Duration of Fades Associated with Radar Clutter

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The fluctuating envelope of the pulse-to-pulse radar echoes from a range cell consisting of a stationary target along with many independent, randomly moving scatterers is assumed to behave like a stationary Rayleigh process. In radar terminology this fluctuating or fading envelope of the pulse-to-pulse radar echoes is called signal plus clutter. The envelope of the pulse-to-pulse radar echoes may fade below some critical threshold level for a duration such that the performance of the radar becomes unsatisfactory. Theoretical approximations for the probability densities of both the duration of fades and the interval between fades of the underlying Rayleigh process are presented in graphs for various threshold levels and various signal-to-clutter power ratios. The corresponding exact results are at present unknown. The results of this paper apply to all other fields of science and technology for which a stationary Rayleigh process characterizes a fading phenomenon.

I. INTRODUCTION

Consider a pulsed radar system "viewing" a range cell consisting of a stationary target along with many independent, randomly moving scatterers as shown in Fig. 1. Each received echo consists of the vector sum of all the elementary echoes originating from within the range cell. The contributions from the randomly moving scatterers arrive at the radar receiver with random phases. As a result, each received echo will consist of a steady signal, from the stationary target, plus a Gaussian perturbation. Accordingly, samples of the envelope of the pulse-to-pulse radar echoes can be considered as samples of an underlying Rayleigh process. Thus, the effect of the randomly moving scatterers is to cause the envelope of the pulse-to-pulse radar echoes to fluctuate or fade in an irregular manner. The envelope of the pulse-to-pulse radar echoes may fade below some critical threshold level for a duration such that the performance of the radar becomes unsatisfactory.

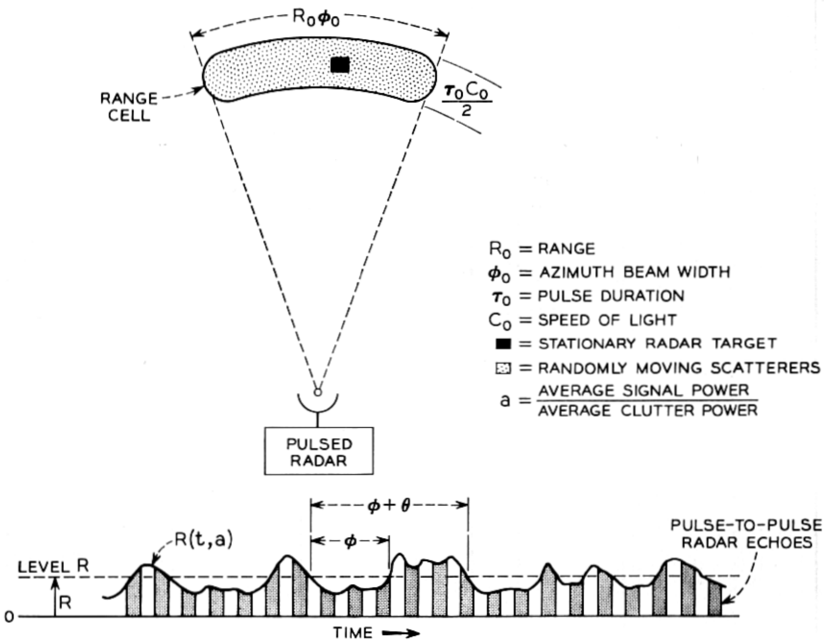

Fig. 1 — A model for studying the duration of fades associated with signal plus clutter. $R(t,a)$ represents the envelope of the pulse-to-pulse radar echoes from the range cell. At the level R, ϕ and $\phi + \theta$ represent the duration of a fade and the interval between fades, respectively.

In radar terminology the fluctuating or fading envelope of these pulse-to-pulse radar echoes is called signal plus clutter. Classical discussions of signal plus clutter were given by H. Goldstein and A. J. F. Siegert and can be found in Refs. 1 and 2. A well-known example of signal plus clutter is the envelope of the pulse-to-pulse radar echoes from a target surrounded by a great deal of "chaff". Some other examples may be the envelope of the pulse-to-pulse radar echoes from the aurora, the ionosphere, the sea, the ground, meterological precipitation, and a hypersonic object during reentry of the earths atmosphere.

A natural assumption for studying the duration of fades associated with signal plus clutter is that the random process underlying the fading is a Rayleigh process. However, only a few theoretical results are available concerning the duration of fades associated with Rayleigh processes. Thus, one is often unable to determine how well a Rayleigh process actually characterizes the duration of fades observed experimentally.

The purpose of this paper is to present some additional theoretical results which characterize approximately the duration of fades one would expect when the random process underlying the fading is indeed a Rayleigh process. We shall assume that the envelope of the pulse-to-pulse radar echoes behaves like the Rayleigh process $R(t,a)$ sketched in Fig. 1. $R(t,a)$ represents the envelope of a stationary random process consisting of a sinusoidal signal of amplitude $\sqrt{2a}$ and frequency f_o plus a Gaussian process of unit variance having a narrowband power spectral density $W_b(f - f_o)$ which is symmetrical about f_o. We assume that the radar pulse repetition frequency is several times greater than the bandwidth of the Rayleigh process $R(t,a)$ in order that an adequate number of radar echoes are used to form $R(t,a)$. Also, we shall assume that the variance of the receiver noise is negligible in comparison with the variance of the clutter.

Using notation consistent with Refs. 3, 4, and 5 we shall present theoretical approximations for the following probability functions for arbitrary signal-to-clutter power ratio "a":

(*i*) $P_o^-(\tau,R,a)$, the probability density of the duration of a fade of the Rayleigh process below the level R.
(*ii*) $P_1(\tau,R,a)$, the probability density of the interval between fades of the Rayleigh process below the level R.
(*iii*) $F_o^-(\tau,R,a)$, the probability that the duration of a fade of the Rayleigh process below the level R lasts longer than τ.
(*iv*) $F_1(\tau,R,a)$, the probability that the interval between fades of the Rayleigh process below the level R lasts longer than τ.

The model considered in this paper also has application in the study of the duration of fades in radio transmission. In fact Rice[6,7] led the way by analyzing the duration of fades in radio transmission assuming that the underlying random process was $R(t,0)$.

II. INTEGRAL EQUATIONS AND EXPECTATIONS

Let us define the following auxiliary probability functions for arbitrary level R and arbitrary signal-to-clutter power ratio "a":

(*i*) $Q^-(\tau,R,a)\,d\tau$, the conditional probability that an upward level-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward level-crossing at t.
(*ii*) $[U(\tau,R,a) - Q(\tau,R,a)]\,d\tau$, the conditional probability that an upward level-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward level-crossing at t.

Explicit expressions for these auxiliary probability functions were presented in Ref. 5.

Approximate theoretical results for $P_o^-(\tau,R,a)$ and $P_1(\tau,R,a)$ are given by the following integral equations:

$$P_o^-(\tau,R,a) = Q^-(\tau,R,a) - P_o^-(\tau,R,a) * [U(\tau,R,a) - Q(\tau,R,a)] \quad (1)$$

$$P_1(\tau,R,a) = [U(\tau,R,a) - Q(\tau,R,a)] - P_1(\tau,R,a) * [U(\tau,R,a) - Q(\tau,R,a)] \quad (2)$$

where $*$ denotes the convolution operator, that is,

$$f * g \equiv \int_{-\infty}^{\infty} f(t)g(\tau - t)dt.$$

Equations (1) and (2) were derived in Ref. 3 by applying McFadden's[8] "quasi-independence" idea to the Rayleigh process $R(t,a)$. Also, by definition we have that

$$F_o^-(\tau,R,a) = 1 - \int_0^{\tau} P_o^-(\tau,R,a)d\tau \quad (3)$$

and

$$F_1(\tau,R,a) = 1 - \int_0^{\tau} P_1(\tau,R,a)d\tau. \quad (4)$$

The exact expectations $E_o^-(\tau,R,a)$ and $E_1(\tau,R,a)$ associated with the respective densities $P_o^-(\tau,R,a)$ and $P_1(\tau,R,a)$ can be computed from the following equations:

$$E_o^-(\tau,R,a) \equiv \frac{Pr\{R(t,a) < R\}}{N_R} = \frac{\sum_{n=1}^{\infty} (R^{2n}/2^n n!)\,{}_1F_1\left(n + \tfrac{1}{2};\ 2n + 1;\ -2R\sqrt{2a}\right)}{\sqrt{(\beta/2\pi)}\ R\,{}_1F_1\left(\tfrac{1}{2};\ 1;\ -2R\sqrt{2a}\right)} \quad (5)$$

$$E_1(\tau,R,a) \equiv \frac{1}{N_R} = \frac{\exp\left[(R - \sqrt{2a})^2/2\right]}{\sqrt{(\beta/2\pi)}\ R\,{}_1F_1\left(\tfrac{1}{2};\ 1;\ -2R\sqrt{2a}\right)} \quad (6)$$

where

$$\beta = (2\pi)^2 \int_0^{\infty} W_b(f - f_o)(f - f_o)^2\, df$$

$W_b(f - f_o)$ = narrowband power spectral density of the Gaussian process involved in the definition of $R(t,a)$

$$a = \frac{\text{average signal power}}{\text{average clutter power}}$$

$${}_1F_1(\alpha;\beta;x) = \text{the confluent hypergeometric function}$$

$$= 1 + \frac{\alpha}{\beta} x + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)} \frac{x^2}{2!} + \cdots .$$

$\Pr\{\ \}$ = probability of the event inside the brace

N_R = average number of upward (or downward) crossings of the level R per second.

Equations (5) and (6) were developed in Ref. 4, and they follow directly from some well-known results reported by Rice and Bennett. Each ${}_1F_1$ function appearing in (5) and (6) can be expressed in terms of a Bessel function of imaginary argument.

Thus, with the aid of a digital computer one can compute theoretical approximations for the probability functions of interest in this paper along with the exact theoretical expectations given by (5) and (6).

III. RESULTS FOR A GAUSSIAN AUTOCORRELATION FUNCTION

In order to define the Rayleigh process $R(t,a)$ underlying the fading phenomenon we need to specify both $W_b(f - f_o)$ and the signal-to-clutter ratio "a". The normalized autocorrelation function $m(\tau)$ associated with $W_b(f - f_o)$ is given by

$$m(\tau) = \int_0^\infty W_b(f - f_o) \cos 2\pi(f - f_o)\tau df. \tag{7}$$

Thus, $m(\tau)$, rather than $W_b(f - f_o)$, can be used to define the Rayleigh process $R(t,a)$ underlying the fading phenomenon. Notice that β appearing in (5) and (6) is merely $-m''(0)$. The primes denote differentiations with respect to τ.

Ref. 1 points out that it is convenient to measure the normalized autocorrelation function of the fluctuating low frequency power $P(t) = R^2(t,a)$ and denotes this normalized autocorrelation function by $\rho(P,\tau)$. In explicit terms $\rho(P,\tau)$ is defined as

$$\begin{aligned}\rho(P,\tau) &= \frac{E\{[P(t + \tau) - EP(t)][P(t) - EP(t)]\}}{Var\ P(t)} \\ &= \frac{EP(t + \tau)P(t) - E^2P(t)}{Var\ P(t)}\end{aligned} \tag{8}$$

where

$$E = \text{Expectation}$$

$$Var = \text{Variance.}$$

Refs. 1 and 2 relate $m(\tau)$ and $\rho(P,\tau)$ as follows:

$$m(\tau) = \sqrt{a^2 + (1 + 2a)\rho(P,\tau)} - a. \tag{9}$$

Ref. 1 also points out that the appropriate value of "a" can be estimated by measuring the probability density of $P(t)$ and comparing the result with the theoretical probability density of $P(t)$. Thus, (9) indicates that the Rayleigh process $R(t,a)$ underlying the fading phenomenon can also be defined from measurements of the normalized autocorrelation function $\rho(P,\tau)$ and the value of "a."

For purposes of computation we shall take $W_b(f - f_o)$ and $m(\tau)$ as

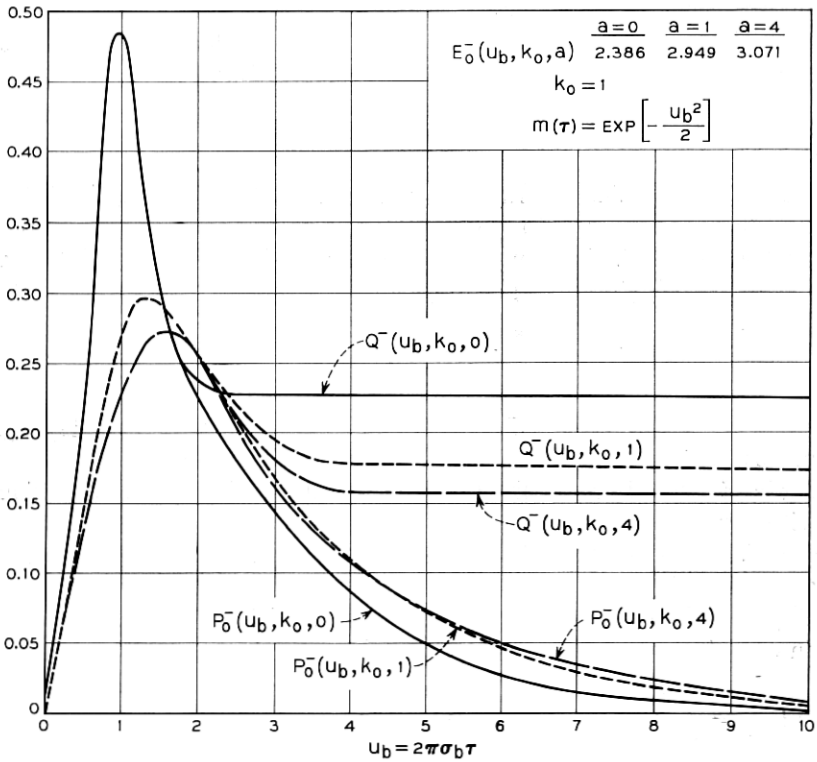

Fig. 2 — $P_0^-(u_b, k_0, a)$ is the probability density of the duration of a fade of the Rayleigh process below the normalized level k_0. The autocorrelation function of the Gaussian process involved in the definition of the Rayleigh process is $m(\tau)$ and the signal-to-clutter power ratio equals "a."

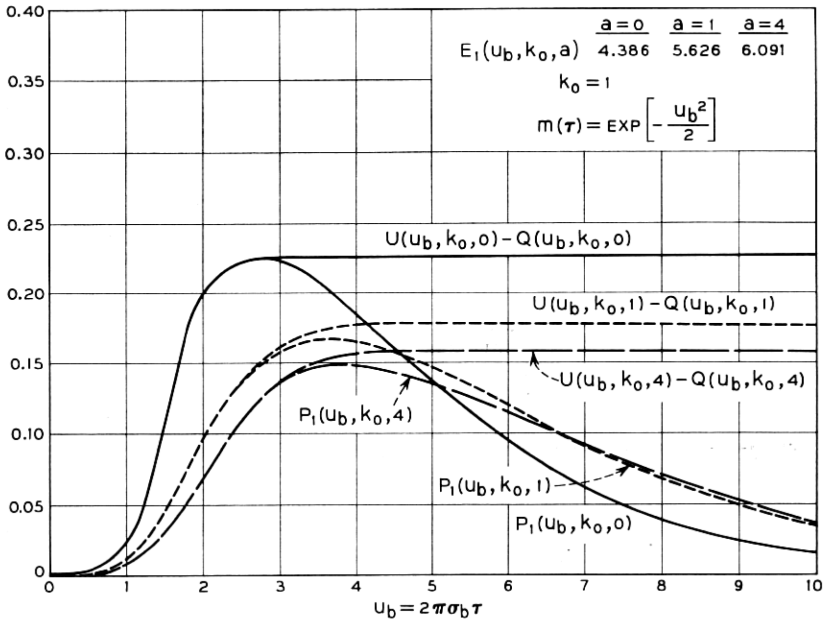

Fig. 3 — $P_1(u_b, k_0, a)$ is the probability density of the interval between fades of the Rayleigh process below the normalized level k_0. The autocorrelation function of the Gaussian process involved in the definition of the Rayleigh process is $m(\tau)$ and the signal-to-clutter power ratio equals "a."

follows:

$$W_b(f - f_o) = \frac{1}{\sigma_b\sqrt{2\pi}} \exp\left[\frac{-(f - f_o)^2}{2\sigma_b^2}\right] \tag{10}$$

and

$$m(\tau) = \exp\left[\frac{-(2\pi\sigma_b\tau)^2}{2}\right]. \tag{11}$$

This particular choice tends to characterize the radar clutter fluctuations observed experimentally.[1,2] From (11) we see that it is convenient to define normalized time as $u_b = 2\pi\sigma_b\tau$.

For the experimenter it is convenient to normalize the threshold level with respect to the average value, $ER(t,a)$, of the Rayleigh process. We shall consider three such normalized levels k_o

$$\frac{R}{ER(t,a)} \equiv k_o = 1, \sqrt{\frac{2}{\pi}}, \frac{1}{\sqrt{2\pi}}. \tag{12}$$

The expectation $ER(t,a)$ was derived by Rice[9] and is given by

$$ER(t,a) = \sqrt{\frac{\pi}{2}}\, {}_1F_1\left(-\frac{1}{2}; 1; -a\right). \tag{13}$$

When $a = 0$ we have that $R = \sqrt{(\pi/2)}$, 1, $\frac{1}{2}$. These latter two values of R were also considered by Rice[7] for the case $a = 0$.

Figs. 2 through 10 present the computed results for $a = 0, 1, 4$, and $k_o = 1, \sqrt{2/\pi}, 1/\sqrt{2\pi}$. The numerical evaluation of $Q^-(\tau,R,a)$ and $U(\tau,R,a) - Q(\tau,R,a)$ was carried out by using Simpson's rule. Integral equations (1) and (2) were solved numerically by using the trapezoidal rule. All results are plotted with respect to normalized time u_b. The corresponding experimental results for $k_o = 1$ and $a = 0, 1, 4$ were presented in Ref. 4, and they agree well with the approximate theoretical results presented in Figs. 2, 3, and 4.

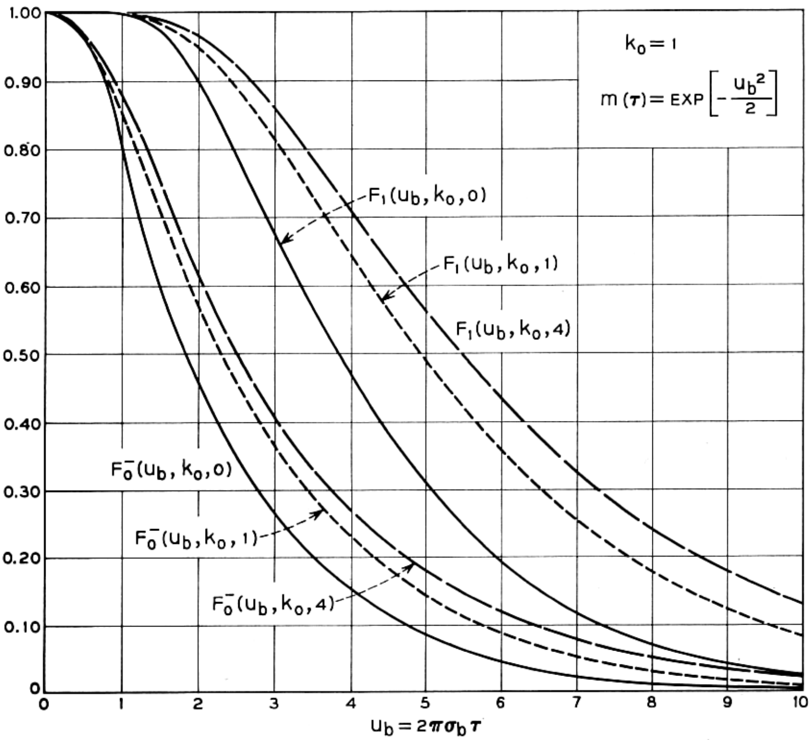

Fig. 4 — $F_0^-(u_b, k_0, a)$ is the probability that the duration of a fade of the Rayleigh process below the normalized level k_0 lasts longer than u_b. $F_1(u_b, k_0, a)$ is the probability that the interval between fades of the Rayleigh process below the normalized level k_0 lasts longer than u_b.

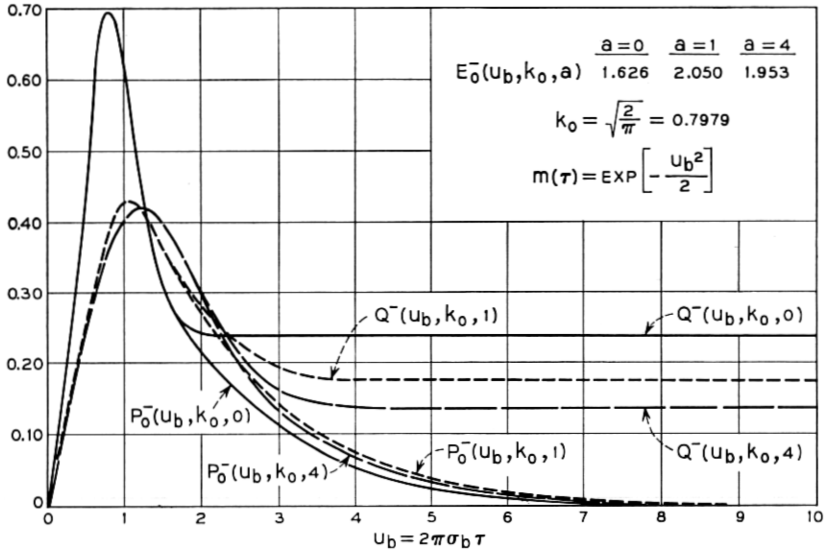

Fig. 5 — $P_0^-(u_b, k_0, a)$ is the probability density of the duration of a fade of the Rayleigh process below the normalized level k_0. The autocorrelation function of the Gaussian process involved in the definition of the Rayleigh process is $m(\tau)$ and the signal-to-clutter power ratio equals "a".

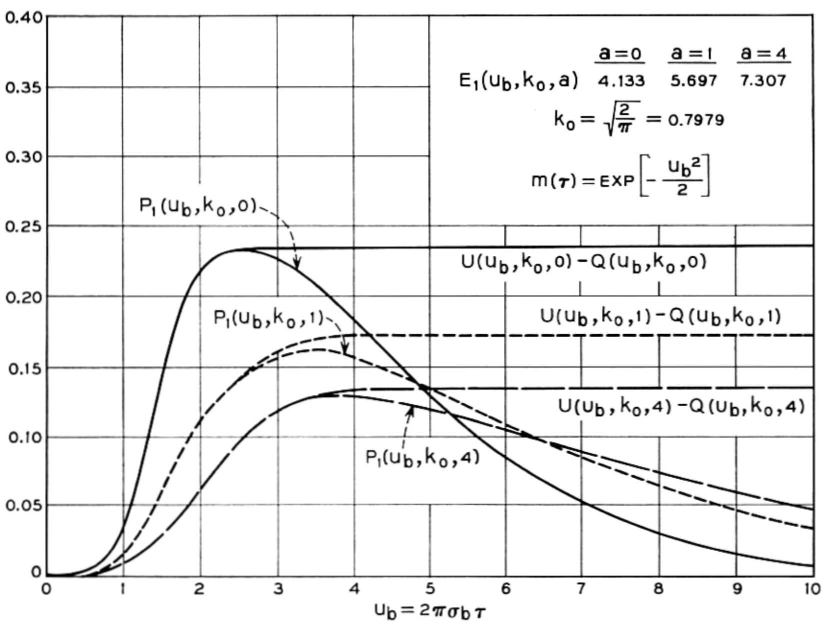

Fig. 6 — $P_1(u_b, k_0, a)$ is the probability density of the interval between fades of the Rayleigh process below the normalized level k_0. The autocorrelation function of the Gaussian process involved in the definition of the Rayleigh process is $m(\tau)$ and the signal-to-clutter power ratio equals "a".

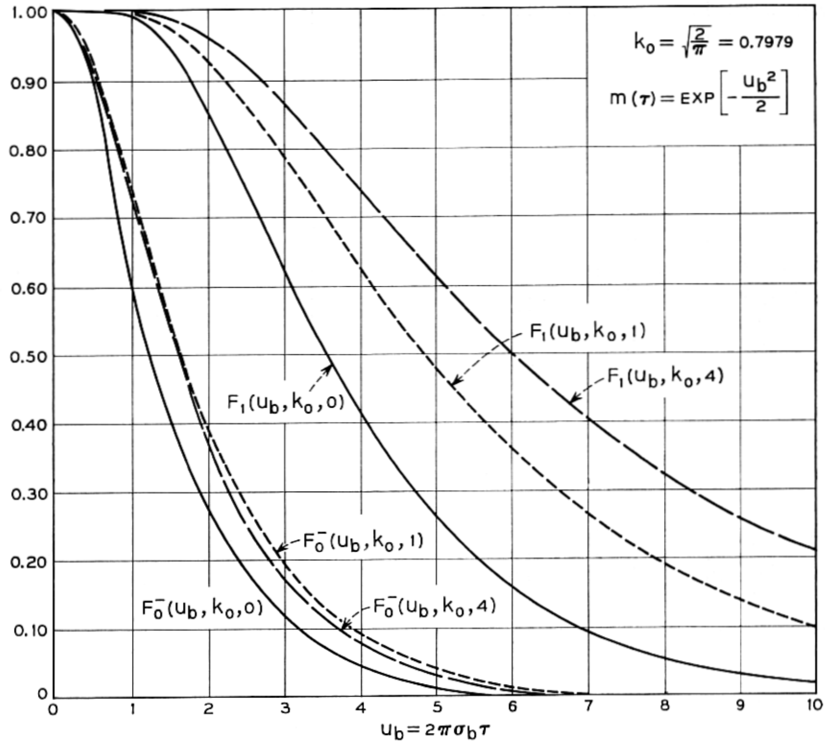

Fig. 7 — $F_0^-(u_b, k_0, a)$ is the probability that the duration of a fade of the Rayleigh process below the normalized level k_0 lasts longer than u_b. $F_1(u_b, k_0, a)$ is the probability that the interval between fades of the Rayleigh process below the normalized level k_0 lasts longer than u_b.

For deep fades and large signal-to-clutter power ratio "a", one would expect $P_o^-(\tau,R,a)$ to approach a Rayleigh probability density. For as "a" gets large the Rayleigh process $R(t,a)$ tends to behave much like a Gaussian process, see (3.6) of Rice,[9] and the durations of deep fades of Gaussian processes are known to be characterized by a Rayleigh probability density.[7,10] Figs. 8 and 10 show that this is, approximately, the case when $k_o = 1/\sqrt{2\pi}$ and $a = 4$. Thus, for $k_o \leqq 1/\sqrt{2\pi}$ and $a \geqq 4$ we have the following approximate results:

$$P_o^-(\tau,R,a) = \frac{\pi}{2E_o^-}\left(\frac{\tau}{E_o^-}\right)\exp\left[-\frac{\pi}{4}\left(\frac{\tau}{E_o^-}\right)^2\right] \tag{14}$$

and

$$F_o^-(\tau,R,a) = \exp\left[-\frac{\pi}{4}\left(\frac{\tau}{E_o^-}\right)^2\right]. \tag{15}$$

The value of E_o^- appearing in (14) and (15) is given by (5) with $R = k_oER(t,a)$.

Equations (14) and (15) are useful approximations when k_o is small and "a" is large for an arbitrary normalized autocorrelation function $m(\tau)$ such that $m'''(0^+) = 0$, although we have been treating the restrictive Gaussian autocorrelation function defined by (11). The condition $m'''(0^+) = 0$ leads to $Q^-(0^+,R,a) = 0$, and thus the approximation given by (14) is exact at $\tau = 0^+$. As a partial check on this generalization we also verified that (14) and (15) begin to be useful approximations when $k_o = 1/\sqrt{2\pi}$, $a = 4$ for the normalized autocorrelation functions

$$m(\tau) = \frac{\sin 2\pi f_c\tau}{2\pi f_c\tau} \tag{16}$$

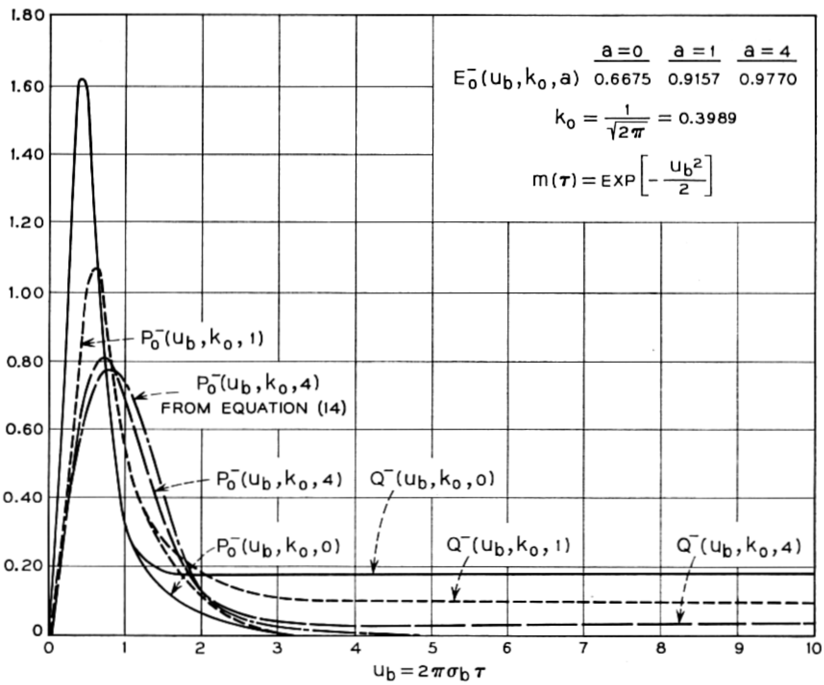

Fig. 8 — $P_0^-(u_b, k_0, a)$ is the probability density of the duration of a fade of the Rayleigh process below the normalized level k_0. The autocorrelation function of the Gaussian process involved in the definition of the Rayleigh process is $m(\tau)$ and the signal-to-clutter power ratio equals "a."

and

$$m(\tau) = \left[1 + \omega_2 \mid \tau \mid + \frac{(\omega_2 \tau)^2}{3}\right] \exp\,(-\omega_2 \mid \tau \mid). \tag{17}$$

Equation (16) corresponds to an ideal bandpass power spectral density $W_b(f - f_o)$ given by

$$W_b(f - f_o) = \begin{cases} (2f_c)^{-1} & \text{for} \quad f_o - f_c \leqq f \leqq f_o + f_c \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Equation (17) corresponds to a power spectral density $W_b(f - f_o)$ given by

$$W_b(f - f_o) = \frac{8/(3\pi f_2)}{\left[1 + \left(\dfrac{f - f_o}{f_2}\right)^2\right]^3}, \tag{19}$$

where

$$\omega_2 = 2\pi f_2 .$$

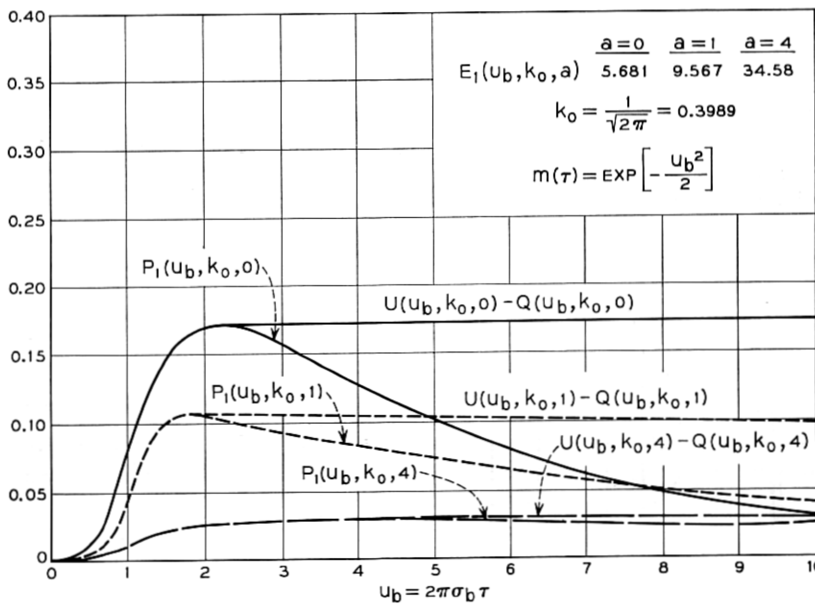

Fig. 9 — $P_1(u_b, k_0, a)$ is the probability density of the interval between fades of the Rayleigh process below the normalized level k_0. The autocorrelation function of the Gaussian process involved in the definition of the Rayleigh process is $m(\tau)$ and the signal-to-clutter power ratio equals "a".

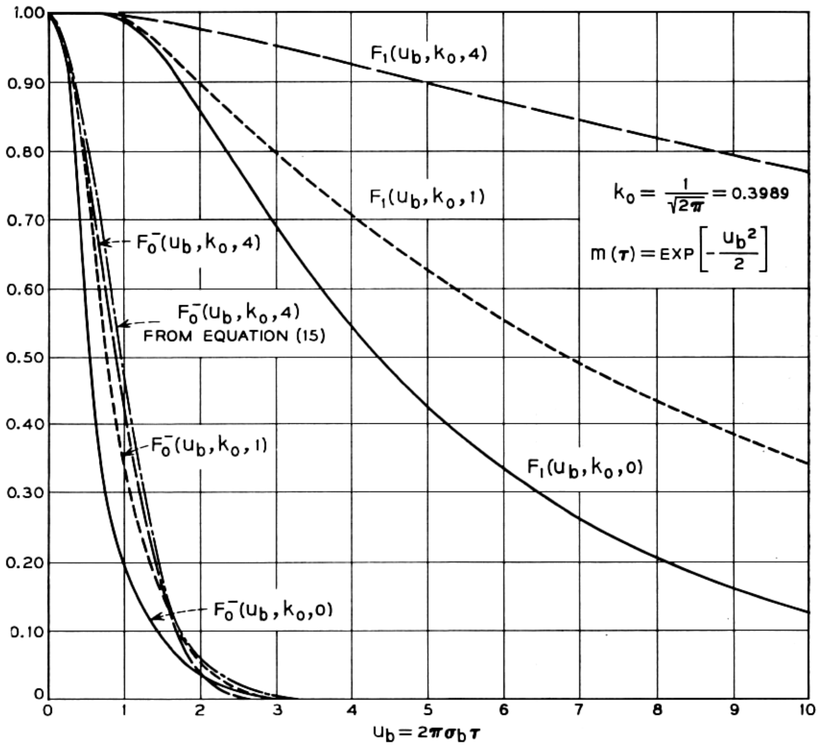

Fig. 10 — $F_0^-(u_b, k_0, a)$ is the probability that the duration of a fade of the Rayleigh process below the normalized level k_0 lasts longer than u_b. $F_1(u_b, k_0, a)$ is the probability that the interval between fades of the Rayleigh process below the normalized level k_0 lasts longer than u_b.

For a given $m(\tau)$ with $m'''(0^+) = 0$ along with $a \geqq 4$, $k_o = 1$, the duration of fades and the interval between fades of the Rayleigh process $R(t,a)$ behave as if they were generated at the mean value level of a Gaussian process having a normalized autocorrelation function of $m(\tau)$. For example, compare $P_o^-(u_b, k_o, 4)$ of Fig. 2 with the experimental points plotted in Fig. 2 of Ref. 3. Also compare $P_1(u_b, k_o, 4)$ of Fig. 3 with the experimental points plotted in Fig. 3 of Ref. 3.

IV. CONCLUSIONS

Assuming that the random process underlying a fading phenomenon is a stationary Rayleigh process, one can compute useful theoretical approximations for the probability functions which characterize the duration of fades and the interval between fades. The corresponding exact results are at present unknown.

For deep fades and large signal-to-clutter power ratio the duration of fades is characterized, approximately, by a Rayleigh probability density.

For large signal-to-clutter power ratio the duration of fades and the interval between fades of the Rayleigh process below the mean value level behave as if they were generated at the mean value level of a certain Gaussian process.

The results of this paper apply to all fields of science and technology for which a stationary Rayleigh process characterizes a fading phenomenon.

V. ACKNOWLEDGMENT

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