Noise in an FM System Due to an Imperfect Linear Transducer

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An approach to the calculation of intermodulation noise in FM systems due to imperfect transmission media is presented in this paper. The technique is essentially that originating with Carson and Fry. In this paper we extend a formulation due to Rice to include an arbitrary continuous pre-emphasis characteristic as well as an arbitrary gain and phase shape transmission medium which are representable by low-order polynomial series in radian frequency. Series approximations are carried out far enough to ensure accurate results for transmission characteristics normally encountered in broadband microwave radio systems. In many cases, only the second- and third-order noise is significant in broadband microwave radio systems. Hence, the analysis carried out in this paper considers only the second- and third-order distortion terms. A digital computer program concerning the intermodulation noise has been written. This analysis and the digital computer program are of aid in the design of microwave radio systems. With a slight modification, the calculation of noise due to AM-to-PM conversion caused by transmission deviation can also be accomplished. An optimum design of the pre-emphasis network may be achieved by using the computer programs through an iterative approach.

I. INTRODUCTION

In the course of designing FM systems, intermodulation noise is an important factor which deserves special attention. Many people have made contributions to this subject.[1-10] In this paper, we extend their results to include an arbitrary continuous pre-emphasis characteristic as well as an arbitrary gain and phase shape transmission medium which are representable by low-order polynomial series in radian frequency. Series approximations are carried out far enough to ensure accurate results for transmission characteristics normally encountered in broadband microwave radio systems. The multichannel baseband signal of an

FM system is represented by a band of gaussian random noise with flat power-density spectrum. The noise due to imperfect transmission medium can be calculated at any frequency in the baseband. In many cases, only the second- and third-order noise is significant in broadband microwave radio systems. Hence, the analysis carried out in this paper considers only the second- and third-order distortion terms. Extension to a higher-order distortion becomes unmanageable. A digital computer program concerning the intermodulation noise has been written. A typical problem can be solved at a very low cost. This analysis and the digital computer program are of aid in the design of microwave radio systems. With a slight modification, the calculation of noise due to AM-to-PM conversion caused by transmission deviation can also be accomplished. An optimum design of the pre-emphasis network may be achieved by using the computer programs through an iterative approach. Several examples are given for illustration.

II. DESCRIPTION OF SYSTEM

A portion of an FM system can be represented by the block diagram shown in Fig. 1. An FM baseband signal, $\varphi_i'(t) = d\varphi_i(t)/dt$, is fed into a pre-emphasis network. The output of the pre-emphasis network is the pre-emphasized signal $\varphi'(t)$. This signal passes through an FM modulator to yield the FM wave, $\cos [\omega_c t + \varphi(t)]$ where ω_c is the angular carrier frequency. When this FM wave goes through an imperfect transmission medium, the output is distorted and becomes $V(t) \cos [\omega_c t + \varphi_0(t)]$. Let the transfer function of the transmission medium be

$$Y(\omega) = \exp [-\alpha(f) - i\beta(f)], \tag{1}$$

where $\omega = 2\pi f$. Its impulse response is

$$g(t) = \int_{-\infty}^{\infty} Y(\omega) \exp (i\omega t) \, df. \tag{2}$$

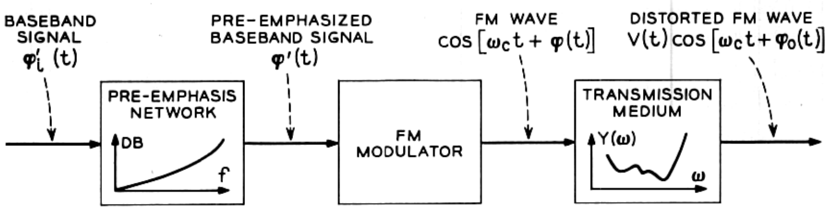

Fig. 1 — Block diagram of a portion of an FM system.

Using the complex notation, the input and output of the transmission medium can be written, respectively, as

$$v_i(t) = \exp[i\omega_c t + i\varphi(t)] \tag{3}$$

and

$$v_0(t) = V(t) \exp[i\omega_c t + i\varphi_0(t)], \tag{4}$$

where $V(t)$ is taken to be positive and $\varphi_0(t)$ is determined to within $2n\pi$, n is an integer.

The output and input of the transmission medium are related by the convolution integral

$$v_0(t) = \int_{-\infty}^{\infty} v_i(t - x)\, g(x)\, dx. \tag{5}$$

From (3), (4), and (5)

$$V(t) \exp[i\varphi_0(t)] = \int_{-\infty}^{\infty} \exp[i\varphi(t - x) - i\omega_c x]\, g(x)\, dx. \tag{6}$$

Let

$$V(t) = \exp[a(t)]. \tag{7}$$

Then

$$a(t) = \operatorname{Re} \ln \int_{-\infty}^{\infty} \exp[i\varphi(t - x) - i\omega_c x]\, g(x)\, dx \tag{8}$$

$$\varphi_0(t) = \operatorname{Im} \ln \int_{-\infty}^{\infty} \exp[i\varphi(t - x) - i\omega_c x]\, g(x)\, dx. \tag{9}$$

The AM distortion term expressed in dB is $20 \times 0.4343\, a(t)$ and the PM distortion term expressed in radians (or degrees) is $\varphi_0(t) - \varphi(t)$.

Assume that the transmission medium passes only frequencies in the neighborhood of the carrier frequency, $\pm f_c \pm b$, with $b/f_c \ll 1$. Thus, to a high degree of approximation, we have

$$\exp(-i\omega_c x)\, g(x) \cong \int_{-b}^{b} Y(\omega_c + \omega) \exp(i\omega x)\, df. \tag{10}$$

Let

$$k(x) = \frac{1}{Y(\omega_c)} \exp(-i\omega_c x)\, g(x). \tag{11}$$

Substituting (1) and (11) into (8) and (9) we obtain

$$a(t) = -\alpha(f_c) + \text{Re}\ln \int_{-\infty}^{\infty} \exp\,[i\varphi(t - x)]\, k(x)\, dx \tag{12}$$

$$\varphi_0(t) = -\beta(f_c) + \text{Im}\ln \int_{-\infty}^{\infty} \exp\,[i\varphi(t - x)]\, k(x)\, dx. \tag{13}$$

The quantity $k(x)$ may be regarded as the normalized envelope function of the impulse response $g(x)$. We shall discuss now the logarithm of the integral in (12) and (13) in detail.

III. DERIVATION OF DISTORTION TERMS

Let

$$\exp\,[i\varphi(t - x)] = M(t,x). \tag{14}$$

A delay t_d is often introduced in order to improve the degree of approximation of various series with the first few terms. Then $M(t,x)$ can be expanded about $x = t_d$ as

$$M(t,x) = M(t,t_d) + \frac{(x - t_d)}{1!}\left[\frac{\partial}{\partial x} M(t,x)\right]_{x=t_d} + \frac{(x - t_d)^2}{2!}\left[\frac{\partial^2}{\partial x^2} M(t,x)\right]_{x=t_d} + \cdots. \tag{15}$$

One choice of t_d is

$$t_d = \text{Re}\int_{-\infty}^{\infty} xk(x)\, dx = \frac{1}{2\pi}\left[\frac{d\beta(f)}{df}\right]_{f=f_c}. \tag{16}$$

Using (14) and (15), the integral in (12) and (13) can thus be expressed as

$$\int_{-\infty}^{\infty} \exp\,[i\varphi(t - x)]\, k(x)dx = \sum_{n=0}^{\infty} \frac{m_n}{n!}\left[\frac{\partial^n}{\partial x^n} M(t,x)\right]_{x=t_d}, \tag{17}$$

where m_n is the nth moment of $k(x)$ defined as

$$m_n = \int_{-\infty}^{\infty} (x - t_d)^n\, k(x)dx, \qquad m_0 = 1 \tag{18}$$

or equivalently,

$$m_n = \frac{(-1)^n}{Y(\omega_c)}\left[\frac{d^n}{d(i\omega)^n} Y(\omega_c + \omega)\exp\,(i\omega t_d)\right]_{\omega=0}. \tag{19}$$

The series (17) is equivalent to the Carson-Fry series with delay t_d. It may not converge in certain cases. However, when the characteristic of a transmission medium can be truly represented by a polynomial, from (19), the higher moments become zero and the series reduces to a polynomial. The logarithm of (17) can be written as

$$\ln \int_{-\infty}^{\infty} \exp[i\varphi(t - x)]\, k(x)dx = i\varphi(t - t_d) + \ln\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n m_n}{n!} F_n(t - t_d)\right], \tag{20}$$

where

$$F_n(t - t_d) = (-1)^n \exp[-i\varphi(t - t_d)] \left[\frac{\partial^n}{\partial x^n} M(t,x)\right]_{x=t_d}.$$

Using Taylor's series expression, we have*

$$\ln\left(1 + \sum_1^{\infty} \alpha_n(t)x^n/n!\right) = \frac{x}{1!}\alpha_1 + \frac{x^2}{2!}(\alpha_2 - \alpha_1^2) + \frac{x^3}{3!}(\alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3) + \frac{x^4}{4!}(\alpha_4 - 4\alpha_1\alpha_3 - 3\alpha_2^2 + 12\alpha_1^2\alpha_2 - 6\alpha_1^4) + \cdots .$$

With $x = -1$ and $\alpha_n(t) = m_n F_n(t - t_d)$, substituting the above series into (20), after considerable algebra, one obtains an asymptotic series which has been written by S. O. Rice in an unpublished work as

$$\begin{aligned}\ln \int_{-\infty}^{\infty} \exp[i\varphi(t - x)]\, k(x)dx = i\Big[&\varphi - \frac{m_1}{1!}\varphi' + \frac{m_2}{2!}\varphi'' - \frac{m_3}{3!}\varphi''' \\ &+ \frac{m_4}{4!}\varphi'''' - \cdots + \tfrac{1}{6}\lambda_3\varphi'^3 \\ &- \tfrac{1}{4}(m_4 - 2m_1m_3 - m_2^2 \\ &+ 2m_1^2m_2)\varphi'^2\varphi'' + \cdots\Big] \\ &+ [-\tfrac{1}{2}\lambda_2\varphi'^2 + \tfrac{1}{2}(m_3 \\ &- m_1m_2)\varphi'\varphi'' - \tfrac{1}{6}(m_4 \\ &- m_1m_3)\varphi'\varphi''' - \tfrac{1}{8}(m_4 \\ &- m_2^2)\varphi''^2 + \tfrac{1}{24}\lambda_4\varphi'^4 + \cdots],\end{aligned} \tag{21}$$

* Notice that this is not the approximation $\ln(1 + y) \cong y$ which can be a poor approximation and yet has been quite widely used in FM work.

where φ stands for $\varphi(t - t_d)$ and λ_n's are the semi-invariants which are related to the moments by

$$\lambda_2 = m_2 - m_1^2$$

$$\lambda_3 = m_3 - 3m_1m_2 + 2m_1^3$$

$$\lambda_4 = m_4 - 4m_1m_3 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4.$$

Taking the real and imaginary parts of (21) and substituting the result into (12) and (13) gives, respectively,

$$\begin{aligned} a(t) = &-\alpha(f_c) + m_{1i}\varphi' - \frac{m_{2i}}{2!}\varphi'' + \frac{m_{3i}}{3!}\varphi''' - \frac{m_{4i}}{4!}\varphi'''' + \cdots \\ &- \frac{\lambda_{3i}}{6}\varphi'^3 + \frac{l_{1i}}{4}\varphi'^2\varphi'' - \frac{\lambda_{2r}}{2}\varphi'^2 + \frac{l_{2r}}{2}\varphi'\varphi'' \\ &- \frac{l_{3r}}{6}\varphi'\varphi''' - \frac{l_{5r}}{8}\varphi''^2 + \frac{\lambda_{4r}}{24}\varphi'^4 + \cdots \end{aligned} \tag{22}$$

$$\begin{aligned} \varphi_0(t) = &-\beta(f_c) + \varphi - m_{1r}\varphi' + \frac{m_{2r}}{2!}\varphi'' - \frac{m_{3r}}{3!}\varphi''' + \frac{m_{4r}}{4!}\varphi'''' - \cdots \\ &+ \frac{\lambda_{3r}}{6}\varphi'^3 - \frac{l_{1r}}{4}\varphi'^2\varphi'' - \frac{\lambda_{2i}}{2}\varphi'^2 + \frac{l_{2i}}{2}\varphi'\varphi'' - \frac{l_{3i}}{6}\varphi'\varphi''' \\ &- \frac{l_{5i}}{8}\varphi''^2 + \frac{\lambda_{4i}}{24}\varphi'^4 + \cdots, \end{aligned} \tag{23}$$

where the subscripts r and i denote the real and imaginary parts of the corresponding coefficients and

$$l_1 = m_4 - 2m_1m_3 - m_2^2 + 2m_1^2m_2\,,$$

$$l_2 = m_3 - m_1m_2\,,$$

$$l_3 = m_4 - m_1m_3\,,$$

$$l_5 = m_4 - m_2^2.$$

From (22) and (23), amplitude and phase distortions are divided into linear, second- and third-order terms and are shown in Table I. The terms $-\alpha(f_c)$ and $-\beta(f_c)$ in (22) and (23) do not appear in Table I. This is due to the fact that they are constants and introduce only constant amounts of amplitude and phase distortions.

IV. PRE-EMPHASIS CHARACTERISTIC

The multichannel baseband signal of an FM system is represented by a band of random noise. Assuming that the bottom baseband fre-

TABLE I — AMPLITUDE AND PHASE DISTORTIONS DUE TO IMPERFECT TRANSMISSION MEDIUM

Order of Distortion	Amplitude Distortion $a(t)$
Linear	$m_{1i}\varphi' - \frac{m_{2i}}{2!}\varphi'' + \frac{m_{3i}}{3!}\varphi''' - \frac{m_{4i}}{4!}\varphi'''' + \cdots$
Second-order	$-\frac{\lambda_{2r}}{2}\varphi'^2 + \frac{l_{2r}}{2}\varphi'\varphi'' - \frac{l_{3r}}{6}\varphi'\varphi''' - \frac{l_{5r}}{8}\varphi''^2 + \cdots$
Third-order	$-\frac{\lambda_{3i}}{6}\varphi'^3 + \frac{l_{1i}}{4}\varphi'^2\varphi'' + \cdots$
	Phase Distortion $\varphi_0(t) - \varphi(t)$
Linear	$-m_{1r}\varphi' + \frac{m_{2r}}{2!}\varphi'' - \frac{m_{3r}}{3!}\varphi''' + \frac{m_{4r}}{4!}\varphi'''' - \cdots$
Second-order	$-\frac{\lambda_{2i}}{2}\varphi'^2 + \frac{l_{2i}}{2}\varphi'\varphi'' - \frac{l_{3i}}{6}\varphi'\varphi''' - \frac{l_{5i}}{8}\varphi''^2 + \cdots$
Third-order	$\frac{\lambda_{3r}}{6}\varphi'^3 - \frac{l_{1r}}{4}\varphi'^2\phi'' + \cdots$

quency is much smaller than the top baseband frequency, the power-density spectrum of the baseband FM signal is expressed as

$$S_{\varphi_i'}(\omega) = P_0, \qquad |f| \leqq f_b,$$

where f_b is the top baseband frequency.

A pre-emphasis network is used in an FM system in order to optimize the noise across the baseband. Let $Z(\omega)$ be the transfer function of the pre-emphasis network, we write

$$|Z(\omega)|^2 = a_0 + a_2f^2 + a_4f^4 + a_6f^6, \qquad |f| \leqq f_b,$$

where the a's are real constants either given *a priori* or determined by the least squares fitting from an actual curve. The power-density spectrum of the pre-emphasized baseband FM signal is

$$\begin{aligned} S_{\varphi'}(\omega) &= |Z(\omega)|^2 S_{\varphi_i'}(\omega) \\ &= P_0(a_0 + a_2f^2 + a_4f^4 + a_6f^6), \qquad |f| \leqq f_b. \end{aligned} \tag{24}$$

In the case when the pre-emphasis coefficients a_0, a_2, a_4, and a_6 are determined by the least squares fitting from an actual curve, the following weighting function of normalized scale has been found useful for better approximations near the bottom channels

$$W(f/f_b) = 10^{-(z/10)[(f/f_b)-1]},$$

where Z is the difference of relative power (in dB) of top and bottom channels of the given pre-emphasis characteristic.

The rms frequency deviation, σ, due to noise loading can be expressed as

$$(2\pi\sigma)^2 = \text{ave}\,[\varphi'^2(t)] = \int_{-\infty}^{\infty} S_{\varphi'}(\omega)\, df.$$

Using (24), the relation between P_0 and σ can be expressed as

$$P_0 = \frac{(2\pi\sigma)^2}{2f_b\left[a_0 + \left(\frac{a_2 f_b^2}{3}\right) + \left(\frac{a_4 f_b^4}{5}\right) + \left(\frac{a_6 f_b^6}{7}\right)\right]}, \qquad (\text{rad/sec})^2/\text{Hz},$$

where the units of σ and f_b are Hz.

V. TRANSMISSION MEDIUM

Within the band of interest, $f_c \pm b$, let the gain and phase of the transmission medium be, respectively,

$$\exp\,[-\alpha(f + f_c)] = 1 + g_1\omega + g_2\omega^2 + g_3\omega^3 + g_4\omega^4 + \sum_{k=1}^{N} u_k \cos\,(p_k\omega + \theta_k) \tag{25}$$

$$-\beta(f + f_c) = b_2\omega^2 + b_3\omega^3 + b_4\omega^4 + \sum_{k=1}^{N} v_k \sin\,(q_k\omega + \sigma_k), \tag{26}$$

where the g's and b's represent coarse shape transmission deviation; the u's and v's represent fine shape transmission deviation. It should be emphasized that the fine shape transmission deviation is restricted to be a slowly varying ripple characteristic, hence this analysis does not apply to the noise due to single echo in general. The fine shape representation is useful when we study the effect on the noise of a given system due to the shift of the carrier. Since a constant delay does not introduce intermodulation noise, the linear term in ω is not included in (26). The transmission medium coefficients g, b, u, v, p, q, θ, and σ in (25) and (26) are either given directly or obtained approximately by a curve-fitting computer program.

Substituting (26) into (16), we have

$$t_d = -\sum_{k=1}^{N} v_k q_k \cos\sigma_k\,.$$

From (19), we can evaluate various moments, hence, the coefficients associated with the distortion terms in Table I in terms of transmission medium coefficients, as given in Appendix A. In the following sections we shall derive expressions for intermodulation noise calculation due to second- and third-order distortion terms.

VI. NOISE POWER DUE TO SECOND-ORDER DISTORTION TERM

Since

$$\frac{d}{dt}\varphi'^2 = 2\varphi'\varphi'',$$

$$\frac{d^2}{dt^2}\varphi'^2 = 2\varphi'\varphi''' + 2\varphi''^2,$$

the second-order PM distortion in Table I can be written approximately as

$$\varphi_2(t) = \left(-\tfrac{1}{2}\lambda_{2i} + \tfrac{1}{4}l_{2i}\frac{d}{dt} - \tfrac{1}{12}l_{3i}\frac{d^2}{dt^2}\right)\varphi'^2 + \tfrac{1}{24}l_{4i}\varphi''^2, \qquad (27)$$

where

$$l_{4i} = 4l_{3i} - 3l_{5i}\,.$$

The second-order PM distortion term of (27) can be represented by the block diagram shown in Fig. 2, where

$$H_1(\omega) = (\tfrac{1}{12}l_{3i}\omega^2 - \tfrac{1}{2}\lambda_{2i}) + i(\tfrac{1}{4}l_{2i}\omega),$$

$$H_2(\omega) = \tfrac{1}{24}l_{4i}\,.$$

The power-density spectrum of $\varphi_2(t)$ is[11]

$$S_{\varphi_2}(\omega) = H_1(-\omega)H_1(\omega)S_{\varphi'^2}(\omega) + H_1(-\omega)H_2(\omega)S_{\varphi'^2\varphi''^2}(\omega) + H_2(-\omega)H_1(\omega)S_{\varphi''^2\varphi'^2}(\omega) + H_2(-\omega)H_2(\omega)S_{\varphi''^2}(\omega), \qquad (28)$$

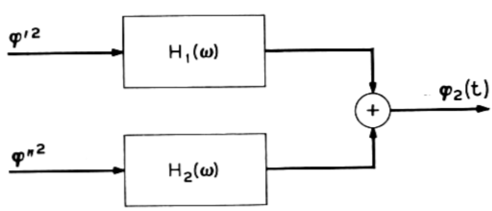

Fig. 2 — Block diagram representation of the second-order PM distortion term.

where $S_{\varphi'^2}(\omega)$ and $S_{\varphi''^2}(\omega)$ are the power-density spectra of φ'^2 and φ''^2, respectively, and $S_{\varphi'^2\varphi''^2}(\omega)$ [or $S_{\varphi''^2\varphi'^2}(\omega)$] is the cross-power-density spectrum of φ'^2 and φ''^2 (or φ''^2 and φ'^2). These quantities can be derived as[12]

$$\begin{aligned} S_{\varphi'^2}(\omega) &= \mathfrak{F}[2R_{\varphi'}^2(\tau) + R_{\varphi'}^2(0)], \\ S_{\varphi'^2\varphi''^2}(\omega) &= \mathfrak{F}[2R_{\varphi'\varphi''}^2(\tau) + R_{\varphi'}(0)R_{\varphi''}(0)], \\ S_{\varphi''^2\varphi'^2}(\omega) &= \mathfrak{F}[2R_{\varphi''\varphi'}^2(\tau) + R_{\varphi'}(0)R_{\varphi''}(0)], \\ S_{\varphi''^2}(\omega) &= \mathfrak{F}[2R_{\varphi''}^2(\tau) + R_{\varphi''}^2(0)], \end{aligned}$$

where $R_{\varphi'}(\tau)$ and $R_{\varphi''}(\tau)$ are the autocorrelation functions of φ' and φ'', respectively, and $R_{\varphi'\varphi''}(\tau)$ [or $R_{\varphi''\varphi'}(\tau)$] is the cross-correlation function of φ' and φ'' (or φ'' and φ'), and $\mathfrak{F}$ stands for "the Fourier transform of".

The Fourier transform of a constant function is a delta function at zero frequency. In the situation of evaluating noise power in the base-band, this quantity is not of interest. Also, it can be shown that

$$S_{\varphi'^2\varphi''^2}(\omega) = S_{\varphi''^2\varphi'^2}(\omega).$$

Thus, (28) can be simplified as

$$\begin{aligned} S_{\varphi_2}(\omega) = {} & 2\,|\,H_1(\omega)\,|^2\mathfrak{F}[R_{\varphi'}^2(\tau)] + 2\,|\,H_2(\omega)\,|^2\mathfrak{F}[R_{\varphi''}^2(\tau)] \\ & + 2[H_1(-\omega)H_2(\omega) + H_2(-\omega)H_1(\omega)]\mathfrak{F}[R_{\varphi'\varphi''}^2(\tau)]. \end{aligned} \tag{29}$$

The intermodulation noise to signal power ratio due to the second-order distortion term expressed in dB is, therefore,

$$N_2/S(\text{dB}) = 10 \log [\omega^2 S_{\varphi_2}(\omega)/S_{\varphi'}(\omega)]. \tag{30}$$

VII. NOISE POWER DUE TO THIRD-ORDER DISTORTION TERM

Since

$$\frac{d}{dt}\varphi'^3 = 3\varphi'^2\varphi'',$$

the third-order PM distortion in Table I can be written approximately as

$$\varphi_3(t) = \left(\tfrac{1}{6}\lambda_{3r} - \tfrac{1}{12}l_{1r}\frac{d}{dt}\right)\varphi'^3.$$

The above equation can be represented by the block diagram shown in Fig. 3, where

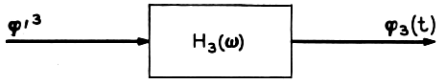

Fig. 3 — Block diagram representation of the third-order PM distortion term.

$$H_3(\omega) = (\tfrac{1}{6}\lambda_{3r}) + i(-\tfrac{1}{12}l_{1r}\omega).$$

The power-density spectrum of $\varphi_3(t)$ is

$$S_{\varphi_3}(\omega) = | H_3(\omega) |^2 S_{\varphi'^3}(\omega),$$

where $S_{\varphi'^3}(\omega)$ is the power-density spectrum of φ'^3 which can be derived as[12]

$$S_{\varphi'^3}(\omega) = 6\mathfrak{F}[R_{\varphi'}^{\ 3}(\tau)] + 9R_{\varphi'}^{\ 2}(0)S_{\varphi'}(\omega).$$

In the above equation, the term $9R_{\varphi'}^{\ 2}(0)S_{\varphi'}(\omega)$ is merely a scaled power-density spectrum of the input baseband FM signal, hence, it does not contribute to the intermodulation noise and can be neglected in the computation.

The intermodulation noise to signal power ratio due to the third-order distortion term expressed in dB is, therefore,

$$N_3/S(\text{dB}) = 10 \log [\omega^2 S_{\varphi_3}(\omega)/S_{\varphi'}(\omega)], \tag{31}$$

where

$$S_{\varphi_3}(\omega) = 6 | H_3(\omega) |^2 \mathfrak{F}[R_{\varphi'}^{\ 3}(\tau)]. \tag{32}$$

In (29) and (32), the Fourier transforms of $R_{\varphi'}^{\ 2}(\tau)$, $R_{\varphi''}^{\ 2}(\tau)$, $R_{\varphi'\varphi''}^{\ 2}(\tau)$, and $R_{\varphi'}^{\ 3}(\tau)$ may be obtained by taking the convolutions in the frequency domain. However, this requires numerical integration. For given pre-emphasis characteristics and P_0 (or σ), these Fourier transforms can be expressed in algebraic forms as shown in Appendix B. Hence, no numerical integration is necessary. A digital computer program has been written to calculate the second- and third-order intermodulation noise due to second- and third-order distortion terms in dB. A typical problem can be solved at a very low cost.

VIII. EXAMPLES

Several examples are considered in this paper. Calculated results are compared with measured data when they are available. Expressions for noise calculation are derived for simple cases. For more complicated situations, the noise calculation is best carried out by using a digital computer.

8.1 *Example 1*

In this example, we wish to demonstrate how the intermodulation noise across the baseband can be optimized using an appropriate pre-emphasis network. To simplify the calculation, we assume that the transmission characteristic consists of linear delay distortion (b_2) only. From Appendix A, we obtain

$$\lambda_{2i} = -2b_2 , \; l_{1r} = -8b_2^2$$

and all the other coefficients are equal to zero. Hence,

$$H_1(\omega) = b_2 , \qquad H_2(\omega) = 0, \qquad H_3(\omega) = i\tfrac{2}{3}b_2^2\omega.$$

In practical cases, the third-order distortion term $[H_3(\omega)]$ is negligible (say, 50 dB less) compared with the second-order distortion term. From (30) we write

$$\frac{N_2}{S}\,(\mathrm{dB}) = 10 \log \frac{2b_2^2\omega^2\, \mathfrak{F}[R_{\varphi'}^2(\tau)]}{S_{\varphi'}(\omega)} .$$

For simplicity, we let

$$S_{\varphi'}(\omega) = P_0(1 + a_2f^2), \qquad |f| \leqq f_b .$$

From Appendix B, we obtain

$$\mathfrak{F}[R_{\varphi'}^2(\tau)] = P_0^2 f_b \eta(f),$$

where

$$\begin{aligned} \eta(f) = & -\tfrac{1}{30}A_2^2\Omega^5 - \tfrac{2}{3}A_2\Omega^3 + (\tfrac{2}{3}A_2 + 2)A_2\Omega^2 \\ & - (1 + A_2)^2\Omega + (\tfrac{6}{15}A_2^2 + \tfrac{4}{3}A_2 + 2) \end{aligned}$$

$$A_2 = a_2f_b^2$$

$$\Omega = f/f_b .$$

Since

$$P_o = \frac{(2\pi\sigma)^2}{2f_b\left(1 + \dfrac{A_2}{3}\right)},$$

consequently,

$$\frac{N_2}{S}\,(\mathrm{dB}) = 10 \log \frac{(2\pi)^4(b_2f_b^2)^2\left(\dfrac{\sigma}{f_b}\right)^2 \Omega^2\eta(f)}{\left(1 + \dfrac{A_2}{3}\right)(1 + A_2\Omega^2)} .$$

Specifically, we let

$$b_2 = 7.962 \times 10^{-17} \text{ (linear delay of 1 nanosec/MHz)}$$

$$f_b = 1 \text{ MHz}$$

$$\sigma = 1 \text{ MHz}.$$

After several computer runs, the optimal choice of a_2 is 7. The intermodulation noise with no pre-emphasis and with the optimal pre-emphasis are plotted in Fig. 4. The noise has been reduced to more than 5 dB and is evenly distributed across the baseband by using the optimal pre-emphasis network.

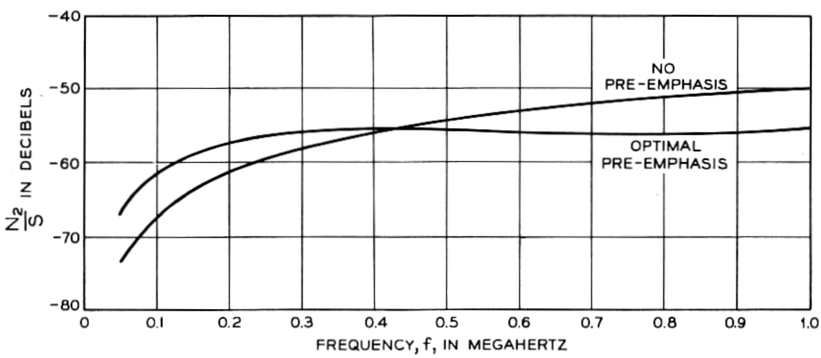

Fig. 4 — Intermodulation noise due to linear delay with and without pre-emphasis.

8.2 *Example 2*

In this example, we use a typical radio system pre-emphasis characteristic shown in Fig. 5. The top baseband frequency is $f_b = 5.772$ MHz. Since the given pre-emphasis characteristic is expressed as relative power in dB versus baseband frequency, it is first converted to ratio versus normalized baseband frequency, $\Omega = f/f_b$.

The weighting function is

$$W(\Omega) = 10^{-(9.5/10)(\Omega-1)}.$$

A least squares approximation program is used to obtain the approximating polynomial

$$a_0 + a_2 f^2 + a_4 f^4 + a_6 f^6,$$

where

$$a_0 = 0.99894166$$

$$a_2 = 11.944252/f_b^2$$

$$a_4 = -5.5771705/f_b^4$$

$$a_6 = 1.4396088/f_b^6.$$

Consider a transmission characteristic consisting of a linear, a parabolic, and a slowly varying sinusoidal delay as shown in Fig. 6. The expressions for the noise due to second- and third-order distortion are too complicated to write down. However, by using a digital computer, the results are plotted in Fig. 7 for $\sigma = 0.771$ MHz. Clearly, a better pre-emphasis network should be used to optimize the noise across the base-band for this particular transmission characteristic.

8.3 *Example 3*

As a final example, we consider a single pole IF filter with

$$Y(\omega + \omega_c) = \frac{1}{1 + i\dfrac{f}{w}},$$

where w is the 3-dB half-bandwidth of the filter. Using a least squares approximation with appropriate weighting function, the magnitude and phase of $Y(\omega + \omega_c)$ are expressed by

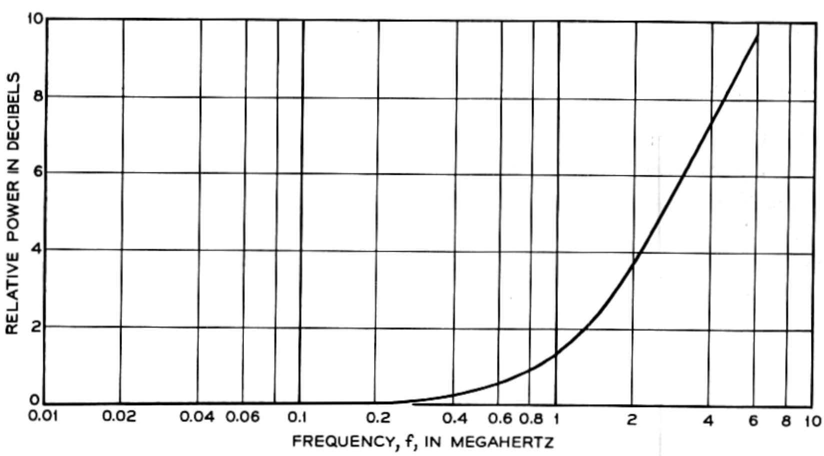

Fig. 5.—Pre-emphasis characteristic of a typical radio system.

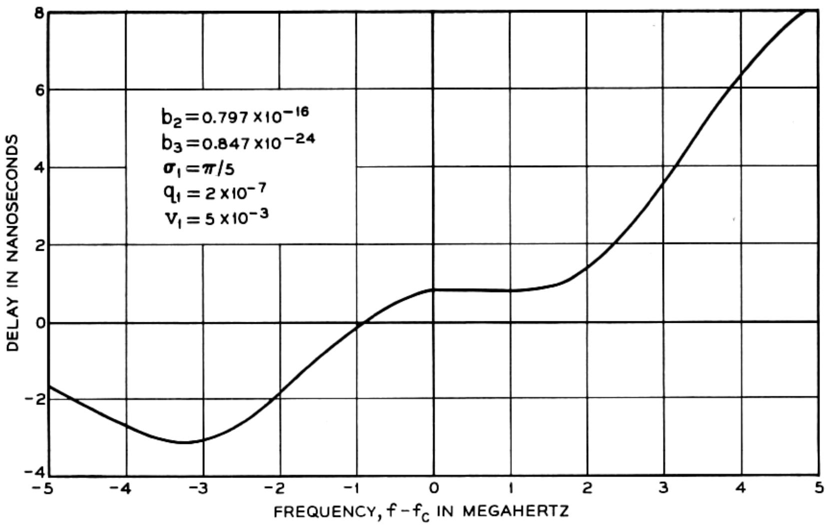

Fig. 6—An arbitrary delay characteristic of a transmission medium.

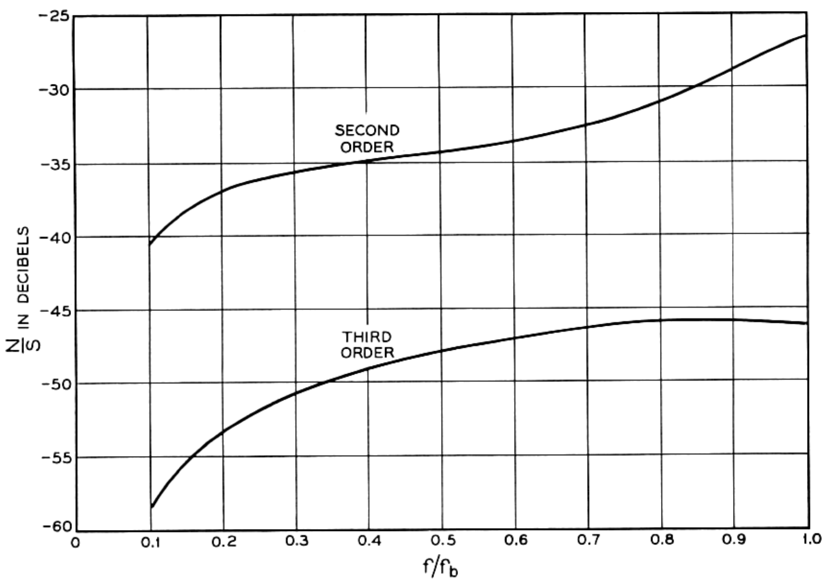

Fig. 7 — Intermodulation noise due to the delay distortion of Fig. 6.

$$\exp\,[-\alpha(f + f_c)] \cong 1 + g_2\omega^2 + g_4\omega^4 = 1 + G_2(f/w)^2 + G_4(f/w)^4$$

$$-\beta(f + f_c) \cong b_1\omega + b_3\omega^3 = B_1(f/w) + B_3(f/w)^3,$$

where

$$G_2 = g_2(2\pi w)^2 = -0.4209, \qquad G_4 = g_4(2\pi w)^4 = 0.08027$$

$$B_1 = b_1(2\pi w) = -0.9529, \qquad B_3 = b_3(2\pi w)^3 = 0.1294.$$

The actual and approximated transmission characteristic are plotted in Fig. 8. Since a constant delay does not introduce intermodulation noise,

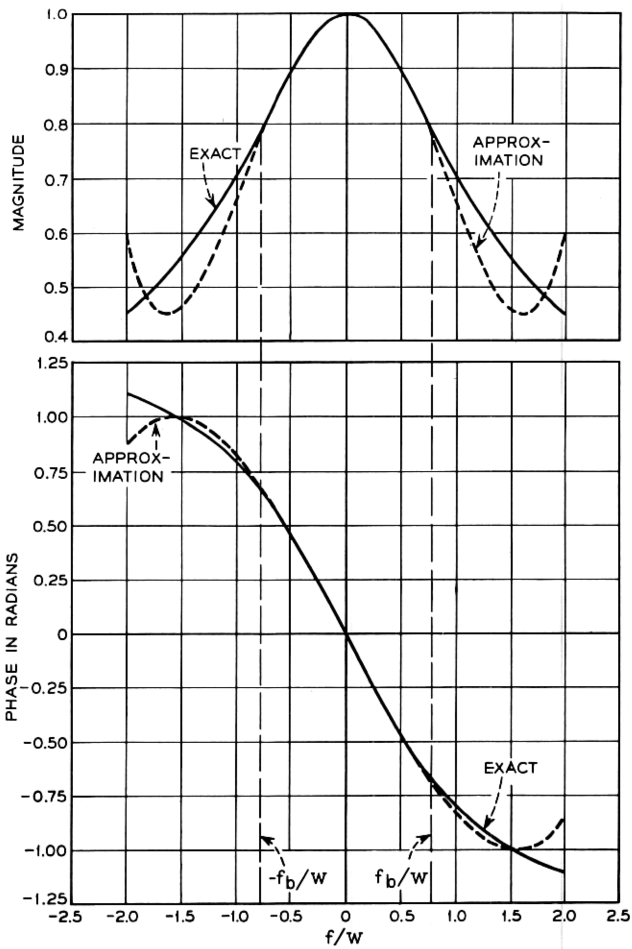

Fig. 8 — Gain and phase characteristic of a single pole filter.

b_1 is not included for calculation. Using the expressions in Appendix A, the coefficients associated with the second-order distortion term are

$$\lambda_{2i} = 0, \qquad l_{2i} = 0, \qquad l_{3i} = 0, \qquad l_{4i} = 0.$$

Hence,

$$H_1(\omega) = H_2(\omega) = 0.$$

The noise contribution due to second-order distortion is, therefore, zero.
The coefficients associated with the third-order distortion term are

$$\lambda_{3r} = 6b_3 , \qquad l_{1r} = 24g_4 - 4g_2^2.$$

Assuming no pre-emphasis, that is, $a_2 = a_4 = a_6 = 0$, from Appendix B, we have

$$\mathfrak{F}[R_{\varphi'}{}^3(\tau)] = P_0^3 f_b^2 a_0^3 (3 - \Omega^2), \qquad |\Omega| \leqq 1$$

where

$$\Omega = f/f_b .$$

Using the relation

$$P_o = \frac{(2\pi\sigma)^2}{2f_b\, a_o},$$

the intermodulation noise to signal power ratio due to the third-order distortion term can be derived from (31) as

$$\frac{N_3}{S}\,(\text{dB}) = 10 \log \frac{3}{2} \left(\frac{\sigma}{f_b}\right)^4 \left(\frac{f_b}{w}\right)^6 \Omega^2 \,(3 - \Omega^2) \left[B_3^2 + \left(\frac{f}{w}\right)^2 \left(2G_4 - \frac{G_2^2}{3}\right)^2 \right].$$

For $f_b = 1$ MHz and $w = 1.25$ MHz, N_3/S (dB) is calculated at $f = 0.084$, 0.36, and 1 MHz as a function of (σ/f_b). The dotted lines in Fig. 9 represent the calculated value (S/N_3) while the solid curves represent the measured data taken by W. F. Bodtmann.[13] The discrepancy between the measured and calculated values can be attributed to several reasons: (*i*) The power-density spectrum of the multichannel baseband signal used in the experiment was not perfectly rectangular, (*ii*) During the measurement, a non-ideal limiter was used which caused some AM-to-PM conversion, (*iii*) The actual transmission characteristic was approximated by few parameters in a limited region, and (*iv*) the formulas (30) and (31) derived in this paper involved approximations. Nevertheless, the measured and calculated results are close enough to show the utility of the analysis even for cases beyond the application for which it was originally intended.

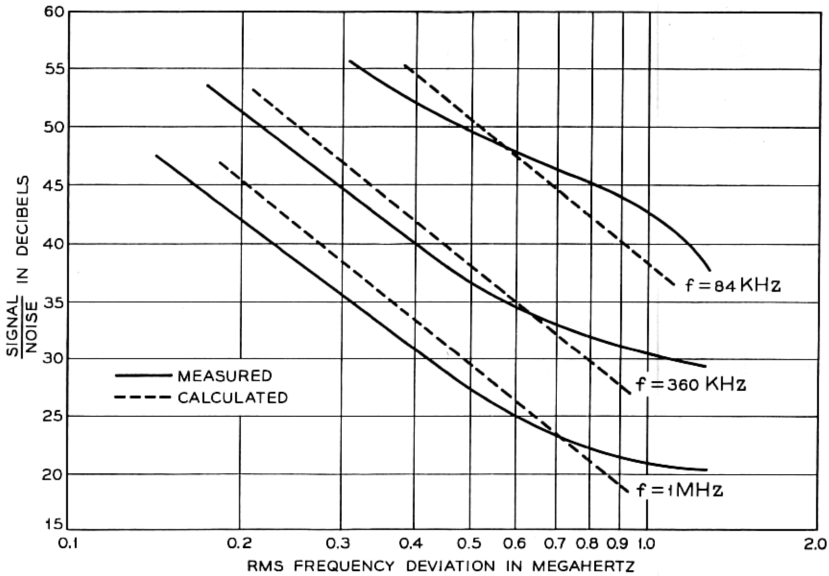

Fig. 9 — Measured and calculated intermodulation noise due to the single pole filter.

IX. DISTORTION DUE TO AM-TO-PM CONVERSION

Let K be the AM-PM conversion factor of a device expressed in degrees/dB, then the phase distortion due to AM-PM conversion is

$$\text{Phase Distortion due to AM-PM Conversion} = 20 \times 0.4343K\, a(t)$$
$$= 8.686K\, a(t) \text{ degrees,}$$

where $a(t)$ is given by (22).

Similarly,

$$\text{Frequency Distortion due to AM-PM Conversion} = \frac{8.686}{2\pi} K \frac{d}{dt} a(t)$$
$$= 1.382\, K \frac{d}{dt} a(t)\ Hz.$$

In Table I, the second- and third-order terms of amplitude distortion cause intermodulation noise while the linear term causes video roll-off or enhancement. Using the same approach discussed previously in this paper, noise calculation due to AM-to-PM conversion which is caused by transmission deviation can also be accomplished.

X. ACKNOWLEDGMENTS

The analysis carried out in this paper is based on an unpublished work of S. O. Rice. His permission to present his result here is greatly appreciated. The author is grateful to J. T. Bangert for his encouragement and direction and also wishes to thank J. J. Lang, F. J. Witt, L. H. Enloe, C. L. Ruthroff, and W. T. Barnett for many constructive suggestions. The author is indebted to Miss J. D. Witkowski for the digital computer programming and to many other colleagues who have helped to carry out this analysis.

APPENDIX A

Coefficients Associated with the Distortion Terms

The moments defined in (19) can be expressed as

$$m_0 = 1$$

$$m_1 = i\left(\frac{A'}{A}\right)$$

$$m_2 = \left(-\frac{A''}{A}\right) + i\,(-B'')$$

$$m_3 = \left(B''' + \frac{3A'B''}{A}\right) + i\left(-\frac{A'''}{A}\right)$$

$$m_4 = \left(-3B''^2 + \frac{A''''}{A}\right) + i\left(B'''' + \frac{4A'B'''}{A} + \frac{6A''B''}{A}\right),$$

where

$$A = 1 + \sum_{k=1}^{N} u_k \cos\theta_k$$

$$A' = g_1 - \sum_{k=1}^{N} u_k p_k \sin\theta_k$$

$$A'' = 2g_2 - \sum_{k=1}^{N} u_k p_k^2 \cos\theta_k$$

$$A''' = 6g_3 + \sum_{k=1}^{N} u_k p_k^3 \sin\theta_k$$

$$A'''' = 24g_4 + \sum_{k=1}^{N} u_k p_k^4 \cos\theta_k$$

$$B'' = 2b_2 - \sum_{k=1}^{N} v_k q_k^2 \sin \sigma_k$$

$$B''' = 6b_3 - \sum_{k=1}^{N} v_k q_k^3 \cos \sigma_k$$

$$B'''' = 24b_4 + \sum_{k=1}^{N} v_k q_k^4 \sin \sigma_k .$$

The coefficients associated with the second-order distortion are

$$\lambda_{2i} = -B''$$

$$l_{2i} = -\frac{A'''}{A} + \frac{A'A''}{A^2}$$

$$l_{3i} = B'''' + \frac{3A'B'''}{A} + \frac{6A''B''}{A} - \frac{3A'^2B''}{A^2}$$

$$l_{4i} = B'''' + \frac{12A''B''}{A} - 12\frac{A'^2B''}{A^2} .$$

The coefficients associated with the third-order distortion are

$$\lambda_{3r} = B'''$$

$$l_{1r} = -2B''^2 + \frac{A''''}{A} - \frac{A''^2}{A^2} - 2\frac{A'A'''}{A^2} + 2\frac{A'^2A''}{A^3} .$$

APPENDIX B

Fourier Transforms of $R_{\varphi'^2}(\tau)$, $R_{\varphi'\varphi''^2}(\tau)$, $R_{\varphi''^2}(\tau)$ and $R_{\varphi'^3}(\tau)$

The Fourier transforms of $R_{\varphi'^2}(\tau)$, $R_{\varphi'\varphi''^2}(\tau)$, $R_{\varphi''^2}(\tau)$ and $R_{\varphi'^3}(\tau)$ can be derived in a straightforward manner. However, considerable amount of algebra has been involved during the derivation. Without writing out the details here we merely present the final results.

$$\begin{aligned}\mathfrak{F}[R_{\varphi'^2}(\tau)] = P_0^2 f_b\{ & D_1(f-2f_b) + 2D_1(f) + D_1(f+2f_b) \\ & - D_2(f-2f_b) + 2D_2(f) - D_2(f+2f_b) \\ & + 2D_3(f-2f_b) - 2D_3(f+2f_b)\}\end{aligned}$$

$$\begin{aligned}\mathfrak{F}[R_{\varphi'^3}(\tau)] = P_0^3 f_b^2\{ & 3E_1(f-f_b) + E_1(f-3f_b) + E_1(f+3f_b) \\ & + 3E_1(f+f_b) + 3E_4(f-f_b) - 3E_4(f-3f_b) \\ & - 3E_4(f+3f_b) + 3E_4(f+f_b) + 3E_3(f-f_b)\end{aligned}$$

$$+ 3E_3(f - 3f_b)$$
$$- 3E_3(f + 3f_b) - 3E_3(f + f_b) + 3E_2(f - f_b)$$
$$- E_2(f - 3f_b) + E_2(f + 3f_b) - 3E_2(f + f_b)\}$$

$$\mathfrak{F}[R_{\varphi'\varphi''}{}^2(\tau)] = 4\pi^2 P_0^2 f_b^3\{J_1(f - 2f_b) + 2J_1(f) + J_1(f + 2f_b) - J_2(f - 2f_b) + 2J_2(f) - J_2(f + 2f_b) - 2J_3(f - 2f_b) + 2J_3(f + 2f_b)\}$$

$$\mathfrak{F}[R_{\varphi''}{}^2(\tau)] = 16\pi^4 P_0^2 f_b^5\{K_1(f - 2f_b) + 2K_1(f) + K_1(f + 2f_b) - K_2(f - 2f_b) + 2K_2(f) - K_2(f + 2f_b) + 2K_3(f - 2f_b) - 2K_3(f + 2f_b)\},$$

where

$$D_1(f) = \sum_{n=2}^{6} (-1)^n \frac{d_{1n}}{2(2n-1)!}\left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$D_2(f) = \sum_{n=1}^{7} (-1)^n \frac{d_{2n}}{2(2n-1)!}\left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$D_3(f) = \sum_{n=1}^{6} (-1)^{n+1} \frac{d_{3n}}{2(2n)!}\left(\frac{f}{f_b}\right)^{2n} \operatorname{sgn} f$$

$$E_1(f) = \sum_{n=3}^{9} (-1)^n \frac{e_{1n}}{2(2n-1)!}\left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$E_2(f) = \sum_{n=1}^{10} (-1)^{n+1} \frac{e_{2n}}{2(2n)!}\left(\frac{f}{f_b}\right)^{2n} \operatorname{sgn} f$$

$$E_3(f) = \sum_{n=2}^{9} (-1)^{n+1} \frac{e_{3n}}{2(2n)!}\left(\frac{f}{f_b}\right)^{2n} \operatorname{sgn} f$$

$$E_4(f) = \sum_{n=2}^{10} (-1)^n \frac{e_{4n}}{2(2n-1)!}\left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$J_1(f) = \sum_{n=1}^{7} (-1)^n \frac{j_{1n}}{2(2n-1)!}\left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$J_2(f) = \sum_{n=2}^{8} (-1)^n \frac{j_{2n}}{2(2n-1)!}\left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$J_3(f) = \sum_{n=1}^{7} (-1)^{n+1} \frac{j_{3n}}{2(2n)!}\left(\frac{f}{f_b}\right)^{2n} \operatorname{sgn} f$$

$$K_1(f) = \sum_{n=2}^{8} (-1)^n \frac{k_{1n}}{2(2n-1)!}\left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$K_2(f) = \sum_{n=1}^{9} (-1)^n \frac{k_{2n}}{2(2n-1)!} \left(\frac{f}{f_b}\right)^{2n-1} \operatorname{sgn} f$$

$$K_3(f) = \sum_{n=1}^{8} (-1)^{n+1} \frac{k_{3n}}{2(2n)!} \left(\frac{f}{f_b}\right)^{2n} \operatorname{sgn} f.$$

The coefficients in the above equations are given as

$$d_{12} = c_1^2, \quad d_{13} = 2c_1c_2, \quad d_{14} = c_2^2 + 2c_1c_3$$

$$d_{15} = 2c_2c_3, \quad d_{16} = c_3^2$$

$$d_{21} = c_4^2, \quad d_{22} = 2c_4c_5, \quad d_{23} = c_5^2 + 2c_4c_6$$

$$d_{24} = 2c_4c_7 + 2c_5c_6, \quad d_{25} = c_6^2 + 2c_5c_7$$

$$d_{26} = 2c_6c_7, \quad d_{27} = c_7^2$$

$$d_{31} = c_1c_4, \quad d_{32} = c_1c_5 + c_2c_4, \quad d_{33} = c_1c_6 + c_2c_5 + c_3c_4$$

$$d_{34} = c_1c_7 + c_2c_6 + c_3c_5, \quad d_{35} = c_2c_7 + c_3c_6, \quad d_{36} = c_3c_7$$

$$e_{13} = c_1^3, \quad e_{14} = 3c_1^2c_2, \quad e_{15} = 3c_1c_2^2 + 3c_1^2c_3$$

$$e_{16} = 6c_1c_2c_3 + c_2^3, \quad e_{17} = 3c_1c_3^2 + 3c_2^2c_3, \quad e_{18} = 3c_2c_3^2$$

$$e_{19} = c_3^3, \quad e_{21} = c_4^3, \quad e_{22} = 3c_4^2c_5$$

$$e_{23} = 3c_4c_5^2 + 3c_4^2c_6, \quad e_{24} = 3c_4^2c_7 + 6c_4c_5c_6 + c_5^3$$

$$e_{25} = 3c_4c_6^2 + 6c_4c_5c_7 + 3c_5^2c_6, \quad e_{26} = 6c_4c_6c_7 + 3c_5c_6^2 + 3c_5^2c_7$$

$$e_{27} = 3c_4c_7^2 + 6c_5c_6c_7 + c_6^3, \quad e_{28} = 3c_5c_7^2 + 3c_6^2c_7$$

$$e_{29} = 3c_6c_7^2, \quad e_{210} = c_7^3, \quad e_{32} = c_1^2c_4$$

$$e_{33} = 2c_1c_2c_4 + c_1^2c_5, \quad e_{34} = c_2^2c_4 + 2c_1c_3c_4 + 2c_1c_2c_5 + c_1^2c_6$$

$$e_{35} = 2c_2c_3c_4 + c_2^2c_5 + 2c_1c_3c_5 + 2c_1c_2c_6 + c_1^2c_7$$

$$e_{36} = c_3^2c_4 + 2c_2c_3c_5 + c_2^2c_6 + 2c_1c_3c_6 + 2c_1c_2c_7$$

$$e_{37} = c_3^2c_5 + 2c_2c_3c_6 + c_2^2c_7 + 2c_1c_3c_7$$

$$e_{38} = c_3^2c_6 + 2c_2c_3c_7, \quad e_{39} = c_3^2c_7, \quad e_{42} = c_1c_4^2$$

$$e_{43} = 2c_1c_4c_5 + c_2c_4^2, \quad e_{44} = c_1c_5^2 + 2c_1c_4c_6 + 2c_2c_4c_5 + c_3c_4^2$$

$$e_{45} = 2c_1c_4c_7 + 2c_1c_5c_6 + c_2c_5^2 + 2c_3c_4c_5 + 2c_2c_4c_6$$

$$e_{46} = c_1c_6^2 + 2c_1c_5c_7 + 2c_2c_4c_7 + 2c_2c_5c_6 + c_3c_5^2 + 2c_3c_4c_6$$

$$e_{47} = 2c_1c_6c_7 + c_2c_6^2 + 2c_2c_5c_7 + 2c_3c_4c_7 + 2c_3c_5c_6$$

$$e_{48} = c_1c_7^2 + 2c_2c_6c_7 + c_3c_6^2 + 2c_3c_5c_7$$

$$e_{49} = c_2c_7^2 + 2c_3c_6c_7\,, \quad e_{410} = c_3c_7^2$$

$$j_{11} = c_4^2, \quad j_{12} = 2c_4(c_5 - 2c_1)$$

$$j_{13} = (c_5 - 2c_1)^2 + 2c_4(c_6 - 4c_2)$$

$$j_{14} = 2c_4(c_7 - 6c_3) + 2(c_5 - 2c_1)(c_6 - 4c_2)$$

$$j_{15} = (c_6 - 4c_2)^2 + 2(c_5 - 2c_1)(c_7 - 6c_3)$$

$$j_{16} = 2(c_6 - 4c_2)(c_7 - 6c_3), \quad j_{17} = (c_7 - 6c_3)^2$$

$$j_{22} = (c_1 + c_4)^2, \quad j_{23} = 2(c_1 + c_4)(c_2 + 3c_5)$$

$$j_{24} = (c_2 + 3c_5)^2 + 2(c_1 + c_4)(c_3 + 5c_6)$$

$$j_{25} = 14c_7(c_1 + c_4) + 2(c_2 + 3c_5)(c_3 + 5c_6)$$

$$j_{26} = (c_3 + 5c_6)^2 + 14c_7(c_2 + 3c_5), \quad j_{27} = 14c_7(c_3 + 5c_6)$$

$$j_{28} = 49c_7^2, \quad j_{31} = c_4(c_1 + c_4)$$

$$j_{32} = c_4(c_2 + 3c_5) + (c_1 + c_4)(c_5 - 2c_1)$$

$$j_{33} = c_4(c_3 + 5c_6) + (c_2 + 3c_5)(c_5 - 2c_1) + (c_1 + c_4)(c_6 - 4c_2)$$

$$j_{34} = 7c_4c_7 + (c_5 - 2c_1)(c_3 + 5c_6) + (c_6 - 4c_2)(c_2 + 3c_5) + (c_7 - 6c_3)(c_1 + c_4)$$

$$j_{35} = 7c_7(c_5 - 2c_1) + (c_3 + 5c_6)(c_6 - 4c_2) + (c_2 + 3c_5)(c_7 - 6c_3)$$

$$j_{36} = 7c_7(c_6 - 4c_2) + (c_3 + 5c_6)(c_7 - 6c_3), \quad j_{37} = 7c_7(c_7 - 6c_3)$$

$$k_{12} = (c_1 + 2c_4)^2, \quad k_{13} = -2(c_1 + 2c_4)(6c_1 - c_2 - 6c_5)$$

$$k_{14} = (6c_1 - c_2 - 6c_5)^2 - 2(c_1 + 2c_4)(20c_2 - c_3 - 10c_6)$$

$$k_{15} = 2(6c_1 - c_2 - 6c_5)(20c_2 - c_3 - 10c_6) - 2(c_1 + 2c_4)(42c_3 - 14c_7)$$

$$k_{16} = (20c_2 - c_3 - 10c_6)^2 + 2(6c_1 - c_2 - 6c_5)(42c_3 - 14c_7)$$

$$k_{17} = 2(20c_2 - c_3 - 10c_6)(42c_3 - 14c_7), \quad k_{18} = (42c_3 - 14c_7)^2$$

$$k_{21} = c_4^2, \quad k_{22} = -2c_4(4c_1 - c_5 + 2c_4)$$

$$k_{23} = (4c_1 - c_5 + 2c_4)^2 - 2c_4(8c_2 + 12c_5 - c_6)$$

$$k_{24} = 2(4c_1 - c_5 + 2c_4)(8c_2 + 12c_5 - c_6) - 2c_4(12c_3 + 30c_6 - c_7)$$

$$k_{25} = (8c_2 + 12c_5 - c_6)^2 - 112c_4c_7 + 2(4c_1 - c_5 + 2c_4)(12c_3 + 30c_6 - c_7)$$

$$k_{26} = 112c_7(4c_1 - c_5 + 2c_4) + 2(8c_2 + 12c_5 - c_6)(12c_3 + 30c_6 - c_7)$$

$$k_{27} = (12c_3 + 30c_6 - c_7)^2 + 112c_7(8c_2 + 12c_5 - c_6)$$

$$k_{28} = 112c_7(12c_3 + 30c_6 - c_7), \quad k_{29} = (56c_7)^2$$

$$k_{31} = c_4(c_1 + 2c_4)$$

$$k_{32} = -(c_1 + 2c_4)(4c_1 - c_5 + 2c_4) - c_4(6c_1 - c_2 - 6c_5)$$

$$k_{33} = -(c_1 + 2c_4)(8c_2 + 12c_5 - c_6) + (6c_1 - c_2 - 6c_5)(4c_1 - c_5 + 2c_4) - c_4(20c_2 - c_3 - 10c_6)$$

$$k_{34} = -(c_1 + 2c_4)(12c_3 + 30c_6 - c_7) + (6c_1 - c_2 - 6c_5)(8c_2 + 12c_5 - c_6) + (20c_2 - c_3 - 10c_6)(4c_1 - c_5 + 2c_4) - c_4(42c_3 - 14c_7)$$

$$k_{35} = -56c_7(c_1 + 2c_4) + (6c_1 - c_2 - 6c_5)(12c_3 + 30c_6 - c_7) + (20c_2 - c_3 - 10c_6)(8c_2 + 12c_5 - c_6) + (42c_3 - 14c_7)(4c_1 - c_5 + 2c_4)$$

$$k_{36} = 56c_7(6c_1 - c_2 - 6c_5) + (20c_2 - c_3 - 10c_6)(12c_3 + 30c_6 - c_7) + (42c_3 - 14c_7)(8c_2 + 12c_5 - c_6)$$

$$k_{37} = 56c_7(20c_2 - c_3 - 10c_6) + (42c_3 - 14c_7)(12c_3 + 30c_6 - c_7)$$

$$k_{38} = 56c_7(42c_3 - 14c_7),$$

where

$$c_1 = 2(a_2f_b^2 + 2a_4f_b^4 + 3a_6f_b^6)$$

$$c_2 = -24(a_4f_b^4 + 5a_6f_b^6), \quad c_3 = 720a_6f_b^6$$

$$c_4 = a_0 + a_2f_b^2 + a_4f_b^4 + a_6f_b^6$$

$$c_5 = -2(a_2f_b^2 + 6a_4f_b^4 + 15a_6f_b^6)$$

$$c_6 = 24(a_4f_b^4 + 15a_6f_b^6), \quad c_7 = -720a_6f_b^6.$$

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