Iterative Solution of Waveguide Discontinuity Problems

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The application of matrix iterative analysis to the solution of waveguide discontinuity problems is discussed. It is concluded that the "Gauss-Seidel" or "point-single-step" method offers several advantages over more conventional invertive procedures, particularly in the speed of execution. Two examples are presented as illustrations: analysis of an H-plane discontinuity in a rectangular waveguide and conversion from TE_{11} *to* TM_{11} *modes at an abrupt discontinuity in a circular waveguide. The latter results are shown to be in good agreement with measured values obtained in a previous investigation.*

I. INTRODUCTION

The analysis of waveguide discontinuities, for application to the design of antennas and microwave networks, continues to offer challenging problems in electromagnetic theory and microwave engineering. Thus far, the solution of these problems has depended to a large extent on various approximate techniques, such as variational and quasi-static methods,[1] which are extremely useful but nevertheless limited in applicability.

The shortcomings of classical analysis have been surmounted to a large extent by our ability to solve electromagnetic boundary value problems by numerical methods, making extensive use of digital computers. Computational techniques are not only an abundant source of engineering data, which might otherwise require elaborate construction and experiment, but they can also provide a unique analytical laboratory in which to evaluate approximate theoretical methods under easily controlled conditions. In this paper, we shall be concerned with these numerical methods as they apply to certain waveguide discontinuity problems.

For the sake of simplicity we shall consider, as an example, the problem of two waveguides with similar cross sections connected together at the plane $z = 0$, as illustrated in Fig. 1. A wave is shown incident from the smaller waveguide impinging on the discontinuity. The result will, of course, be to excite an infinite number of normal modes in each guide, some of which carry real power away from the junction, with the remainder being evanescent and contributing to the electromagnetic field only in the vicinity of the connecting aperture. It must be recognized that these evanescent modes play an important role since they, in part, determine the amplitudes and phases of the propagating modes. It is the fact that an infinite number of waves must, in principle, be considered that makes this type of problem so difficult.

The contents of the paper may be summarized as follows: We begin by establishing an appropriate form of the uniqueness theorem for Maxwell's equations as they apply to boundary value problems of this type. In numerical analysis, the criteria for uniqueness are of more than academic interest since they provide meaningful and practical methods by which to assess the accuracy of results. Next, the normal mode representation of the fields is discussed, the object being to arrive at a matrix equation formulation of the problem in which the components of the unknown vector are the modal coefficients. It is

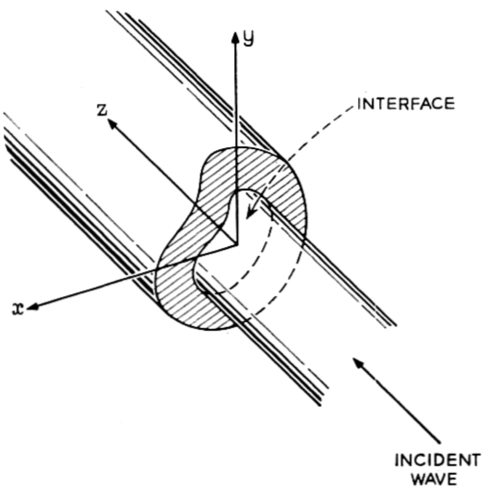

Fig. 1 — Waveguides of similar cross section connected at the plane $z = 0$ by an abrupt discontinuity. A wave is assumed incident from the smaller guide.

suggested that this matrix equation may be solved by an iterative procedure and, upon studying the convergence properties of such methods we find a critical dependence on the particular algorithm used. Two examples will be presented as illustrations, analysis of an H-plane discontinuity in a rectangular waveguide, and conversion from TE_{11} to TM_{11} modes at an abrupt discontinuity in a circular waveguide. The latter results are shown to be in good agreement with measured values obtained in a previous investigation.

Rationalized MKS units and the (suppressed) harmonic time dependence exp $(-i\omega t)$ will be used, unless otherwise specified.

II. UNIQUENESS AND ERROR CRITERIA

A representation of the discontinuity is shown in Fig. 2. It is assumed that the regions to the left (denoted by $-$) and to the right (denoted by $+$) are each filled with homogeneous material, but with possibly different constitutive parameters. Maxwell's curl equations in the respective regions are thus given by

$$\begin{aligned} \nabla \times \mathbf{E}^{\pm} &= -\mu^{\pm} \frac{\partial \mathbf{H}^{\pm}}{\partial t} \\ \nabla \times \mathbf{H}^{\pm} &= \epsilon^{\pm} \frac{\partial \mathbf{E}^{\pm}}{\partial t}. \end{aligned} \tag{1}$$

As usual for uniqueness theorems, we begin with two solutions in each region *presumed* to be correct, and denote the differences respectively by $\mathbf{E}^{\pm}$, $\mathbf{H}^{\pm}$.* Then from the Poynting theorem,[2] it follows that

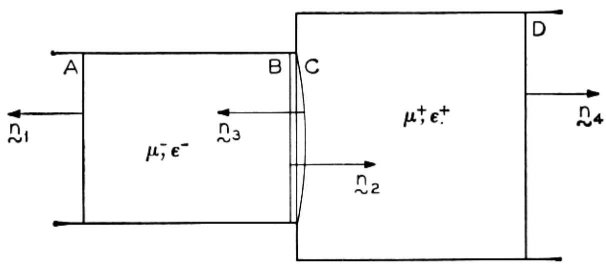

Fig. 2 — Waveguide discontinuity showing boundary surfaces A, B, C, D and respective normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, $\mathbf{n}_4$.

* Physically, these fields would correspond to a waveguide discontinuity problem without excitation.

in the − region

$$\iint_A (\mathbf{E}_t^- \times \mathbf{H}_t^-)\cdot\mathbf{n}_1 \, dS + \iint_B (\mathbf{E}_t^- \times \mathbf{H}_t^-)\cdot\mathbf{n}_2 \, dS$$
$$= -\epsilon^- \frac{\partial}{\partial t} \iiint_{V-} \mathbf{E}^-\cdot\mathbf{E}^- \, dV - \mu^- \frac{\partial}{\partial t} \iiint_{V-} \mathbf{H}^-\cdot\mathbf{H}^- \, dV \tag{2}$$

and in the + region that

$$\iint_C (\mathbf{E}_t^+ \times \mathbf{H}_t^+)\cdot\mathbf{n}_3 \, dS + \iint_D (\mathbf{E}_t^+ \times \mathbf{H}_t^+)\cdot\mathbf{n}_4 \, dS$$
$$= -\epsilon^+ \frac{\partial}{\partial t} \iiint_{V+} \mathbf{E}^+\cdot\mathbf{E}^+ \, dV - \mu^+ \frac{\partial}{\partial t} \iiint_{V+} \mathbf{H}^+\cdot\mathbf{H}^+ \, dV, \tag{3}$$

where $\partial/\partial t$ denotes differentiation with respect to time and the subscript t denotes the field transverse to the generatrix of the cylinder. The unit normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, and $\mathbf{n}_4$ are shown in Fig. 2.

One must also take into account the fact that certain physical considerations will limit the class of admissible solutions. For example, if we let the surfaces A and D recede to infinity, then all evanescent modes will have decayed to zero and the respective surface integrals then represent the power flow away from the discontinuity. Assuming no loss, the total power must vanish. Furthermore, it can be shown from Maxwell's equations that the transverse components of electric and magnetic field at the interface must be continuous. Adding (2) and (3), we find that the following time derivative must vanish,

$$\frac{\partial}{\partial t}\left[\epsilon^- \iiint_{V-} \mathbf{E}^-\cdot\mathbf{E}^- \, dV + \mu^- \iiint_{V-} \mathbf{H}^-\cdot\mathbf{H}^- \, dV \right.$$
$$\left. + \epsilon^+ \iiint_{V+} \mathbf{E}^+\cdot\mathbf{E}^+ \, dV + \mu^+ \iiint_{V+} \mathbf{H}^+\cdot\mathbf{H}^+ \, dV \right] = 0, \tag{4}$$

We may, however, regard the quantity in brackets as having had a zero value at some time, say at $t = 0$, the excitation time. The term in brackets therefore, vanishes for all time and, since each of the integrands is positive semi-definite, they must vanish separately. Thus, at each point in the + and − regions,

$$\begin{aligned} \mathbf{E}_1^+ - \mathbf{E}_2^+ &= \mathbf{H}_1^+ - \mathbf{H}_2^+ \equiv 0 \\ \mathbf{E}_1^- - \mathbf{E}_2^- &= \mathbf{H}_1^- - \mathbf{H}_2^- \equiv 0 \end{aligned} \tag{5}$$

and the solution is thereby shown to be unique. We may now state the following uniqueness theorem for waveguide discontinuity problems.

Theorem: The solution to a waveguide discontinuity problem is uniquely specified if it can be shown to have the following properties:

(*i*) *It satisfies Maxwell's equations and the appropriate boundary conditions in the regions on each side of the discontinuity.*

(*ii*) *The components of electric and magnetic fields tangent to the interface are continuous.*

(*iii*) *In the case of a lossless discontinuity, energy is conserved.*

These three conditions obviously play an important theoretical role in the solution; where numerical methods are used, they also provide fundamental criteria by which the accuracy of computed results can be assessed. Accordingly, we shall define the following quantities to be used as error criteria: First, there is the parameter ε_P, which indicates how well the solution conserves energy, given by

$$\varepsilon_P = \frac{P_r + P_t}{P_{\text{inc}}} - 1, \tag{6}$$

where P_r, P_t, and P_{inc} are the reflected, transmitted, and incident powers, respectively. Second, the mean square error in the tangential electric field is defined by

$$\varepsilon_E = \frac{\iint_{\text{Aperture}} |(\mathbf{E}_t^+ - \mathbf{E}_t^-)|^2 \, dA}{\iint_{\text{Aperture}} |\mathbf{E}_t^{(\text{inc})}|^2 \, dA} \tag{7}$$

and third, for the magnetic field,

$$\varepsilon_H = \frac{\iint_{\text{Aperture}} |(\mathbf{H}_t^+ - \mathbf{H}_t^-)|^2 \, dA}{\iint_{\text{Aperture}} |\mathbf{H}_t^{(\text{inc})}|^2 \, dA}, \tag{8}$$

where $\mathbf{E}^{(\text{inc})}$ and $\mathbf{H}^{(\text{inc})}$ refer to the incident wave.

The smaller the quantities ε_P, ε_E, and ε_H, the more closely the boundary conditions are satisfied, at least in the mean square sense, and the more accurate we shall consider the solution to be.

III. MATRIX FORMULATION OF THE BOUNDARY VALUE PROBLEM

The most convenient format for numerical solution of waveguide discontinuity problems is a matrix representation, in which the modal coefficients form the unknown column vectors and the discontinuity is characterized by a square matrix. We recognize that there is also an analogous integral equation in terms of the aperture electric or magnetic field. However, since the numerical solution of the integral equation is generally carried out by reducing it to a matrix equation, we shall proceed to the matrix formulation directly from the physical characterization of the boundary value problem. This matrix equation will then be solved by an iterative method, the theory of which is discussed in Section IV.

It is assumed that in each of the waveguides, the electromagnetic fields may be characterized by a denumerable set of known vector eigenfunctions which may be ordered according to some index. We shall be concerned only with the *transverse* fields,* denoted as follows:

${}^{+}\mathbf{E}_p'(\mathbf{r})$ $(p = 1, 2, 3, \cdots)$ denotes the transverse electric field for the pth TM mode in the $+$ waveguide, with $\mathbf{r}$ as the position vector in the transverse plane.

${}^{+}\mathbf{H}_p'(\mathbf{r})$ = transverse magnetic field for the pth TM mode in the $+$ waveguide.

${}^{+}\mathbf{E}_p''(\mathbf{r})$ = transverse electric field for the pth TE mode in the $+$ waveguide.

${}^{+}\mathbf{H}_p''(\mathbf{r})$ = transverse magnetic field for the pth TE mode in the $+$ waveguide.

By replacing the $+$ by $-$ we have the analogous notation for the other waveguide. An important point concerning sign convention is that the unknown modes in the $-$ waveguide will all be taken to propagate away from the discontinuity, i.e., in the $-z$ direction. Although the electric field does not change sign when the direction of propagation is reversed, the magnetic field does, and this fact must be carefully taken into account.

In order to define the amplitudes of the respective vector wave functions, we adopt the following normalization,[3] written in terms of integrals over the waveguide cross sections:

$$\int_{\pm A} {}^{\pm}\mathbf{E}_p' \cdot {}^{\pm}\mathbf{E}_q'^{*} \, dA = | \, {}^{\pm}h_p' \, |^2 \, \delta_{pq} \tag{9}$$

$$\int_{\pm A} {}^{\pm}\mathbf{E}_p'' \cdot {}^{\pm}\mathbf{E}_q''^{*} \, dA = \omega^2 \mu^2 \, \delta_{pq} \tag{10}$$

in which h_p is the respective characteristic wavenumber, and μ is the permeability, which in our case will be the permeability of vacuum,

* It is assumed that the individual waveguides can support pure TE and TM modes, which is the case for applications of interest here.

since both waveguides will be assumed empty.† By introducing the Kronecker delta δ_{pq}, we have also expressed the fact that the transverse fields in the individual waveguides are orthogonal.

Once the normalizations for the electric wave functions are defined, those for the magnetic field are also specified since, for both the TE and TM modes, the transverse electric and magnetic fields are uniquely related. In particular, for a TM mode

$$^{\pm}\mathbf{H}_p' = \pm\frac{\omega\epsilon}{^{\pm}h_p'}\,\mathbf{e}_z \times {}^{\pm}\mathbf{E}_p' \tag{11}$$

and for a TE mode

$$^{\pm}\mathbf{H}_p'' = \pm\frac{^{\pm}h_p''}{\omega\mu}\,\mathbf{e}_z \times {}^{\pm}\mathbf{E}_p'', \tag{12}$$

where $\mathbf{e}_z$ is a unit vector in the z direction. Note again that the sign convention is such that a field in the $-$ waveguide is taken to be a reflected wave, travelling away from the discontinuity. The magnetic field normalization is thus given by

$$\int_{\pm A} {}^{\pm}\mathbf{H}_p' \cdot {}^{\pm}\mathbf{H}_q'^{*}\, dA = \omega^2\epsilon^2\,\delta_{pq} \tag{13}$$

$$\int_{\pm A} {}^{\pm}\mathbf{H}_p'' \cdot {}^{\pm}\mathbf{H}_q''^{*}\, dA = |\,{}^{\pm}h_p''\,|^2\,\delta_{pq}\,. \tag{14}$$

Both sets of transverse wave functions have the property of completeness, which is to say that any transverse electric (or magnetic) field can be synthesized from a set of TE and TM vector wave functions, provided that the directions of propagation of the normal modes are known. For the problems to be considered here, this latter information is available from physical considerations, since all modes propagate away from the junction with the exception of the incident wave whose amplitude is known. This amplitude will be taken to be that of a normalized mode.

We now derive the appropriate matrix representation for the discontinuity problem. Assume a dominant (TE) mode wave ($\mathbf{E}_1''$, $\mathbf{H}_1''$) is incident from the $-$ guide, setting up a transverse electric field in the aperture just to the left of the junction. This field, referred to as $^{-}\mathbf{E}_t$, may be synthesized as follows:

$$^{-}\mathbf{E}_t = {}^{-}\mathbf{E}_1'' + \sum_{p=1}^{\infty} {}^{-}A_p\,{}^{-}\mathbf{E}_p' + \sum_{q=1}^{\infty} {}^{-}B_q\,{}^{-}\mathbf{E}_q'' \tag{15a}$$

† The asterisk (*) denotes the complex conjugate.

with the modal coefficients ${}^{-}A_p$ and ${}^{-}B_q$ as yet undetermined. The corresponding transverse electric field on the + side, denoted by ${}^{+}\mathbf{E}_t$, would then be given in terms of normal modes on the + side by

$$ {}^{+}\mathbf{E}_t = \sum_{p=1}^{\infty} {}^{+}A_p {}^{+}\mathbf{E}_p' + \sum_{q=1}^{\infty} {}^{+}B_q {}^{+}\mathbf{E}_q''. \tag{15b} $$

Since the transverse electric field is continuous across the aperture, we have that

$$ {}^{+}\mathbf{E}_t = {}^{-}\mathbf{E}_t \quad \text{on} \quad C $$

$$ {}^{+}\mathbf{E}_t \equiv 0 \quad \text{on} \quad D\text{-}C. $$

As shown in Fig. 2, D-C represents the conducting wall which makes up the remainder of the junction, and on which the transverse electric field must vanish. Expanding (15a) in a Fourier series of modes in the + waveguide, we find that the modal coefficients are related by

$$ {}^{+}A_p = \frac{1}{|{}^{+}h_p'|^2} \int_C {}^{-}\mathbf{E}_1'' \cdot {}^{+}\mathbf{E}_p'^{*} \, dA + \frac{1}{|{}^{+}h_p'|^2} \sum_{q=1}^{\infty} {}^{-}A_q \int_C {}^{-}\mathbf{E}_q' \cdot {}^{+}\mathbf{E}_p'^{*} \, dA $$
$$ + \frac{1}{|{}^{+}h_p'|^2} \sum_{q=1}^{\infty} {}^{-}B_q \int_C {}^{-}\mathbf{E}_q'' \cdot {}^{+}\mathbf{E}_p'^{*} \, dA \tag{16a} $$

$$ {}^{+}B_p = \frac{1}{\omega^2 \mu^2} \int_C {}^{-}\mathbf{E}_1'' \cdot {}^{+}\mathbf{E}_p''^{*} \, dA + \frac{1}{\omega^2 \mu^2} \sum_{q=1}^{\infty} {}^{-}A_q \int_C {}^{-}\mathbf{E}_q' \cdot {}^{+}\mathbf{E}_p''^{*} \, dA $$
$$ + \frac{1}{\omega^2 \mu^2} \sum_{q=1}^{\infty} {}^{-}B_q \int_C {}^{-}\mathbf{E}_q'' \cdot {}^{+}\mathbf{E}_p''^{*} \, dA \tag{16b} $$

or, more succinctly, using partitioned matrix representations,

$$ \begin{bmatrix} {}^{+}\mathcal{A} \\ {}^{+}\mathcal{B} \end{bmatrix} = \begin{bmatrix} {}^{+}\mathfrak{D}' & 0 \\ 0 & \frac{1}{\omega^2 \mu^2} \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathfrak{F}_1 \\ \mathfrak{F}_2 \end{bmatrix} $$
$$ + \begin{bmatrix} {}^{+}\mathfrak{D}' & {}^{+}\mathfrak{D}' \\ \frac{1}{\omega^2 \mu^2} \mathcal{I} & \frac{1}{\omega^2 \mu^2} \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathcal{E}^T \begin{pmatrix} -, & + \\ ', & ' \end{pmatrix} & \mathcal{E}^T \begin{pmatrix} -, & + \\ '', & ' \end{pmatrix} \\ \mathcal{E}^T \begin{pmatrix} -, & + \\ ', & '' \end{pmatrix} & \mathcal{E}^T \begin{pmatrix} -, & + \\ '', & '' \end{pmatrix} \end{bmatrix} \begin{bmatrix} {}^{-}\mathcal{A} \\ {}^{-}\mathcal{B} \end{bmatrix} \tag{17} $$

in which the vectors and submatrices are defined as follows:

$$
{}^{\pm}\mathcal{A} = \begin{bmatrix} {}^{\pm}A_1 \\ {}^{\pm}A_2 \\ \vdots \end{bmatrix} \qquad {}^{\pm}\mathcal{B} = \begin{bmatrix} {}^{\pm}B_1 \\ {}^{\pm}B_2 \\ \vdots \end{bmatrix} \tag{18}
$$

$$
\begin{aligned}
(\mathfrak{F}_1)_p &= \int_C {}^{-}\mathbf{E}_1'' \cdot {}^{+}\mathbf{E}_p'^{*}\, dA \\
(\mathfrak{F}_2)_p &= \int_C {}^{-}\mathbf{E}_1'' \cdot {}^{+}\mathbf{E}_p''^{*}\, dA
\end{aligned} \tag{19}
$$

$$
{}^{\pm}\mathcal{D}' = \begin{bmatrix} \dfrac{1}{|{}^{+}h_1'|^2} & 0 & 0 & \cdots \\ 0 & \dfrac{1}{|{}^{+}h_2'|^2} & 0 & \cdots \\ 0 & 0 & & \\ \vdots & \vdots & & \ddots \end{bmatrix} \tag{20}
$$

and $\mathcal{J}$ is the identity matrix. The matrix $\mathcal{E}$, whose *transpose* appears in (17), is the matrix of coupling coefficients, defined as the scalar products of electric transverse vector wave functions for the waveguides on each side of the discontinuity. The four index notation is interpreted as:

$$
\mathcal{E}_{pq}\begin{pmatrix} -, & + \\ ', & '' \end{pmatrix} = \int_C {}^{-}\mathbf{E}_p' \cdot {}^{+}\mathbf{E}_q''^{*}\, dA \tag{21}
$$

with analogous definitions for other combinations.

The system of equations given in (17) is clearly underdetermined since the number of unknowns is twice the number of equations. However, an additional set can be derived by employing the boundary condition that the transverse magnetic field must also be continuous across the interface. The matrix equation, analogous to (17), but corresponding to this second boundary condition, is given by

$$
\begin{bmatrix} {}^{-}\mathcal{A} \\ {}^{-}\mathcal{B} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{G} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{\omega^2\varepsilon^2}\mathcal{J} & \dfrac{1}{\omega^2\varepsilon^2}\mathcal{J} \\ {}^{-}\mathcal{D}'' & {}^{-}\mathcal{D}'' \end{bmatrix} \begin{bmatrix} \mathcal{H}^T\begin{pmatrix} +, & - \\ ', & ' \end{pmatrix} & \mathcal{H}^T\begin{pmatrix} +, & - \\ '', & ' \end{pmatrix} \\ \mathcal{H}^T\begin{pmatrix} +, & - \\ ', & '' \end{pmatrix} & \mathcal{H}^T\begin{pmatrix} +, & - \\ '', & '' \end{pmatrix} \end{bmatrix} \begin{bmatrix} \mathcal{A}^{+} \\ \mathcal{B}^{+} \end{bmatrix} \tag{22}
$$

in which

$$\mathcal{G} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{23}$$

$$^{-}\mathfrak{D}'' = \begin{bmatrix} \dfrac{1}{|\,^{-}h_1''\,|^2} & 0 & 0 & \cdots \\ 0 & \dfrac{1}{|\,^{-}h_2''\,|^2} & 0 & \cdots \\ 0 & 0 & \cdot & \\ \vdots & \vdots & & \ddots \end{bmatrix} \tag{24}$$

and the matrix $\mathcal{H}$, whose *transpose* appears in (22), is the matrix of scalar products of magnetic transverse vector wave functions. The four-index notation is interpreted in the same way as in (21).

It should be noted once again that all the matrices which appear in (17) and (22) are infinite matrices, corresponding to the fact that in general an infinite number of modes are excited in the neighborhood of the discontinuity. In practice, of course, there must be a truncation and the problem then becomes one of solving a set of matrix equations whose order, N, depends on the accuracy required. Unfortunately there is, as yet, no way in which the number of modes required to produce a given accuracy can be predicted. We can only emphasize the need for meaningful error criteria which will act as a guide in choosing a number of modes which will be large enough to give sufficiently accurate results but at the same time not be so large as to require excess computation. It is expected that the criteria given in Section II will prove very useful in this respect.

IV. MATRIX ITERATIVE METHODS

It was shown in the previous section that the waveguide discontinuity problem of interest here can be formulated in terms of a system of linear algebraic equations. This is a recurrent theme in mathematical physics, so that an extensive theory concerned with the efficient solution of matrix equations has evolved. In this section, we shall be concerned with some of the elements of this theory, placing particular emphasis on the solution of matrix equations by iteration.

The system of linear algebraic equations which results from satisfying the aperture boundary conditions on the transverse electric and magnetic fields can be written in the matrix forms

$$^{+}\alpha = \mathcal{U} + \mathcal{R} \cdot {}^{-}\alpha \tag{25}$$

$$^{-}\alpha = \mathcal{V} + \mathcal{S} \cdot {}^{+}\alpha, \tag{26}$$

where

$$^{\pm}\alpha = \begin{bmatrix} ^{\pm}\mathcal{A} \\ ^{\pm}\mathcal{B} \end{bmatrix}, \tag{27}$$

the vectors $\mathcal{U}$ and $\mathcal{V}$ and the matrices $\mathcal{R}$ and $\mathcal{S}$ being correspondingly identified from (17) and (22). Equations (25) and (26) are easily uncoupled to give

$$[\mathcal{I} - \mathcal{R}\,\mathcal{S}] \cdot {}^{+}\alpha = \mathcal{U} + \mathcal{R}\mathcal{V} \tag{28}$$

$$[\mathcal{I} - \mathcal{S}\mathcal{R}] \cdot {}^{-}\alpha = \mathcal{V} + \mathcal{S}\mathcal{U} \tag{29}$$

both of which are seen to have the general form

$$\mathfrak{M}x = y. \tag{30}$$

In (30), x is an N-dimensional complex vector whose components are the coefficients of the normal modes in the two waveguides, $\mathfrak{M}$ is an $N \times N$ complex matrix characterizing the discontinuity, and y is an excitation vector due to the incident wave.

The obvious method of solving (30) is to compute the inverse of $\mathfrak{M}$ and thus directly obtain

$$x = \mathfrak{M}^{-1}y. \tag{31}$$

However, we should recognize that it is the solution vector x which is required, and that computing the inverse is not always the best equation solving technique. For example, because the modal coefficients may decrease slowly with mode index, an accurate approximation of the physical problem often requires that $\mathfrak{M}$ be a very large matrix, and inversion procedures for large complex matrices require considerable computational effort. An alternate approach is therefore suggested, namely the solution of (30) by a method of iteration.

In an iterative algorithm, we begin with an initial "guess" for the solution and, from this, generate a supposedly improved solution, repeating the process until successive iterations give results which

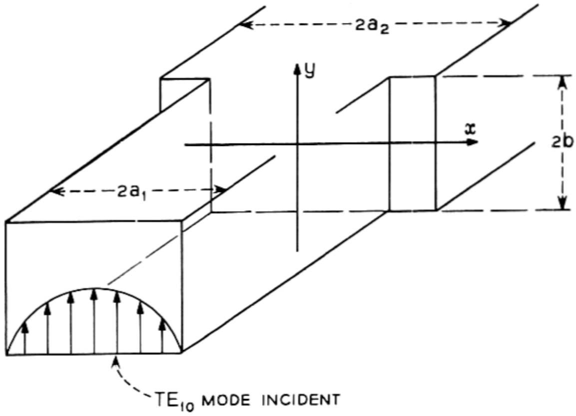

Fig. 3 — H-plane discontinuity in a rectangular waveguide. The incident mode is a TE_{10} mode (electric field vertical).

agree to within some prescribed norm. The solution to which the procedure converges must, of course, be *independent of the initial assumption.*

A tempting iterative procedure for the present problem is suggested by writing (28) in the form

$$^{+}\alpha = \mathfrak{U} + \mathfrak{R}\mathfrak{V} + \mathfrak{R}\,\mathfrak{S}\cdot{}^{+}\alpha \tag{32}$$

with an initial assumption

$$^{+}\alpha^{(0)} = \mathfrak{U} + \mathfrak{R}\mathfrak{V}. \tag{33}$$

Physically this corresponds to first assuming the aperture electric field to be that of the unperturbed incident wave, and calculating the

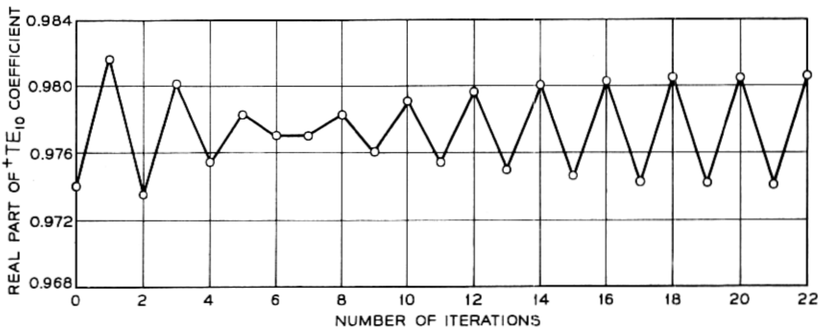

Fig. 4 — Transmission coefficient of an H-plane discontinuity.

TE and TM modal coefficients in the + waveguide on this basis. The corresponding magnetic field is then determined on the + side of the aperture and, with the aid of the appropriate continuity condition, is used to find the magnetic field and subsequently an "improved" electric field in the − region. This second guess for the aperture electric field is then used to repeat the process, etc.

As a test, this algorithm was applied to analysis of an H-plane discontinuity in a rectangular waveguide, the incident wave being a TE_{10} mode of normalized amplitude [see (14)] as in Fig. 3. The dimensions were $ka_2 = 4.5$ and $ka_1 = 3.5$ where k is the free space wavenumber. With this choice of parameters, only the TE_{10} modes can propagate in each guide. (This problem is discussed in further detail in Section V.)

Fig. 4 shows the result of calculating the real part of the modal coefficient for the TE_{10} mode in the larger waveguide, plotted as a distribution of points giving the value at each iteration. The Fourier series for this particular example was truncated after twenty five terms. Aside from a small amplitude oscillation of less than one percent rms, the results seem reasonable, especially in view of calculations for the mean-square errors ε_E and ε_H, which are illustrated in Fig. 5. These

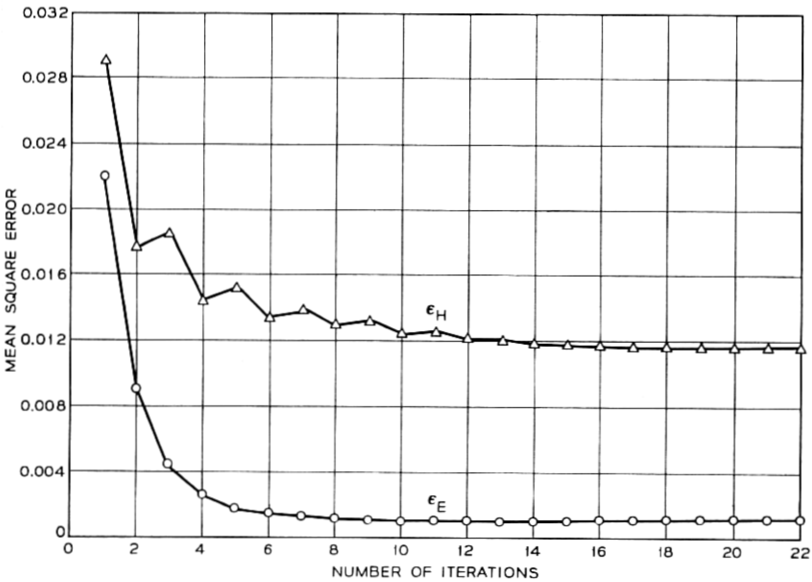

Fig. 5—Mean square errors in transverse electric and magnetic fields for an H-plane discontinuity.

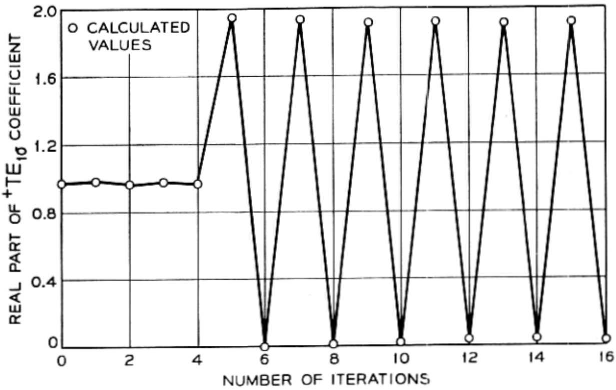

Fig. 6 — Oscillatory instability of a nonconvergent algorithm for the H-plane discontinuity analysis.

decrease monotonically with succeeding iterations, ε_E approaching approximately 0.001 and ε_H a value of about 0.012. The larger error in the magnetic field can be attributed to the singular behavior in H near the corner of the discontinuity. The asymptotic value of the energy parameter ε_P is approximately 0.007.

The apparently accurate results obtained using this algorithm are in fact quite deceptive, and may actually be attributed to the propitious initial choice for the aperture electric field. It will be recalled that a very important criterion for validity of an iterative procedure is that the results be independent of the initial assumption. In order to determine whether such a criterion is satisfied for this particular algorithm, the TE_{10} modal coefficient for the larger waveguide was arbitrarily doubled after the fourth iteration, which is equivalent to deliberately assuming a poor initial choice for the aperture field. The effect, shown in Fig. 6, indicates that the algorithm does not relax to the previous values, but continues to oscillate with a large amplitude. Similarly large fluctuations occur in ε_E , ε_H , and ε_P , the conclusion being that this particular precedure is not satisfactory, and can be expected to give reasonable results only if the initial choice is a very good one. The reason for this instability will become apparent after we consider those aspects of matrix-iterative analysis which are relevant to these problems.

In the usual framework for iterative procedures, the matrix equation satisfied by the unknown vector x is written in the form

$$x = \mathfrak{M}x + f, \tag{34}$$

where $\mathfrak{M}$ and f are appropriate to the particular scheme being used.

This leads very naturally to the recursive formula relating the $m + 1$ to the m iteration,

$$x^{(m+1)} = \mathfrak{M}x^{(m)} + f. \tag{35}$$

We denote the error vector at any iteration by $\varepsilon^{(m)}$, where

$$\varepsilon^{(m)} = x^{(m)} - \bar{x}, \tag{36}$$

and $\bar{x}$ is the exact solution to (34). Then, by substituting (36) into (35), we find that $\varepsilon^{(m+1)}$ is related to $\varepsilon^{(m)}$ by

$$\varepsilon^{(m+1)} = \mathfrak{M}\varepsilon^{(m)}. \tag{37}$$

Therefore, the error at the mth iteration is expressible in terms of the initial error by

$$\varepsilon^{(m)} = \mathfrak{M}^m\varepsilon^{(0)}. \tag{37a}$$

For an absolutely converging solution we thus require that

$$\varepsilon^{(m)} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty \tag{38}$$

regardless of the initial guess $x^{(0)}$. This is equivalent to

$$\mathfrak{M}^m \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty \tag{39}$$

where the 0 in (38) and (39) denotes a null vector or matrix, respectively. It can be shown[4] that an $N \times N$ complex matrix $\mathfrak{M}$ is "convergent", in the sense of (39), if and only if all the eigenvalues λ_i of $\mathfrak{M}$ magnitude less than unity, i.e.,

$$|\lambda_i| < 1 \quad \text{all} \quad i. \tag{40}$$

We can easily see why this requirement will guarantee convergence, at least for the special case where the eigenvectors $\boldsymbol{\alpha}_i$ of $\mathfrak{M}$ span the space of N-dimensional complex vectors. The initial error is then expressible as

$$\varepsilon^{(0)} = \sum_{i=1}^{N} C_i\alpha_i , \tag{41}$$

where the C_i are constants. The error at the mth iteration then becomes, from (37),

$$\varepsilon^{(m)} = \sum_{i=1}^{N} C_i\mathfrak{M}^m\alpha_i = \sum_{i=1}^{N} C_i\lambda_i^m\alpha_i \tag{42}$$

and, as $m \rightarrow \infty$, each term in the sum approaches zero, provided, of course, that (40) is satisfied.

Equation (42) also reveals useful information concerning the speed of convergence, which is seen to depend on how close the magnitudes of the λ_i are to unity. Clearly if the largest eigenvalue is very near one in magnitude, the convergence will be very slow and hence a large number of iterations will be required. It would be logical, in the light of this reasoning, to evaluate the magnitude of the largest eigenvalue for the H-plane discontinuity problem discussed proviously. Unfortunately, because of the geometrical asymmetry, the matrix $\mathfrak{M}$ is not Hermitian and so the usual computational techniques for determining eigenvalues cannot be used. We can, however, find an upper bound for the modulus of the maximum eigenvalue, $\rho(\mathfrak{M}) = \max \{| \lambda_i |\}$, given by[5]

$$\rho(\mathfrak{M}) \leqq [\rho(\mathfrak{M}^\dagger \mathfrak{M})]^{\frac{1}{2}}. \tag{43}$$

where † denotes the conjugate transpose matrix. Note that $\mathfrak{M}^\dagger\mathfrak{M}$ is Hermitian, so that standard computer programs can be used to evaluate its eigenvalues. We find for the previous H-plane problem that $| \lambda_i | \leqq 1.16$ which, although, not conclusive, shows the possibility of such an oscillatory instability.

One technique which is suggested as a means of obtaining a convergent algorithm is called the "Richardson" or "point-Jacobi" method.[5] In this approach, the matrix $\mathfrak{M}$ is first partitioned as

$$\mathfrak{M} = \mathfrak{D} + \mathfrak{L} + \mathfrak{U}, \tag{44}$$

where $\mathfrak{D}$ is a diagonal matrix containing the diagonal terms of $\mathfrak{M}$, $\mathfrak{L}$ is a strictly lower triangular matrix and $\mathfrak{U}$ is a strictly upper triangular matrix. The system (34) is then written as

$$(\mathfrak{I} - \mathfrak{D})x = (\mathfrak{L} + \mathfrak{U})x + f \tag{45}$$

from which

$$\begin{aligned} x &= (\mathfrak{I} - \mathfrak{D})^{-1}(\mathfrak{L} + \mathfrak{U})x + (\mathfrak{I} - \mathfrak{D})^{-1}f \\ &= \mathfrak{M}_R x + f_R , \end{aligned} \tag{46}$$

(46) being the matrix representation of the "Richardson" or "point-Jacobi" method.

A modification of this procedure is referred to as the "Gauss-Seidel" or "point-single-step" iteration method.[6] Note that in solving (46) by iteration, the components of $x^{(m+1)}$ are all computed from the components of $x^{(m)}$. Intuitively, it would seem more attractive to use the latest estimates of x, i.e., in computing $x_i^{(m+1)}$ we should use, wherever they appear, the components $x_k^{(m+1)}$ $(k < j)$ already computed, and in this way utilize the most accurate information available. This procedure is, in fact, easier to implement in a computer program and, in addition to requiring less storage, often has better convergence properties than Richardson's method. It may be shown that the matrix representation, analogous to (46), for the "Gauss-Seidel" method is

$$x = \mathfrak{M}_G x + f_G \tag{47}$$

$$\begin{aligned} \mathfrak{M}_G &= (\mathfrak{I} - \mathfrak{D} - \mathfrak{L})^{-1}\mathfrak{U} \\ f_G &= (\mathfrak{I} - \mathfrak{D} - \mathfrak{L})^{-1} f, \end{aligned} \tag{48}$$

$\mathfrak{D}$, $\mathfrak{L}$, and $\mathfrak{U}$ having been defined previously.

The eigenvalue condition given in (40) is, of course, a very restrictive one, so that iteration procedures cannot be applied with success to every system of equations. However, when a convergent matrix $\mathfrak{M}$ can be found, the methods which have been discussed offer several distinct advantages over a straightforward matrix inversion. For example, if the order of the system is N, then it can be shown that each iteration requires approximately N^2 multiply-add operations.* On the other hand, an inversion requires *at least* N^3 multiply-adds, so that the relative saving is the ratio of the order N to the number of iterations required. It is often the case that the maximum eigenvalue is so small that the number of iterations required for an accuracy equivalent to that obtained by inversion is considerably less than N.

Iterative methods also have the property that the solution accuracy is "adjustable", in the sense that once the solution has converged to the point where some norm, e.g., ε_E or ε_H defined previously, is less than a specified tolerance, the iteration process can be terminated. This property is especially attractive in view of the fact that truncation errors have already been introduced, and it would therefore be superfluous to accurately invert a system which is itself approximate. By having the option of terminating the iterative procedure, we introduce an additional degree of freedom by which we can optimize the computational program.

V. APPLICATIONS

The iterative techniques discussed in the previous section will now be applied to two problems of interest, namely the H-plane discontinuity

* A multiply-add consists of the multiplication of two complex numbers and the adding of the result to a third complex number. For repetitive computational algorithms, the number of multiply-adds is a measure of the computational effort required.

problem mentioned in Section IV and the analysis of $TE_{11} \rightarrow TM_{11}$ mode conversion at a step discontinuity in a circular waveguide. In both of these examples the required matrix elements may be expressed in convenient closed forms, which considerably reduces the required computational effort.

5.1 *H-plane Discontinuity in a Rectangular Guide*

In Section IV, the H-plane discontinuity of Fig. 3 was analyzed using an iterative algorithm which was observed to exhibit an oscillatory instability when initiated with a poor approximation to the actual solution. It was concluded that this was due to the eigenvalues of the iteration matrix $\mathfrak{M}$ being close to, or perhaps greater than unity. We now consider the same problem using the Gauss-Seidel method which, on the basis of the previous discussion, is expected to improve matters substantially.

We assume a normalized TE_{10} mode incident from the smaller guide. Because of the symmetry of the junction, such a wave excites only TE modes in both the + and − regions. The problem is, of course, to determine the corresponding modal coefficients for the fields on each side of the discontinuity. We shall present results only for the transmitted TE_{10} mode, which is the only mode propagating in the larger waveguide for the present dimensions, $ka_1 = 3.5$, $ka_2 = 4.5$.

It can be shown[9] that the normalized vector wave functions are given by

$$\begin{aligned} {}^{-}\mathbf{E}_p'' &= \mathbf{e}_y \frac{i\omega\mu}{\sqrt{2a_1 b}} \sin\left[\frac{p\pi}{2a_1}(x + a_1)\right] \\ {}^{+}\mathbf{E}_p'' &= \mathbf{e}_y \frac{i\omega\mu}{\sqrt{2a_1 b}} \sin\left[\frac{p\pi}{2a_2}(x + a_2)\right], \end{aligned} \tag{49}$$

where a_1, a_2, and b are the dimensions of the guide as shown and $\mathbf{e}_y$ is a unit vector in the y direction. The corresponding magnetic vector wave functions can be found from (12). The respective propagation constants are

$$\begin{aligned} {}^{-}h_p'' &= \left[k^2 - \left(\frac{p\pi}{2a_1}\right)^2\right]^{\frac{1}{2}} \\ {}^{+}h_p'' &= \left[k^2 - \left(\frac{p\pi}{2a_2}\right)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{50}$$

From (21) and (49) we find that the coupling coefficients for the electric fields are given by

$$\mathcal{E}_{pq}\begin{bmatrix} -, + \\ '' , '' \end{bmatrix} = 2\omega^2\mu^2\sqrt{\frac{a_1}{a_2}}\sin\frac{p\pi}{2}\sin\frac{q\pi}{2}\left\{\frac{1}{p\pi + \frac{a_1}{a_2}q\pi}\sin\left(\frac{p\pi}{2} + \frac{a_1}{a_2}q\frac{\pi}{2}\right)\right.$$

$$\left. + \frac{1}{p\pi - \frac{a_1}{a_2}q\pi}\sin\left(\frac{p\pi}{2} - \frac{a_1}{a_2}q\frac{\pi}{2}\right)\right\}. \qquad (51)$$

The appropriate magnetic field coupling coefficients are easily found from (51) using the relation

$$\mathcal{H}_{pq}\begin{bmatrix} +, - \\ '' , '' \end{bmatrix} = -\frac{1}{{}^{+}Z_p'' \; {}^{-}Z_q''^{*}}\,\mathcal{E}_{qp}\begin{bmatrix} -, + \\ '' , '' \end{bmatrix}, \qquad (52)$$

where Z_p denotes the modal impedance, equal to

$${}^{\pm}Z_p'' = \frac{\omega\mu}{{}^{\pm}h_p''}. \qquad (53)$$

The results for the TE_{10} modal coefficient, ${}^{+}B_1$, as obtained using the Gauss-Seidel iteration method, are conveniently represented in Table I. Also given are the numerical values for the error parameters ε_P, ε_E and ε_H. Truncation for this example was at 25 modes in each waveguide.

We conclude that for most applications, two iterations would probably have been sufficient, corresponding to a saving of greater than 90 percent in actual execution time, compared to a matrix inversion. The reason for this extremely rapid convergence is, as expected, in the magnitude of the largest eigenvalue, which was found, using (43), to be less than 0.078.

As a means of establishing the convergence of the normal mode solution, we have plotted in Figs. 7 and 8, the mean square errors ε_E and ε_H, respectively, as a function of the number of modes taken.

TABLE I — RESULTS USING THE GAUSS-SEIDEL METHOD FOR ANALYSIS OF THE H-PLANE DISCONTINUITY

Iteration number	${}^{+}B_1$		Energy coefficient ϵ_P	*rms* error ϵ_P	*rms* error ϵ_H
	Real	Imaginary			
1	0.97445	0.00951	-0.62×10^{-2}	0.5×10^{-5}	0.01445
2	0.97747	0.00455	-0.74×10^{-6}	0.49×10^{-5}	0.01355
3	0.97747	0.00426	0.85×10^{-5}	0.49×10^{-5}	0.01350
4	0.97747	0.00424	0.71×10^{-6}	0.49×10^{-5}	0.01349
5	0.97747	0.00424	0.51×10^{-7}	0.49×10^{-5}	0.01349

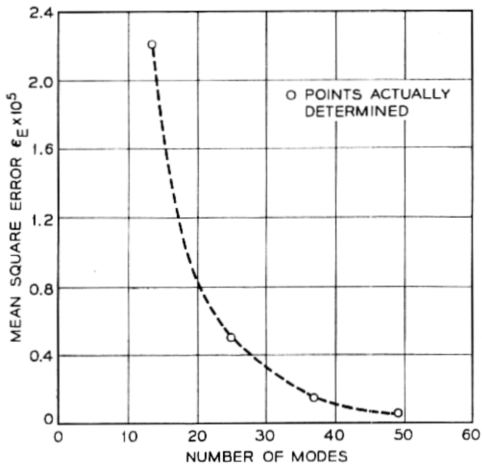

Fig. 7—Mean square error ε_E, as function of the number of modes used, for H-plane discontinuity.

These figures give genuine significance to the term "convergence in mean square", since they indicate that the use of more terms leads to better agreement with the boundary conditions in the mean square sense. Again, the error is uniformly higher for the magnetic field than for the electric field, due to the singularity at the corner of the discontinuity.

It is finally of interest to determine the effect of a poor initial estimate

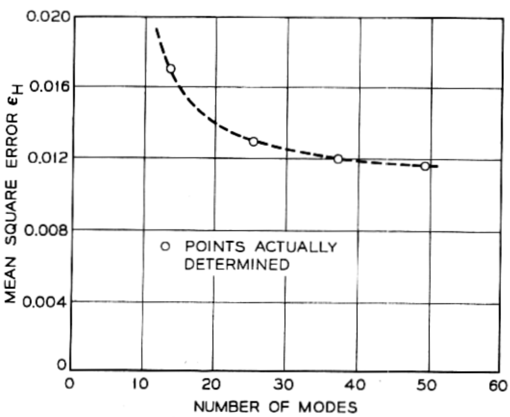

Fig. 8—Mean square error ε_H as function of the number of modes used, for H-plane discontinuity.

TABLE II—RESULTS OF SPOILING ELECTRIC FIELD AT 4TH ITERATION

Iteration number	TE_{10} coefficient	
	Real	Imaginary
3	0.97747	0.00426
4	1.94165	0.02572
5	0.98052	−0.00063
6	0.97747	0.00396
7	0.97746	0.00423

of the solution. We find that, unlike the simple algorithm discussed in the previous section, the Gauss-Seidel procedure is very stable, returning to the correct "steady state" solution within a few iterations after the spoiling was introduced. The results are shown in Table II, again for a truncation of 25 modes in each waveguide.

5.2 *Mode Conversion at a Step Discontinuity in a Circular Waveguide*

The second problem to which these techniques were applied is that of calculating the $TE_{11} \rightarrow TM_{11}$ mode conversion at an abrupt discontinuity in a circular waveguide. Recent studies have indicated that this configuration is a very efficient transducer for use in dual mode conical horns.[10] The discontinuity is illustrated in Fig. 9 which shows the TE_{11} mode, incident from the smaller guide, being converted to a combination of TE_{11} and TM_{11} modes propagating in the larger guide.

The normalized TE and TM vector wave functions are known[11] and, fortunately, it is possible to determine the appropriate coupling coefficients. For the elements of the matrix we find that

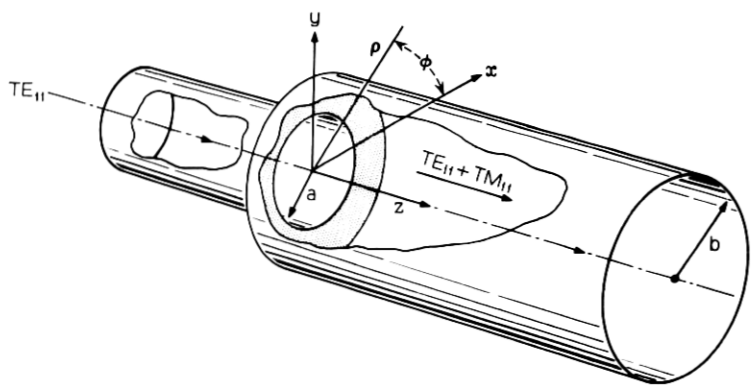

Fig. 9—TE_{11} – TM_{11} mode conversion at a discontinuity in a circular waveguide.

$$\mathcal{E}_{pq}\begin{bmatrix} -, & + \\ ', & ' \end{bmatrix} = \frac{2\ {}^{-}h_p'\ {}^{+}h_q'^{*} x_q J_1\left(x_q \dfrac{a}{b}\right)}{\left(\dfrac{b^2}{a^2} x_p^2 - x_q^2\right) J_2(x_q)} \tag{54}$$

$$\mathcal{E}_{pq}\begin{bmatrix} -, & + \\ '', & '' \end{bmatrix} = \frac{2\omega^2\mu^2 b y_p^2 y_q \left[J_0\left(\dfrac{a}{b} y_q\right) - \dfrac{b}{a} J_1\left(\dfrac{a}{b} y_q\right)\right]}{\left(\dfrac{b^2}{a^2} y_p^2 - y_q^2\right)\sqrt{(y_p^2 - 1)(y_q^2 - 1)}\ J_1(y_q)} \tag{55}$$

$$\mathcal{E}_{pq}\begin{bmatrix} -, & + \\ '', & ' \end{bmatrix} = \frac{2\omega\mu\ {}^{+}h_q'^{*} J_1\left(x_q \dfrac{a}{b}\right)}{x_q J_2(x_q)\sqrt{(y_p^2 - 1)}} \tag{56}$$

$$\mathcal{E}_{pq}\begin{bmatrix} -, & + \\ ', & '' \end{bmatrix} = 0. \tag{57}$$

In (59) through (62) we have used the following notation:

a – radius of smaller waveguide,
b – radius of larger waveguide,
x_p – pth zero of $J_1(x)$,
y_p – pth zero of $J_1'(y)$.

The elements of the matrix $\mathcal{H}$ can easily be found from the impedance relations of (11) and (12).

One parameter which has been found to be useful in characterizing the mode conversion properties of the discontinuity is the conversion coefficient C, defined as the ratio of the ρ-components of electric field for the two modes evaluated at the wall of the larger waveguide, i.e.,

$$C = 20 \log_{10} \left| \frac{E_\rho^{TM}}{E_\rho^{TE}} \right|_{\rho=b} \text{dB}. \tag{58}$$

This quantity was calculated for the particular discontinuity $a = 1.05''$, $b = 1.4''$ over the frequency range 5.2 to 7.0 kHz, these parameters having been chosen for purposes of comparison with available experimental data. Truncation of the normal mode expansion was made after twenty-five TE and twenty-five TM modes in each waveguide. The iterative sequence was terminated when successive values of the modal amplitudes differed by less than 10^{-6}. It was found that typical values for the error criteria are $\varepsilon_P \approx 10^{-7}$, and ε_E, $\varepsilon_H \approx 0.015$. These results indicate that for a given accuracy, a much lower value can be expected for ε_P, which is a function only of the lower-order modes,

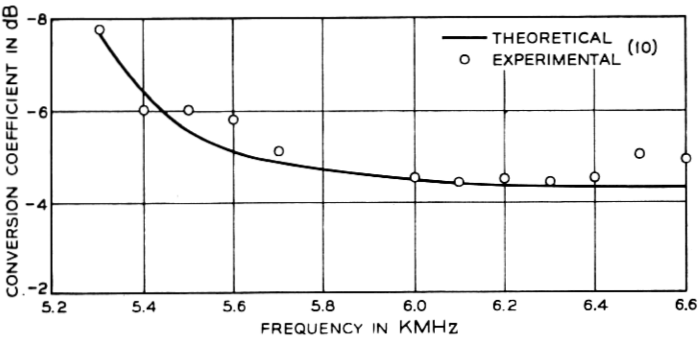

Fig. 10—$TE_{11} \rightarrow TM_{11}$ conversion coefficient of a step discontinuity in a circular waveguide. $a = 1.05''$, $b = 1.4''$.

than for ε_E and ε_H, which depend on the higher-order terms as well.

In Fig. 10 we have plotted the computed values of the conversion coefficient, defined in (58), as a function of frequency. Also shown, as discrete points, are experimental results obtained previously.[10] The theoretical values are seen to be in very good agreement.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have considered the solution of those matrix equations which arise in the analysis of a class of waveguide discontinuity problems. In searching for criteria to estimate the accuracy of computed results, we have found that the uniqueness theorem itself yields a convenient set of error parameters which are easily implemented in the computational program.

It is suggested that an iterative technique, particularly the "Gauss-Seidel" or "point-single-step" method often leads to a rapidly converging solution, thus offering several advantages over the usual invertive procedures, particularly in the speed of execution. Of particular interest is the fact that when this method is applied to the analysis of $TE_{11} \rightarrow TM_{11}$ mode conversion at an abrupt discontinuity in a circular waveguide, it yields a rapidly convergent and accurate solution. This has been established not only on the basis of theoretical error criteria, but also by comparison with experimental results previously obtained.

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