascaded Structures: Lossless Transmission Lines

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*Cascaded structures play a major role in many signal processing and signal propagating systems. The universality of such structures is particularly evident when the signals are of a wave nature, i.e., the components of the structure are representable by transmission lines rather than lumped elements. Transmission media with discontinuities are examples of such structures. Other examples include integrated, microwave, and optical circuits.*

*Theory of distributed structures has, so far, been successfully developed only for structures whose components are lossless (or RC) transmission lines of equal electrical lengths. It is the purpose of this paper to present a theory of cascaded structures when the component elements are lossless transmission lines of arbitrary electrical lengths. Extensions of the theory developed here to other structures will be discussed in a subsequent paper.*

## I. INTRODUCTION

### 1.1 *Purpose*

A large class of signal processing and signal propagating systems takes the form of a cascade of elementary two-port, linear transducers. For example, in the classical filter theory cascades of constant-$k$, $m$-derived sections, etc. and in the modern network synthesis cascades of transmission-zero sections form the conceptual basis. Integrated circuits utilize *RC* transmission lines in cascade. In microwave filter theory, the structure takes the form of a cascade of quarter-wave transformers. Optical filters incorporate the same idea in multilayer dielectric thin-film structures. In propagation problems, one typically encounters waves (electromagnetic, acoustic, etc.) travelling in cascades of transmission media and discontinuities. These are but a few examples to indicate the importance and universality of such structures.

It is the purpose of this paper to present a theory of such structures when the component two-ports are representable by uniform lossless transmission lines of arbitrary electrical lengths. Extensions of the theory to include other structures as well as lumped network elements will be discussed in a subsequent paper. A secondary aim of this paper is to incorporate into the theory those algorithms for analysis and synthesis that are most appropriate for computing purposes.

The distinguishing feature of this study is the novel formulation for the transmission matrix specifying each transmission line. The total signal quantities at input and output of the line are related to each other in terms of the forward and backward travelling waves in the line. In the past, results have been obtained only for those structures in which the component transmission lines are of equal electrical length. The new formulation presented here leads to a complete theory of lossless lines in cascade. The analysis and synthesis algorithms obtained here are particularly simple and straightforward. They appear to be quite promising for computation.

### 1.2 *Background*

In the theory of lumped networks, extensive literature exists on cascaded (lumped) structures. The difficulty arises when some or all of the component two-ports consist of distributed elements. In the case of lumped elements, the system functions are defined by rational functions of the complex frequency variable for which there exist many well-known mathematical results. When distributed elements are present, the system functions involve transcedental functions of the complex variable with a consequent increase in complexity. It has been possible in the past to obtain significant results only for certain classes of transmission line structures by applications of Richards' transformation. In particular, Richards[1] showed that distributed structures consisting only of *uniform*, *lossless* transmission lines of *equal electrical lengths* are equivalent, under a change of variable, to lumped networks. Many techniques and results of the lumped network theory can thus be carried over to such a class of distributed structures. Ozaki and Ishii[2] applied such a transformation to obtain physical realizability conditions for such (i.e., uniform, lossless, and equal electrical lengths) transmission lines in cascade. The same results have been better formulated and extended by Riblet.[3] An interesting root-locus approach has been used by Seidel[4] to derive the realizability of insertion loss functions. Finally, Shih[5] has recently used the same idea to obtain some results in the time domain. All of these results are obviously directly applicable to cascades of *RC* transmission lines of equal electrical lengths again by a simple change of variable.

### 1.3 *Results*

An entirely new formulation in terms of the forward and backward waves in the component transmission lines of arbitrary lengths is developed in this paper. Such a formulation is then used to obtain several significant results. Specifically, these results include:

(*i*) A method of analysis which allows one to write down, by inspection, the system functions of the cascaded structures. The expressions for these functions are obtained explicitly in terms of the physical parameters (characteristic impedances, propagation constants, etc.) of the component lines.

(*ii*) Physical realizability conditions for system functions of cascaded transmission lines.

(*iii*) A synthesis method which is simple and appears to have the distinction of minimizing computational errors.

### 1.4 *Organization*

We begin with a statement of the problem in complete generality but we end up with restricting it to the case of interest here (i.e., cascades of uniform, lossless transmission lines). A summary of results for the equal length case follows. We then proceed to introduce our new formulation and discuss the lossless case in detail. The ideas developed for the lossless case will be extended to other structures in a subsequent paper.

## II. STATEMENT OF PROBLEM

In its completely general form, a cascade of linear two-ports may be represented as in Fig. 1. Each component two-port may be characterized by any one of numerous relationships between the various signal parameters at the two ports. The most convenient one for a cascade structure relates all the signal parameters at one port to those at the other. The signal parameters that we shall use are the voltages and the currents at the several ports. Other parameters (such as forward and backward waves and many others) can also be used; but, these are not so con-

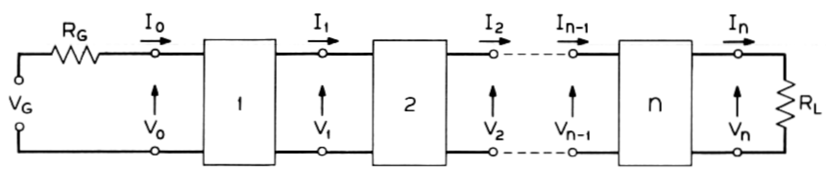


Fig. 1 — The cascaded structure.

venient. The conventions of positive directions for voltages and currents are also shown in the figure.

The input and output signal parameters for each two-port are then related by a transmission matrix for that two-port. Thus,

$$S_{k-1} = T_k S_k \tag{1}$$

where $S_k$ is the signal vector whose elements are $\{V_k, I_k\}$ and $T_k$ is the transmission matrix for the $k$th two-port. It follows that

$$S_o = T_1 T_2 \cdots T_n S_n = T S_n \tag{2}$$

and

$$T = T_1 T_2 \cdots T_n . \tag{3}$$

Equation (3) allows us to study the properties of the composite transmission matrix $T$ in terms of those of the component matrices $T_k$. Our interest is in the methods of analysis and synthesis of such structures. These are carried out conveniently in terms of some scalar system function of the complex frequency variable $s = \sigma + j\omega$. The system functions that we shall be concerned with are the impedance function

$$Z_o(s) = \frac{V_0(s)}{I_0(s)}, \tag{4}$$

and the transmission (or insertion) loss function

$$\Theta(s) = \frac{V_G(s)}{2V_n(s)} \cdot \sqrt{\frac{R_L}{R_G}} \tag{5}$$

where $V_G$ is the voltage of the source and appropriate resistive source and load terminations ($R_G$ and $R_L$) are assumed. The above functions are simply related to the elements $t_{ij}(s)$ of the matrix $T$.

$$Z_o(s) = \frac{t_{11}R_L + t_{12}}{t_{21}R_L + t_{22}} \tag{6}$$

and

$$\Theta(s) = \frac{t_{11}R_L + t_{12} + t_{21}R_G R_L + t_{22}R_G}{2\sqrt{R_G R_L}} . \tag{7}$$

In this paper, our interest is limited primarily to those structures that are representable as cascades of uniform lossless transmission lines. The component two-ports are thus lossless transmission lines whose ends are the ports (Fig. 2). The component matrix $T_k$ can now be obtained from the transmission line equations which, under zero initial

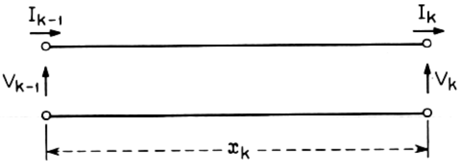


Fig. 2 — A single section of lossless transmission line.

conditions and Laplace transformation with respect to the time variable, are:[6]

$$\begin{bmatrix} \dot{V}(s) \\ \dot{I}(s) \end{bmatrix} = \begin{bmatrix} 0 & -sL_k \\ -sC_k & 0 \end{bmatrix} \begin{bmatrix} V(s) \\ I(s) \end{bmatrix} \tag{8}$$

where $\dot{V}$ and $\dot{I}$ are the derivatives with respect to the distance variable $x$. For the $k$th line, it follows that

$$\begin{bmatrix} V_{k-1}(s) \\ I_{k-1}(s) \end{bmatrix} = \begin{bmatrix} \cosh s\tau_k & R_k \sinh s\tau_k \\ R_k^{-1} \sinh s\tau_k & \cosh s\tau_k \end{bmatrix} \begin{bmatrix} V_k(s) \\ I_k(s) \end{bmatrix}, \tag{9}$$

where $L_k$ and $C_k$ are the inductance and capacitance per unit length of the line,

$$\tau_k = \sqrt{L_k C_k}\, x_k = \text{electrical length},$$

$$R_k = \sqrt{L_k / C_k} = \text{characteristic impedance},$$

and $x_k$ = physical length of the $k$th line. We thus have

$$T_k = \begin{bmatrix} \cosh s\tau_k & R_k \sinh s\tau_k \\ R_k^{-1} \sinh s\tau_k & \cosh s\tau_k \end{bmatrix}. \tag{10}$$

We shall use (10) to derive most of our results.

## III. EQUAL ELECTRICAL LENGTHS (LOSSLESS)

In this section, we briefly summarize the known results that have been obtained for the case of transmission lines of equal electrical lengths, i.e.,

$$\tau_k = \tau \qquad \text{for all } k. \tag{11}$$

Actually, $\tau$ has the dimension of time and it is the time of propagation in each line. It is commonly expressed as a fraction of the wavelength, hence it is called the electrical length.*

* In the sequel, it will simply be called "the length"; when physical length is meant, it will be so specified.

The transmission matrix $T_k$ is now dependent only on the parameter $R_k$. Now, it is clear from (6) that any factor common to all $t_{ij}$ cancels out in the expression for $Z_o$. If we make all matrices $T_k$ rational in some variable, except for a scalar multiplier, then the function $Z_o$ will also be rational. From (10) and (11), it is apparent that either the hyperbolic cosine or sine is the scalar multiplier if we use the transformation

$$p = \tanh s\tau,$$

or

$$= \coth s\tau. \tag{12}$$

The matrices $T_k$ are then all rational in $p$ (except, of course, for the scalar multipliers) and so, $Z_o$ will also be a rational function in $p$.

It should be observed that (12) maps the real and imaginary axes into the real and imaginary axes, respectively, and the right half-plane into the right half-plane. It is this fact together with the rational $Z_o$ that allows us to draw upon the theory of lumped networks. We summarize some of the important conclusions. First, we choose $p = \coth s\tau$ and observe that

$$Z_{k-1}(p) = \frac{pZ_k(p) + R_k}{R_k^{-1}Z_k(p) + p}, \qquad k = 1,2,\cdots,n \tag{13}$$

where $Z_k(p)$ is the impedance

$$Z_k(s) = \frac{V_k(s)}{I_k(s)} \tag{14}$$

under the above change of variables.

A basic theorem for the physical realizability of cascaded lossless equal length lines is as follows.

*Theorem:*[2,3] *The necessary and sufficient conditions that $Z_o$, a real rational function of $p$ of degree $n$ be the input impedance of cascaded lossless equal-length lines terminated in a resistor are: (i) $Z_o$ is a positive-real function of $p$, and (ii) even part of $Z_o$ has only the $n$-fold zeros at $p = \pm 1$.*

The necessity of condition (*i*) follows from (13) by observing that the real part of $Z_{k-1}$ is non-negative for all values of $p$ with non-negative real parts whenever the real part of $Z_k$ is also non-negative for those values of $p$. By iteration of (13) the first condition is seen to follow. The second condition follows from the determinant of $T_k$ which is $(p^2 - 1)$ if we neglect the scalar multiplier $\sinh s\tau_k$. From (3), the determinant of $T$ is $(p^2 - 1)^n$, neglecting the scalar multiplier again. But the determinant of (10), is given by the difference of products of the even and the odd polynomials in the numerator and the denominator of the impedance function $Z_o$. This difference is also the numerator of the even part of $Z_o$ and the second condition is seen to be necessary. Sufficiency of these conditions can be shown by an actual constructive synthesis procedure using the inverse relationship of (13), *viz.*,

$$Z_k = \frac{pZ_{k-1} - R_k}{p - R_k^{-1}Z_{k-1}}. \tag{15}$$

From (15), one can show that if $Z_{k-1}$ satisfies the above conditions, then so does $Z_k$ and it is of a lower degree. The process ultimately terminates yielding the load resistance.

Other results in the $p$-domain include explicit expressions for the coefficients of the input impedance in terms of the characteristic impedances of the lines and vice versa.[3] There is also some discussion on the realizability of the transmission loss functions.[3] These results follow from the basic theorem above.

An interesting departure from the above is the time domain investigation of the same structure.[5] No physical realizability conditions are available in the time domain; however, the synthesis procedure is conceptually quite simple. The system function used is the (impulse) reflection function in the time domain which takes the form of an infinite series of equally spaced impulses. The first impulse at $t = 0$ can only result from the first discontinuity thereby yielding $R_1$. The second impulse ($t = 2\tau$) results from the second discontinuity and yields $R_2$ (since we know $R_1$). The third impulse ($t = 4\tau$) results from the third discontinuity as well as multiple reflections encountering the first and second discontinuities. Since the only unknown in all these discontinuities is the third one, it is uniquely determined and yields $R_3$. The process continues and every new impulse determines the next characteristic impedance until all the junctions are specified. The rest of the impulses are then sums of the multiple reflections from all the junctions and the synthesis is complete.

The major drawback of the time domain approach is that there are no concise physical realizability conditions available.

## IV. LOSSLESS CASE (GENERAL)

In this section, we consider the general case of lossless transmission lines of arbitrary lengths in cascade. The transformation (12) no longer reduces the system functions into rational functions. In fact, it is no longer possible to think in terms of rational functions. We must, therefore, abandon the previous approach and start fresh.

We begin with some physical observations. The structure is certainly passive and so the impedance function $Z_o$ must be a positive-real function of $s$. The component two-ports as well as the cascade of them represent reciprocal structures and so the determinant of $T_k$ as well as that of $T$ must be unity. This follows from the reciprocal property in general and can be verified from (3) and (10) directly. The next observation stems from the time delay property of transmission lines. As mentioned earlier, the length of the transmission line $\tau_k$ represents in reality a time delay of $\tau_k$ seconds between the input and output signals for the $k$th line. The cascade structure, of course, distorts the signal but we can still speak of the time delay as the time interval between the start of the input signal and that of the output signal. This is the time delay that an impulse will undergo, *viz.*,

$$\tau = \sum_{k=1}^{n} \tau_k . \tag{16}$$

In this same cascade structure, however, each component line may be viewed as a delay line of length $\tau_k$. To bring the parameter $\tau_k$ in prominence, we can look upon the line with its discontinuities at the two ends as a spatial resonator for an impulse. If we can make these elementary resonators $\tau_k$ explicitly apparent in the system functions, we would be able to identify the several lines. It is this fact that motivates the formulation that we shall pursue.

Let

$$z = e^s \tag{17}$$

so that

$$z^{\tau_k} = e^{s\tau_k}. \tag{18}$$

This maps the left half $s$-plane into the unit disc whose boundary $|z| = 1$ corresponds to the imaginary axis of the $s$-plane. Then

$$T_k = \tfrac{1}{2}[A_k^{+} z^{\tau_k} + A_k^{-} z^{-\tau_k}], \tag{19}$$

where

$$A_k^{\pm} = \begin{bmatrix} 1 & \pm R_k \\ \pm R_k^{-1} & 1 \end{bmatrix}. \tag{20}$$

Equation (19) expressed $T_k$ directly in terms of the forward and backward wave delays $z^{-\tau_k}$ and $z^{+\tau_k}$. It would be more meaningful to express

(19) in terms of the delay terms, $z^{-\tau_k}$, and the terms, $z^{-2\tau_k}$, corresponding to the elementary resonator. However, we shall find positive exponents of $z$ more convenient to use and when necessary it is always possible to revert to the negative exponents. Hence, we shall have occasion to use

$$T_k = \frac{1}{2z^{\tau_k}} [A_k^+ z^{2\tau_k} + A_k^-]. \tag{21}$$

V. LOSSLESS CASE — ANALYSIS

It is desirable in many cases to study the behavior of system functions for different values of physical parameters of the system. In such cases, it is necessary to bring out explicitly the dependence of these functions on the system parameters. We proceed to do so by first expressing $T$ in terms of these parameters. From (3) and (19)

$$T = \prod_{k=1}^{n} \frac{1}{2} [A_k^+ z^{\tau_k} + A_k^- z^{-\tau_k}], \tag{22}$$

or

$$T = 2^{-n} \sum A_1^{u_1} A_2^{u_2} \cdots A_n^{u_n} z^{u_1\tau_1 + u_2\tau_2 + \cdots + u_n\tau_n} \tag{23}$$

$$\text{and } u_k = \pm 1 \text{ when a coefficient} \tag{24}$$

$$= \pm \text{ when a superscript.}$$

The summation above is over all possible combinations $(u_1, u_2, \cdots, u_n)$. Thus, there are $2^n$ terms in all. Each of these terms needs to be examined further to make (23) meaningful. First, however, let us observe that the matrix $T$ as well as the functions $Z_o$ and $\Theta$ can be all obtained very simply from

$$y = Tv, \tag{25}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. For example,

$$Z_o = \frac{y_1}{y_2} \tag{26}$$

and

$$2\sqrt{R_G R_L}\,\Theta = y_1 + R_G y_2, \tag{27}$$

where, in (25),

$$v = \begin{pmatrix} R_L \\ 1 \end{pmatrix}. \tag{28}$$

The elements of matrix $T$ are obtained by letting the vector $v$ have elements $\{1,0\}$ and $\{0,1\}$. Our interest will, therefore, be in obtaining $y$ in terms of $v$. It follows from (23) that we need merely obtain

$$v' = A_1^{u_1}A_2^{u_2} \cdots A_n^{u_n}v. \tag{29}$$

We show below that $A_k^{u_k}$ is singular and so $v'$ is obtained by successive projections of a vector onto the appropriate eigenvector. Let

$$A_k^{u_k} = \begin{bmatrix} 1 & u_kR_k \\ u_kR_k^{-1} & 1 \end{bmatrix}. \tag{30}$$

It is obvious that the above matrix is singular ($u_k^2 = +1$). Its nonzero eigenvalue is at $\lambda = 2$ with the eigenvector

$$e_2^{u_k} = \begin{pmatrix} u_kR_k \\ 1 \end{pmatrix}; \tag{31}$$

its other eigenvector is

$$e_0^{u_k} = \begin{pmatrix} -u_kR_k \\ 1 \end{pmatrix}. \tag{32}$$

It should be clear that $A_k^{u_k}$ operating on any vector $v$ results in two times the projection of $v$ onto $e_2^{u_k}$. Or,

$$A_k^{u_k}v = u_kR_k^{-1}(v_1 + v_2u_kR_k)e_2^{u_k}. \tag{33}$$

Then, from (29)

$$v' = e_2^{u_1}\left(v_2 + \frac{v_1}{u_nR_n}\right)\prod_{k=1}^{n-1}\left(1 + \frac{u_{k+1}R_{k+1}}{u_kR_k}\right). \tag{34}$$

Finally, we obtain

$$y = 2^{-n}\sum z^{u_1\tau_1+\cdots+u_n\tau_n}\left[\prod_{k=1}^{n-1}\left(1 + \frac{u_{k+1}R_{k+1}}{u_kR_k}\right)\right]\left(v_2 + \frac{v_1}{u_nR_n}\right)e_2^{u_1}, \tag{35}$$

where the summation is again over combinations $(u_1, u_2, \cdots, u_n)$. Equation (35) expresses the system functions as well as the composite matrix explicitly in terms of the system parameters $\tau_k$ and $R_k$. Further simplifications in (35) are possible for special situations. However, the important thing to be emphasized here is that we have an explicit expression in scalar form for the elements $y_1$ and $y_2$ and therefore for all system functions of interest. For computational purposes, (35) can be expressed in terms of hyperbolic cosine and sine terms. For discussing physical realizability, it would be more convenient to eliminate all negative exponents of $z$ in (35). This is accomplished by using (21) or, equivalently, by considering $z^{\tau}y$ since the highest negative exponent in (35) is $\tau$. The impedance function $Z_o$ will be now a ratio of functions involving only positive exponents of $z$.

$$Z_o(z) = \frac{z^{\tau}y_1}{z^{\tau}y_2}. \tag{36}$$

## VI. LOSSLESS CASE — PHYSICAL REALIZABILITY

The basic results will be derived for the realizability of the impedance function $Z_o(z)$. It is then easy to carry over the results to determine the realizability of other system functions. Let $Z_o$ be expressed in the form

$$Z_o(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{m} a_k z^{2i_k}}{\sum_{k=0}^{m} b_k z^{2i_k}}, \tag{37}$$

where $N$ and $D$ are finite sums as shown and have no common factors, $i_k$ are nonnegative and increasing with $k$. The coefficients $a_k$ and $b_k$ are real and both are not zero for any $k$.

The necessary conditions that must be satisfied have been mentioned before (see Section IV):

(*i*) $Z_o$ must be a positive-real function of $s$, or

$$\text{Re } Z_o(z) \geqq 0 \qquad \text{for} \qquad |z| \geqq 1.$$

(*ii*) Determinant of $T$ is one. Since we are considering $(z^{\tau}T)$

$$\text{det. } (z^{\tau}T) = z^{2\tau} = z^{2i_m}.$$

(see (23), (36), and (37)).

The second condition must be somehow expressed in terms of $N$ and $D$. To do this, observe that except for a constant positive multiplier, $z^{\tau}T$ is a product of matrices of the type

$$T_k' = \begin{bmatrix} (z^{2\tau_k}+1) & R_k(z^{2\tau_k}-1) \\ R_k^{-1}(z^{2\tau_k}-1) & (z^{2\tau_k}+1) \end{bmatrix} = \begin{bmatrix} f_1^k(z) & f_2^k(z) \\ g_2^k(z) & g_1^k(z) \end{bmatrix} \tag{38}$$

where for all $k$

$$\begin{aligned} f_1(z) &= z^{2\tau_k} f_1(1/z); \qquad & g_1(z) &= z^{2\tau_k} g_1(1/z) \\ f_2(z) &= -z^{2\tau_k} f_2(1/z); \qquad & g_2(z) &= -z^{2\tau_k} g_2(1/z) \end{aligned} \tag{39}$$

and

$$\text{det. } T_k' = f_1 g_1 - f_2 g_2 . \tag{40}$$

Observe that the product of $T_k'$ will yield

$$T' = \begin{bmatrix} F_1(z) & F_2(z) \\ G_2(z) & G_1(z) \end{bmatrix} \tag{41}$$

where $F_1$, $F_2$, $G_1$, and $G_2$ satisfy the same type of relations as $f_1$, $f_2$, $g_1$, and $g_2$, respectively, *viz.*,

$$\begin{aligned} F_1(z) &= z^{2\tau} F_1(1/z) \\ F_2(z) &= -z^{2\tau} F_2(1/z), \qquad \text{etc.} \end{aligned} \tag{42}$$

Also, if $c > 0$ is a constant,

$$z^{2\tau} = \text{det. } (z^\tau T) = c \text{ det. } (T') = c(F_1 G_1 - F_2 G_2). \tag{43}$$

Then, if

$$N = N_1 + N_2 \tag{44}$$

and

$$D = D_1 + D_2 ,$$

where

$$\begin{aligned} N_1(z) &= z^{2i_m} N_1(1/z); \qquad & D_1(z) &= z^{2i_m} D_1(1/z) \\ N_2(z) &= -z^{2i_m} N_2(1/z); \qquad & D_2(z) &= -z^{2i_m} D_2(1/z), \end{aligned} \tag{45}$$

we can express, using (41),

$$Z_o = \frac{N_1 + N_2}{D_2 + D_1} = \frac{F_1 R_L + F_2}{G_2 R_L + G_1} . \tag{46}$$

The condition (2) now can be expressed using (43) and (46) as

$$N_1 D_1 - N_2 D_2 = c z^{2i_m} \tag{47}$$

where $c$ is again a positive constant.

It is further possible to simplify the statement of the necessary conditions. $Z_o$ is a positive-real function for $|z| \geqq 1$. Consequently, it is also an analytic function for all $|z| \geqq 1$, hence one need verify the non-negative property of $Z_o$ only on the boundary $|z| = 1$.* On the unit

* For a justification of all such statements, see the Appendix.

circle,

$$\text{Re } Z_o \mid_{|z|=1} = \tfrac{1}{2}[Z_o(z) + Z_o(1/z)]_{|z|=1} . \tag{48}$$

Define

$$EvZ_o(z) = \tfrac{1}{2}[Z_o(z) + Z_o(1/z)];$$

then, using (44)–(46),

$$\begin{aligned} EvZ_o(z) &= \frac{1}{2}\left[\frac{N_1(z) + N_2(z)}{D_2(z) + D_1(z)} + \frac{N_1(z) - N_2(z)}{-D_2(z) + D_1(z)}\right] \\ &= \left[\frac{\{N_1(z)D_1(z) - N_2(z)D_2(z)\}}{D(z)D(1/z)z^{2i_m}}\right]. \end{aligned} \tag{49}$$

Substituting (47) in (49), we have

$$EvZ_o(z) = \frac{c}{D(z)D(1/z)}, \qquad (c \geqq 0). \tag{50}$$

Observe that the real part of $Z_o$ is always positive since $c$ is positive. It is zero only if $c$ is zero and this can happen only if $R_L = 0$ or $\infty$. The two necessary conditions are thus equivalent to:

(*i*) $D(z)$ is Hurwitz-type, i.e., all its zeros lie in the interior of the unit circle.

(*ii*) $EvZ_o = c[D(z)D(1/z)]^{-1}; \qquad c \geqq 0.$

These alternative conditions are easier to check. In any case, we now state and prove the physical realizability conditions in the following theorem.

*Theorem: The necessary and sufficient conditions that $Z_o(z)$ be an impedance function of a resistively terminated cascade of lossless, uniform transmission lines are:*

(*i*) *$Z_o(z)$ is a positive real function for $|z| \geqq 1$.*

(*ii*) $N_1D_1 - N_2D_2 = cz^{2i_m}, \qquad (c \geqq 0, \quad i_m > 0)$.*

*Proof:* The necessity of the conditions has already been shown. The sufficiency will be shown by a constructive method of realization. In fact, we shall show that given a $Z_o$ satisfying these conditions, it represents the impedance of a transmission line terminated in an impedance $Z_1$ satisfying the same conditions and of lower order. $Z_o$ and $Z_1$ are re-

* A lossless structure will result if $c = 0$.

lated by

$$Z_o = \frac{(z^{2\tau_1} + 1)Z_1 + R_1(z^{2\tau_1} - 1)}{(z^{2\tau_1} - 1)Z_1R_1^{-1} + (z^{2\tau_1} + 1)} \tag{51}$$

or

$$Z_1 = -\frac{(z^{2\tau_1} + 1)Z_o - R_1(z^{2\tau_1} - 1)}{(z^{2\tau_1} - 1)Z_oR_1^{-1} + (z^{2\tau_1} + 1)}. \tag{52}$$

To show that $Z_1$ is *p-r* and of lower order for some positive $R_1$ and $\tau_1$ we first observe, from (44) through (46), that

$$Z_o(\infty) = \frac{N_1(0) - N_2(0)}{D_1(0) - D_2(0)} = -\frac{N_2(0)}{D_1(0)},$$

since by condition (2),

$$\frac{N_1(0)}{N_2(0)} = \frac{D_2(0)}{D_1(0)};$$

and

$$Z_o(0) = \frac{N_1(0) + N_2(0)}{D_1(0) + D_2(0)} = \frac{N_2(0)}{D_1(0)}$$
$$= -Z_o(\infty). \tag{53}$$

Next, we observe from (52) that

$$\rho_1' = \frac{\dfrac{Z_1}{R_1} - 1}{\dfrac{Z_1}{R_1} + 1} = \frac{\dfrac{Z_o}{R_1} - 1}{\dfrac{Z_o}{R_1} + 1} z^{2\tau_1} = \rho_0' z^{2\tau}. \tag{54}$$

In the above, both $\rho_1'$ and $\rho_0'$ are reflection coefficients. It is obvious that if $\rho_1'$ is analytic for $|z| \geqq 1$ and bounded by one on the unit circle, than $Z_1$ is *p-r*. It is also true that if $\rho_1'$ is of lower order than $\rho_0'$ then $Z_1'$ is of lower order than $Z_o'$. We, therefore, let $R_1 = Z_o(\infty)$ and note that the highest exponents of $z$ in the numerator and the denominator of $\rho_0'$ are $2i_{m-1}$ and $2i_m$, respectively. The denominator of $\rho_0'$ has no constant term and has the lowest exponent of $2i_1$. If we now choose $\tau_1$ equal to the lesser of $i_1$ and $(i_m - i_{m-1})$, it is assured that $\rho_1'$ is analytic for $|z| \geqq 1$ and well behaved at infinity. This follows from the analyticity of $\rho_0'$ and the cancellation of $z^{2\tau_1}$. It is also clear from (54) that for $|z| = 1$, $|\rho_1'| \leqq |\rho_0'|$. But since $Z_0$ is *p-r*, $|\rho_0'| \leqq 1$, hence $|\rho_1'|$ is bounded by one. We thus have, $Z_1$ is *p-r* if $Z_o$ is *p-r* and the order of $Z_1$ is, $2(i_m - \tau_1)$, when the order of $Z_o$ is $2i_m$.

Next, if we express

$$Z_1 = \frac{N_1' + N_2'}{D_1' + D_2'} \tag{55}$$

where $N_1'$, $N_2'$, $D_1'$, and $D_2'$ are defined in a similar manner as $N_1$, $N_2$, $D_1$, and $D_2$ in (45), except that

$$N_1'(z) = z^{2(i_m-\tau_1)}N_1'(1/z), \qquad \text{etc.} \tag{56}$$

It then follows from (51) and (55) that

$$\begin{aligned} N_1(z) &= (z^{2\tau_1} + 1)N_1'(z) + R_1(z^{2\tau_1} - 1)\,D_2'(z) \\ N_2(z) &= (z^{2\tau_1} + 1)N_2'(z) + R_1(z^{2\tau_1} - 1)\,D_1'(z) \\ D_1(z) &= (z^{2\tau_1} + 1)D_1'(z) + R_1^{-1}(z^{2\tau_1} - 1)N_2'(z) \\ D_2(z) &= (z^{2\tau_1} + 1)\,D_2'(z) + R_1^{-1}(z^{2\tau_1} - 1)N_1'(z). \end{aligned} \tag{57}$$

From condition (2),

$$\begin{aligned} cz^{2i_m} &= N_1D_1 - N_2D_2 \\ &= 4z^{2\tau_1}(N_1'D_1' - N_2'D_2') \qquad (\text{from } (57)). \end{aligned}$$

Thus,

$$N_1'D_1' - N_2'D_2' = c'z^{2(i_m-\tau_1)}, \qquad c' \geqq 0$$

and the second condition is satisfied. This proves the basic theorem.

## VII. LOSSLESS CASE — SYNTHESIS

It is indeed possible to synthesize the cascaded structure using (52) to (54) as discussed in the previous section. We present here an algorithm for synthesis which is much more straightforward. In our discussion here, we shall tacitly assume that the conditions of the realizability theorem are satisfied. Given a $Z_o(z)$ satisfying the realizability conditions there exist $R_1$ and $\tau_1$ such that it is the impedance function of a transmission line of length $\tau_1$ and characteristic impedance $R_1$ terminated in a realizable impedance function $Z_1$ of order lower than that of $Z_o$. Let

$$Z_o = \frac{y_1}{y_2}$$

and

$$Z_1 = \frac{y_1'}{y_2'};$$

then if $y$ is a vector with components $\{y_1, y_2\}$, $v_k$ are some vectors, and

$$y' = \sum_{k=0} v_k z^{2i_k},$$

we have

$$\begin{aligned} y &= \tfrac{1}{2}(A_1{}^{+}z^{2\tau_1} + A_1{}^{-})y' \\ &= \tfrac{1}{2}(A_1{}^{+}y')z^{2\tau_1} + \tfrac{1}{2}(A_1{}^{-}y'). \end{aligned}$$

The ratio of elements in the first term on the right is $+R_1$ and for the second term it is $(-R_1)$. The lowest exponent of $z$ in the first term is $2\tau_1$. Finally, $y'$ is obtained by removing the multiplier $z^{2\tau_1}$ in the first term and adding the terms together since

$$\tfrac{1}{2}(A^{+} + A^{-}) = I,$$

the identity matrix. We thus have a unique algorithm provided we have a nondegenerate structure, i.e., the sum of the lengths of any subset of the lines is not equal to the sum of the lengths of any other subset. This assures us that there are no exponents equal in the two terms above. We shall now specify the algorithm.

Given an impedance function.

$$Z_o = \frac{\sum_{k=0}^{m} a_k z^{2i_k}}{\sum_{k=0}^{m} b_k z^{2i_k}};$$

(*i*) Separate like and unlike signs of the coefficients $a_k$, $b_k$

$$Z_o = \frac{\sum a_l z^{2i_l} + \sum a_u z^{2i_u}}{\sum b_l z^{2i_l} + \sum b_u z^{2i_u}},$$

$$\frac{a_l}{b_l} > 0; \qquad \frac{a_u}{b_u} < 0.$$

(*ii*) Identify $a_l/b_l = R_1$ and the lowest $i_l = \tau_1$.

(*iii*) Obtain

$$Z_1 = \frac{\sum a_l z^{2(i_l-\tau_1)} + \sum a_u z^{2i_u}}{\sum b_l z^{2(i_l-\tau_1)} + \sum b_u z^{2i_u}}.$$

The algorithm is repeated until step (*iii*) leads to a constant representing the terminating resistor $R_L$. It must be observed that this algorithm is valid for nondegenerate structures only (i.e., $a_l/b_l = -(a_u/b_u)$ for all $l$ and $u$).

For degenerate structures, the first step in the algorithm has to be modified. It is known, of course, from our discussion of (53) that

$$\frac{a_m}{b_m} = -\frac{a_0}{b_0} = R_1 .$$

So, if for any $k$, $(a_k/b_k) \neq \pm R_1$, then we must split $a_k$ and $b_k$ such that

$$a_k = a_{kl} + a_{ku}$$

$$b_k = b_{kl} + b_{ku}$$

and

$$\frac{a_{kl}}{b_{kl}} = -\frac{a_{ku}}{b_{ku}} = R_1 ,$$

so that

$$a_{kl} = R_1 b_{kl} = \tfrac{1}{2}(a_k + R_1 b_k)$$

$$a_{ku} = -R_1 b_{ku} = \tfrac{1}{2}(a_k - R_1 b_k).$$

Using the above, we obtain the modified algorithm:

(*i*) Identify

$$\frac{a_m}{b_m} = R_1 .$$

(*ii*) Decompose

$$Z_o = \frac{\sum a_{kl} z^{2i_k} + \sum a_{ku} z^{2i_k}}{\sum b_{kl} z^{2i_k} + \sum b_{ku} z^{2i_k}} .$$

(*iii*) Identify the lowest $i_k$ with nonzero $a_{kl} = \tau_1$ .

(*iv*) Obtain

$$Z_1 = \frac{\sum a_{kl} z^{2(i_k - \tau_1)} + \sum a_{ku} z^{2i_k}}{\sum b_{kl} z^{2(i_k - \tau_1)} + \sum b_{ku} z^{2i_k}} .$$

The synthesis method presented here minimizes algebraic operations on the coefficients $a_k$ and $b_k$, hence it is computationally advantageous.

## VIII. CONCLUSION

We have presented a formulation which allows us to investigate structures involving lossless transmission lines of arbitrary electrical lengths. An analysis method is then developed which explicitly expresses the system functions in terms of the physical parameters of the system.

A basic theorem specifying the physical realizability conditions for such structures has been presented together with a computationally simple method of synthesis of impedance functions satisfying these conditions. The significant characteristic of the results presented so far is the simplicity of the algorithms involved both for analysis as well as synthesis. These algorithms allow one to proceed by inspection in simple problems and are most suitable for computer studies when the problems are more complex.

Extensions of the theory to more general transmission lines and lumped structures have been carried out. These results as well as design approaches to the cascade structures and questions of testing conditions, approximations, etc., will be discussed elsewhere.

## APPENDIX

### *Maximum Modulus Theorem and Transcendental Functions*

Throughout the text, the maximum modulus theorem[7] has been applied to functions which have either essential singularities or are not single-valued in the domain concerned. Some justification for the validity of the theorem for such functions is in order. The theorem has been used to imply that the unit bound on the reflection coefficient (or the positive reality of the impedance function) on the imaginary axis of the $s$-plane is sufficient to ensure the same throughout the semi-infinite right half $s$-plane. Consider the reflection function

$$\rho(s) = \frac{\sum_{n=0}^{m} a_n e^{2s i_n}}{\sum_{n=0}^{m} b_n e^{2s i_n}}, \qquad b_m \neq 0$$

where $i_n$ are nonnegative and increasing with $n$. The above function is, of course, assumed to be analytic in the right half-plane. The function $\rho(s)$ is a meromorphic function with infinite singularities, hence the point at infinity is an essential singularity. This makes it difficult to apply the maximum modulus theorem to the entire right half-plane. The transformation $z = e^s$ eliminates the essential singularity at infinity but makes $\rho(z)$ multi-valued since $i_n$ are not necessarily integers. If the $i_n$ are indeed integers, then $\rho(z)$ is single-valued and the theorem can be applied. If the $i_n$ are not integers, they can be approximated arbitrarily closely by rational numbers (dense in the field of real numbers) and the transformation $z^u = e^s$, where $ui_n =$ integer, will yield a single-valued function to which the theorem can be applied. This discussion should suffice to justify the use of the maximum modulus theorem for our purposes. In fact, the theorem can be applied to the function in its original $s$-domain or under any suitable transformation.

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