Phase and Amplitude Modulation in High-Efficiency Varactor Frequency Multipliers – General Scattering Properties

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The presence of phase and amplitude modulation in the signals of a varactor frequency multiplier is analyzed, and some general multiplier properties are derived. The following summarizes one of the most important results of this paper.

Consider a frequency multiplier which has the following characteristics: (i) a varactor which has a square-law characteristic, (ii) the order of multiplication is $N = 2^n = 2, 4$, etc., (iii) the minimum number of idlers, and (iv) it is lossless and tuned. It is shown that for this multiplier there is no conversion between small-index, low-frequency amplitude and phase modulation. Therefore, since narrow-band noise from external sources will be present at the input and output of the multiplier, the noise components corresponding to phase modulation of the carriers can be treated independently of the noise components corresponding to amplitude modulation.

Consider now the input and output noise sidebands corresponding to phase modulation (PM). It is shown that the multiplier behaves with respect to these sidebands as an amplifier with the following properties: (i) a forward voltage transmission equal to N, (ii) unity reverse transmission, (iii) an output reflection of 1, −1, 3, respectively, for $N = 2, 4, 8$, and (iv) no input reflections. As a consequence of these properties the multiplier is "potentially" unstable with respect to PM.

The utility of the multiplier properties derived in this paper will be illustrated by the discussion in a companion paper in this issue which shows how, in practical cases, instability arises and how it can be avoided.

I. SUMMARY OF RESULTS

The frequency multipliers to be considered are harmonic generators which use varactor diodes as nonlinear elements. Noise produces unwanted amplitude and phase modulation in the signals of a frequency multiplier. In this paper, the presence of these modulations is analyzed and some general multiplier properties are derived. The following summarizes one of the most important results of this paper. Consider a frequency multiplier which has the following characteristics:

(*i*) a varactor which has a square-law characteristic,
(*ii*) the order of multiplication is $N = 2^n$,
(*iii*) the minimum number of idlers, and
(*iv*) it is lossless and tuned.

It is shown that for this multiplier there is no conversion between small-index, low-frequency amplitude, and phase modulation. Therefore, since narrow-band noise from external sources will be present at the input and output of the multiplier, the noise components corresponding to phase modulation of the carriers can be treated independently of the noise components corresponding to amplitude modulation.

Consider now the input and output noise sidebands corresponding to phase modulation (PM). It is shown that the multiplier behaves with respect to these sidebands as an amplifier with the following properties:

(*i*) a forward voltage transmission equal to N,
(*ii*) unity reverse transmission,
(*iii*) an output reflection of 1, -1, 3, respectively, for $N = 2, 4, 8$, and
(*iv*) it has no input reflections.

As a consequence of these properties the multiplier is "potentially" unstable with respect to PM. This summarizes one of the most important results of this paper. Now let us consider the results of this paper in more detail.

This paper is concerned with the presence of amplitude and phase modulation in the multiplier signals. Suppose, for the moment, that these two types of modulation are independent of each other. That is, suppose that the multiplier does not produce AM-to-PM conversion, and vice-versa. Suppose, furthermore, that only PM is present. Then each signal will consist of a carrier and of a pair of sidebands in quadrature with respect to the carrier. Since either sideband can be obtained from a knowledge of the other, then one may consider only one of the two sidebands and ignore the other one. For example, let the upper sideband be chosen as the variable, and let it be described in terms of propagating waves. Then the input variables of the multiplier are the two waves* $\overrightarrow{v}_{\omega_0+p}$, $\overleftarrow{v}_{\omega_0+p}$ which constitute the upper sideband of the input carrier ω_0. Similarly, the two output waves $\overrightarrow{v}_{N\omega_0+p}$, $\overleftarrow{v}_{N\omega_0+p}$ represent the output variables. ω_0 is the input "carrier" frequency of the multiplier and p is the frequency of the fluctuations. At this point, the scattering formalism furnishes a convenient way of describing the properties of the multiplier. More precisely, one may define the PM scattering parameters of the multiplier as the reflection and transmission coefficients which relate the "scattered" waves $\overleftarrow{v}_{\omega_0+p}$, $\overrightarrow{v}_{N\omega_0+p}$ to the "incident" ones $\overrightarrow{v}_{\omega_0+p}$, $\overleftarrow{v}_{N\omega_0+p}$. In this way one obtains the PM scattering matrix $\hat{S}_\theta$ defined by

$$\begin{aligned} \frac{\overleftarrow{v}_{\omega_0+p}}{V_{\omega_0}} &= \overrightarrow{\rho_\theta}\, \frac{\overrightarrow{v}_{\omega_0+p}}{V_{\omega_0}} + \overleftarrow{T_\theta}\, \frac{\overleftarrow{v}_{N\omega_0+p}}{V_{N\omega_0}} \\ \frac{\overrightarrow{v}_{N\omega_0+p}}{V_{N\omega_0}} &= \overrightarrow{T_\theta}\, \frac{\overrightarrow{v}_{\omega_0+p}}{V_{\omega_0}} + \overleftarrow{\rho_\theta}\, \frac{\overleftarrow{v}_{N\omega_0+p}}{V_{N\omega_0}}, \end{aligned} \tag{1}$$

where V_{ω_0} is the Fourier coefficient of the input carrier ω_0, and $V_{N\omega_0}$ is the Fourier coefficient of the output carrier $N\omega_0$. Note that the sidebands have been normalized with respect to the relative carriers. Equations (1) are equivalent to (8).†

In a completely similar way one defines the AM scattering matrix $\hat{S}_a$.

It has been found that a multiplier with characteristics (*i*) through (*iv*) above does not produce AM ⇄ PM conversion and that it has the following scattering matrices:

$$\hat{S}_\theta = \begin{vmatrix} \overrightarrow{\rho_\theta} & \overleftarrow{T_\theta} \\ \overrightarrow{T_\theta} & \overleftarrow{\rho_\theta} \end{vmatrix} = \begin{vmatrix} 0 & (-1)^n \\ N & \dfrac{1-(-1)^n N}{3} \end{vmatrix} \tag{2a}$$

$$\hat{S}_a = \begin{vmatrix} \overrightarrow{\rho_a} & \overleftarrow{T_a} \\ \overrightarrow{T_a} & \overleftarrow{\rho_a} \end{vmatrix} = \begin{vmatrix} \dfrac{N-(-1)^n}{3N} & \dfrac{(-1)^n}{N} \\ 1 & 0 \end{vmatrix} \tag{2b}$$

if the modulation frequency p is small enough.

Notice that (2a) gives the properties stated at the beginning of this summary. As a consequence of (2a), the multiplier is potentially unstable with respect to PM. AM instabilities cannot occur. This follows from (2b).

* $\overleftarrow{v}_\omega$, $\overrightarrow{v}_\omega$ designate the Fourier coefficients of the voltage components of frequency ω. They propagate in the directions indicated by the arrows.

† Note that $\overrightarrow{v}_{\omega_0+p}/V_{\omega_0}$, etc. correspond to the modulation indexes $j\overrightarrow{\theta_1}$, etc. defined in the next section.

Equations (2a), (2b) also have some other interesting consequences. For instance, if one injects a single tone $N\omega_0 + p$ into the output port of a multiplier, then the multiplier will reflect a pure PM wave. That is, the multiplier behaves as an ideal reflection-type limiter in the vicinity of the output carrier. This property is common to all types of multipliers [see (19)] and may be useful if one wants to modulate the output phase of a multiplier without generating AM. Note that $|\rho_\theta^-| > 1$ for $N > 4$. This means that, if $N > 4$, then the output port of the multiplier reflects amplifying the PM components of the output sidebands.

It is also shown that some of the properties expressed by (2a), (2b) are common to all lossless multipliers. A consequence of this is that most of the results obtained for a multiplier with the characteristics listed above can be qualitatively extended to all efficient multipliers.

II. INTRODUCTORY REMARKS

This paper is concerned with the presence of small amplitude and phase fluctuations in the electrical variables of the networks to be considered. By using the familiar terminology and concepts of modulation theory, the problem can be defined as follows.

The electrical signals will consist of sinusoidal carriers which are both amplitude- and phase-modulated. The amplitudes of the modulating waves* will be supposed to be small enough to guarantee superposition to hold, and to allow the modulated waves to be approximated by the sum of the carriers and their first-order sidebands. Furthermore, since superposition holds, consideration will be limited to sinusoidal modulating waves. Then, if p is the frequency of the modulating waves (i.e., of the fluctuations), each carrier frequency ω will be surrounded by two small side-frequencies $\omega + p$, $\omega - p$ representing the sidebands of ω. One can say that the object of this analysis is to study the presence of these side-frequencies around the carriers. A pair of sidebands is completely specified either by its spectrum or by the spectrum of the modulating waves associated with it. More precisely, consider a small voltage $v(t)$ consisting of two side-frequencies† $\omega + p$, $\omega - p$

* The "modulating waves" are the "fluctuations" and will be clarified later.

† Through this analysis a real function $g(t)$ consisting of sinusoids will be represented by the complex Fourier series

$$g(t) = \sum_{-\infty}^{\infty} C(\omega_i) \exp [j\omega_i t] = 2(\text{Re}) \sum_{0}^{\infty} C(\omega_i) \exp [j\omega_i t].$$

The complex coefficients C will be called "the Fourier coefficients" of $g(t)$. Furthermore, only positive frequencies will be considered, as illustrated by the second

$$v(t) = 2(Re)(v_{\omega+p} \exp [j(\omega + p)t] + v_{\omega-p} \exp [j(\omega - p)t]) \tag{3}$$

and let $V_c(t)$ be the carrier associated with $v(t)$. Let ω, V_c be the carrier frequency and the Fourier coefficient of $V_c(t)$, respectively. Then the Fourier coefficients of the modulating waves $a(t)$, $\theta(t)$ associated with $v(t)$ are given by

$$\begin{aligned} a &= \frac{1}{2}\left(\frac{v_{\omega+p}}{V_c} + \frac{v^*_{\omega-p}}{V^*_c}\right) \\ \theta &= \frac{1}{2j}\left(\frac{v_{\omega+p}}{V_c} - \frac{v^*_{\omega-p}}{V^*_c}\right). \end{aligned} \tag{4}$$

In fact, if one uses (4) to express $v(t)$ in terms of a, θ, one obtains*

$$\begin{aligned} v(t) &= 2(\mathrm{Re})[(a + j\theta)V_c \exp [j(\omega + p)t] \\ &\qquad\qquad + (a - j\theta)^* V_c \exp [j(\omega - p)t]] \\ &= a(t)V_c(t) + \frac{\theta(t)}{\omega} V_c'(t). \end{aligned} \tag{5}$$

Since it is supposed $| v_{\omega\pm p} | \ll | V_c |$, one has $| a | \ll 1$, $| \theta | \ll 1$. Therefore, the last equality of (5) gives

$$v(t) + V_c(t) = [1 + a(t)]V_c\left[t + \frac{\theta(t)}{\omega}\right]. \tag{6}$$

Equation (6) shows that $a(t)$, $\theta(t)$ are the modulating waves which produce the sidebands $v(t)$.† $a(t)$ is the amplitude modulating wave, and $\theta(t)$ is the phase modulating wave.

a, θ, the Fourier coefficients of $a(t)$, $\theta(t)$, will be called, respectively, the AM *index* and the PM *index* of $v(t)$. Extensive use will be made of the AM, PM indexes to represent a pair of sidebands.‡

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summation. This convention will not be followed in the Appendix where (and only there) both positive and negative frequencies will have to be considered for convenience. Note that $C(\omega_i) = C^*(-\omega_i)$ because $g(t)$ is real. The asterisk $(\ )^*$ will always indicate the complex conjugate of a complex quantity. (Re) means "real part of".

* $a(t) = 2(\mathrm{Re})(a \exp jpt)$, $\theta(t) = 2(\mathrm{Re})(\theta \exp jpt)$, $V_c(t) = 2(\mathrm{Re})(V_c \exp j\omega t)$ according to the preceding footnote.

† When $a(t)$, $\theta(t)$ are applied to the carrier $V_c(t)$.

‡ As already noted, a, θ, represent the spectra of the modulating waves relative to $v(t)$. It should also be noted that the representation of $v(t)$ by means of a, θ corresponds to the familiar representation of a pair of sidebands by means of the so-called "in-phase" and "quadrature" components about a given carrier. This is shown by (5) in which aV_c, a^*V_c are the "in-phase" components and $j\theta V_c$, $j\theta^* V_c$ are the "quadrature" components. Therefore (a) can also be regarded as the normalized (with respect to jV_c) amplitude of the "quadrature" upper-sideband.

2.1 *The Variables of the Interconnection Between Two Stages*

With reference to Fig. 1 consider now the interconnection between two general stages. Let $V(t)$ be the interstage voltage and assume that $V(t)$ consists of a sinusoidal carrier $V_c(t)$ and a pair of sidebands $v(t)$. Now suppose that a short piece of line of characteristic impedance Z_c is inserted between the two stages. Its electrical length is assumed to be so short that it may be inserted without altering the electrical properties of the cascade connection of the two stages. This should be regarded as an artifice made to allow the signals of the interconnection to be described in terms of propagating waves.* The sidebands $v(t)$

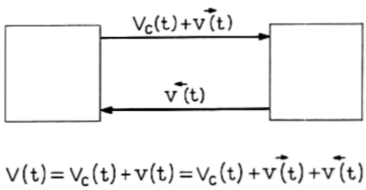


Fig. 1 — Description of the interstage voltage in terms of propagating waves.

now consist of two waves $v^{\rightarrow}(t)$, $v^{\leftarrow}(t)$ propagating in opposite directions, as indicated by the arrows $\rightarrow$, $\leftarrow$, and

$$v(t) = v^{\rightarrow}(t) + v^{\leftarrow}(t). \tag{7}$$

Finally, consider the respresentation of $v^{\rightarrow}(t)$, $v^{\leftarrow}(t)$ by means of their PM, AM indexes with respect to the carrier V_c. By using (4) one obtains† two pairs of modulation indexes

$$a^{\rightarrow},\ \theta^{\rightarrow},\quad \text{the indexes of}\quad v^{\rightarrow}(t)$$
$$a^{\leftarrow},\ \theta^{\leftarrow},\quad \text{the indexes of}\quad v^{\leftarrow}(t)$$

($a^{\rightarrow}$, etc.) provide a complete description of the fluctuations present at the generical interconnection, as illustrated in Fig. 2.

2.2 *Definition of the PM, AM Scattering Matrices of a Stage*

In most cases to be analyzed‡ each stage will have two basic properties:

* Z_c will always be chosen such that only one propagating wave exists at the carrier frequency. Therefore, the voltage of the interconnection consists of a modulated wave propagating in one direction, and of a pair of sidebands without carriers propagating in the other direction, as it is illustrated in Fig. 1. The carrier $V_c(t)$ is used as a reference for deriving the modulation indexes of both sets of sidebands.

† By replacing $v(t)$, $v_{\omega+p}$, $v_{\omega-p}$ with $v^{\rightarrow}(t)$, $\vec{v}_{\omega+p}$, $\vec{v}_{\omega-p}$ one obtains $a^{\rightarrow}$, $\theta^{\rightarrow}$. In a similar way one obtains $a^{\leftarrow}$, $\theta^{\leftarrow}$.

‡ In Ref. 1, AM $\rightleftarrows$ PM conversion is considered.

(*i*) it will be linear with respect to the modulating signals (since they are supposed to be small)
(*ii*) it will not generate any AM-to-PM (or vice versa) conversion.

As a result of property (*ii*) the PM indexes of a stage are independent from the AM indexes, and vice versa. Therefore, the two cases AM, PM can be treated separately.

Consider for instance the PM case. With reference to the stage of Fig. 3, let $\vec{\theta}_i$, $\overleftarrow{\theta}_i$ be the input ($i = 1$) and output ($i = 2$) PM indexes. Then, because of (*i*), the scattered indexes $\overleftarrow{\theta}_1$, $\vec{\theta}_2$ can be related to the incident ones $\vec{\theta}_1$, $\overleftarrow{\theta}_2$ through a set of linear equations of the type

$$\begin{aligned} \overleftarrow{\theta}_1 &= \vec{\rho}_\theta \vec{\theta}_1 + \overleftarrow{T}_\theta \overleftarrow{\theta}_2 \\ \vec{\theta}_2 &= \vec{T}_\theta \vec{\theta}_1 + \overleftarrow{\rho}_\theta \overleftarrow{\theta}_2 . \end{aligned} \tag{8}$$

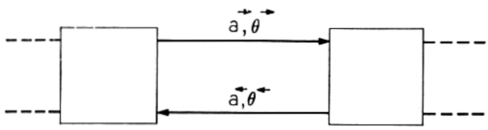


Fig. 2 — Description of the interstage sidebands in terms of modulation indexes.

Notice that (8) are equivalent to (1). The scattering matrix $\hat{S}_\theta$

$$\hat{S}_\theta = \begin{vmatrix} \vec{\rho}_\theta & \overleftarrow{T}_\theta \\ \vec{T}_\theta & \overleftarrow{\rho}_\theta \end{vmatrix} \tag{9}$$

will be called the *PM scattering matrix** of the stage. $\vec{T}_\theta$ will be called the *PM forward transmission coefficient.* $\overleftarrow{T}_\theta$ is the *PM reverse transmission coefficient.* $\vec{\rho}_\theta$ is the *PM input reflection coefficient.* $\overleftarrow{\rho}_\theta$ is the *PM output reflection coefficient.*

In a completely similar way the AM scattering matrix $\hat{S}_a$ of a stage is defined†

$$\hat{S}_a = \begin{vmatrix} \vec{\rho}_a & \overleftarrow{T}_a \\ \vec{T}_a & \overleftarrow{\rho}_a \end{vmatrix} . \tag{10}$$

* Note that the complex coefficients $\rho^{\rightarrow}$, $T^{\rightarrow}$, $T^{\leftarrow}$, $\rho^{\leftarrow}$ correspond to the familiar coefficients S_{11}, S_{12}, S_{21}, S_{22} which are usually employed to represent scattering parameters. (See Ref. 2). As can be directly verified from (8), if a unit index is incident on either port, T gives the index transmitted out of the other port, and ρ gives the index reflected at the same port. The arrows →, ← are applied to T, ρ to indicate the directions of the incident waves to which T, ρ refer.

† The convention of using the letters a, θ for designating the AM, PM cases will always be followed through the analysis.

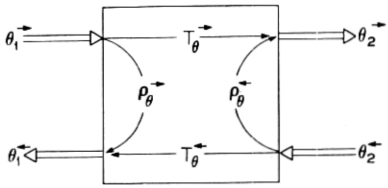


Fig. 3 — Description of the PM properties of a stage by means of the PM scattering coefficients.

In conclusion in this analysis: the *variables* are the PM, AM indexes; the *frequency* of the variables is p; the *parameters* specifying the behavior of the various networks are the *PM, AM scattering coefficients.*

III. GENERAL PROPERTIES OF A LOSSLESS MULTIPLIER WHICH DOES NOT PRODUCE AM $\rightleftarrows$ PM CONVERSION

It is important to emphasize that this analysis applies both to a multiplier consisting of a single stage and to a chain of multipliers.

Furthermore, in this paper, the analysis will be limited to the case of a lossless multiplier which does not produce AM $\rightleftarrows$ PM conversion. This is the most important case for the following two reasons. The first reason is that, in general, a multiplier which is tuned does not produce AM $\rightleftarrows$ PM conversion if the sidebands are close enough to the carriers (see Appendix). Note that it is assumed that the bias network is properly designed so that effects such as those described in Ref. 3 are absent. Furthermore, results obtained for the case of no AM $\rightleftarrows$ PM conversion can be extended to the general case of AM $\rightleftarrows$ PM, as it is shown in the second paper. The second reason is that the question of stability generally arises only when the losses of the multiplier are small. Furthermore, if the losses are small, the results which are obtained by neglecting them can be readily extended to include them, as is pointed out in the second paper.

In this section it is demonstrated that a lossless multiplier of order N has the following properties:

$$\begin{aligned} \overrightarrow{T}_\theta \overrightarrow{T}_a^{\,*} &= N(1 - \overrightarrow{\rho}_\theta \overrightarrow{\rho}_a^{\,*}) \\ \overleftarrow{T}_\theta \overleftarrow{T}_a^{\,*} &= \frac{1}{N}(1 - \overleftarrow{\rho}_\theta \overleftarrow{\rho}_a^{\,*}) \end{aligned} \tag{11}$$

$$\frac{\overleftarrow{\rho}_a^{\,*}}{\overleftarrow{T}_a^{\,*}} + N\frac{\overrightarrow{\rho}_\theta}{\overrightarrow{T}_\theta} = 0, \qquad N\frac{\overrightarrow{\rho}_a^{\,*}}{\overrightarrow{T}_a^{\,*}} + \frac{\overleftarrow{\rho}_\theta}{\overleftarrow{T}_\theta} = 0.$$

The equations of (11) have three important properties.

One is that they depend on only one multiplier parameter: the order of multiplication N. They do not depend in any way on the actual circuit configuration or on the type of varactor, etc.

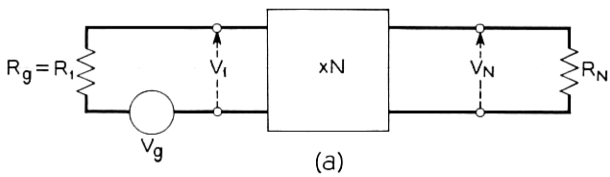
Another property is that they are frequency independent. That is, they do not contain p explicitly even though $T_\theta^{\rightarrow}$ etc. will in general be frequency dependent.

The last important property of (11) is that they represent 4 independent relations among the 8 unknowns $T_\theta^{\rightarrow}$, etc. For instance, if the AM coefficients are known, then the PM coefficients can be calculated by means of (11), and vice versa.

3.1 *Demonstration of (11)*

Consider the multiplier of order N shown in Fig. 4(a) connected to a load R_N and driven by a voltage generator having impedance R_1. Let $V_r(t)$ $(r = 1, N)$ be the voltage of the input port $(r = 1)$ and output port $(r = N)$ of the multiplier in the absence of fluctuations. It will be supposed that the input of the multiplier is matched to the generator. Therefore, the input impedance of the multiplier is R_1. The multiplier is assumed to be lossless and without AM $\rightleftarrows$ PM. The characteristic impedance associated with the rth $(r = 1, N)$ port is R_r, according to the footnote on p. 780.

Assume now that two small waves $v_1^{\rightarrow}(t)$, $v_N^{\leftarrow}(t)$ are arriving towards the multiplier, and that they consist of the frequencies $\omega_0 \pm p$, $N\omega_0 \pm p$, respectively. The problem to be considered is to find the scattered



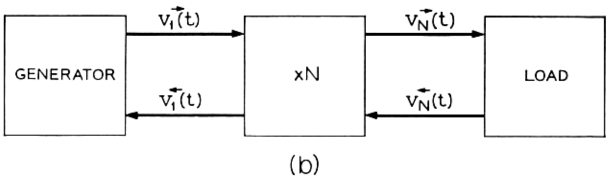


Fig. 4 — (a) Multiplier of order N. (b) Input and output sidebands of the multiplier.

waves $v_1^{\leftarrow}(t)$, $v_N^{\rightarrow}(t)$, as illustrated in Fig. 4 (in which the carriers $V_1(t)$, $V_N(t)$ are not shown).

Let $(\theta_r^{\rightarrow}, \theta_r^{\leftarrow})$, $(a_r^{\rightarrow}, a_r^{\leftarrow})$ be the indexes of $v_r(t)$ with respect to the carrier $V_r(t)$. Then, according to (5),

$$v_r^{\rightarrow}(t) = 2(\mathrm{Re})\{(a_r^{\rightarrow} + j\theta_r^{\rightarrow})V_r \exp[j(r\omega_0 + p)t] + (a_r^{\rightarrow *} + j\theta_r^{\rightarrow *})V_r \exp[j(r\omega_0 - p)t]\} \tag{12}$$

and similarly for $v_r^{\leftarrow}(t)$ (just change the arrows directions). V_r is the Fourier coefficient of $V_r(t)$ $(r = 1, N)$.

Now let $P_{r\omega_0+p}^{\rightarrow}$ be the power carried by the forward wave of frequency $r\omega_0 + p$, and let $P_{r\omega_0-p}^{\leftarrow}$, $P_{r\omega_0-p}^{\rightarrow}$, $P_{r\omega_0+p}^{\leftarrow}$ have similar meanings. Then the total power at the frequency ω $(\omega = r\omega_0 \pm p)$ is the sum of the two components $(\rightarrow)$, $(\leftarrow)$:

$$P_\omega = P_\omega^{\rightarrow} + P_\omega^{\leftarrow}. \tag{13}$$

Consider now the fact that the multiplier is lossless, and that therefore Manley–Rowe relations hold.* That is,

$$\left(\frac{P_{\omega_0+p}}{\omega_0 + p} - \frac{P_{\omega_0-p}}{\omega_0 - p}\right) - \left(\frac{P_{N\omega_0+p}}{N\omega_0 + p} - \frac{P_{N\omega_0-p}}{N\omega_0 - p}\right) = 0. \tag{14}$$

From (12) one can calculate $P_\omega^{\rightarrow}$, $P_\omega^{\leftarrow}$ $(\omega = r\omega_0 \pm p)$. For instance,

$$P_{r\omega_0+p}^{\rightarrow} = |a_r^{\rightarrow} + j\theta_r^{\rightarrow}|^2 \frac{|V_r|^2}{2R_r} = |a_r^{\rightarrow} + j\theta_r^{\rightarrow}|^2 P_0 \tag{15}$$

in which $|\ |$ indicates the absolute value. P_0 is the power delivered by the generator to the multiplier at ω_0 and transferred by the multiplier to the load R_N. (The input carrier power is equal to the output carrier power because the multiplier is lossless.)

After calculating $P_\omega^{\rightarrow}$, $P_\omega^{\leftarrow}$, one obtains P_ω $(\omega = r\omega_0 \pm p)$ by means of (13). Then, by substituting P_ω in (14), one obtains†

$$(\mathrm{IM})[N(\theta_1^{\rightarrow}a_1^{\rightarrow *} - \theta_1^{\leftarrow}a_1^{\leftarrow *}) + \theta_N^{\leftarrow}a_N^{\leftarrow *} - \theta_N^{\rightarrow}a_N^{\rightarrow *}] = 0, \tag{16}$$

where (IM) indicates the imaginary part.

Now if one considers the multiplier PM, AM scattering matrices

* The minus sign in front of the second addend of (14) occurs because power is assumed positive when it is flowing towards the right. This, at the output port, is opposite to the usual convention of assuming positive the power absorbed by the varactor. See Ref. 4.

† In deriving (16) the approximation $r\omega_0 \pm p \cong r\omega_0$ has been made so that (14) becomes:

$$NP_{\omega_0+p} - NP_{\omega_0-p} - P_{N\omega_0+p} + P_{N\omega_0-p} = 0.$$

$\hat{S}_a$, $\hat{S}_\theta$ defined in the previous section

$$\left| \begin{matrix} \theta_1^{\leftarrow} \\ \theta_N^{\rightarrow} \end{matrix} \right| = \hat{S}_\theta \left| \begin{matrix} \theta_1^{\rightarrow} \\ \theta_N^{\leftarrow} \end{matrix} \right|, \qquad \left| \begin{matrix} a_1^{\leftarrow} \\ a_1^{\rightarrow} \end{matrix} \right| = \hat{S}_a \left| \begin{matrix} a_1^{\rightarrow} \\ a_N^{\leftarrow} \end{matrix} \right| \tag{17}$$

and substitutes (17) in (16), one obtains

$$\begin{aligned} (\mathrm{IM})[\theta_1^{\rightarrow} a_1^{\rightarrow *}(N - N\rho_\theta^{\rightarrow}\rho_a^{\rightarrow *} - T_\theta^{\rightarrow} T_a^{\rightarrow *}) \\ + \theta_N^{\leftarrow} a_N^{\leftarrow *}(1 - \rho_\theta^{\leftarrow}\rho_a^{\leftarrow *} - NT_\theta^{\leftarrow} T_a^{\leftarrow *}) \\ + \theta_1^{\rightarrow} a_N^{\leftarrow *}(-N\rho_\theta^{\rightarrow} T_a^{\leftarrow *} - \rho_a^{\leftarrow *} T_\theta^{\rightarrow}) \\ + \theta_N^{\leftarrow} a_1^{\rightarrow *}(-NT_\theta^{\leftarrow}\rho_a^{\rightarrow *} - \rho_\theta^{\leftarrow} T_a^{\rightarrow *})] = 0. \end{aligned} \tag{18}$$

A relation which has to hold for any choice of ($\theta_1^{\rightarrow}$, $\theta_1^{\leftarrow}$, etc.) and therefore it has to hold for any choice of ($\theta_1^{\rightarrow} a_1^{\rightarrow *}$, $\theta_N^{\leftarrow} a_N^{\leftarrow *}$, etc.). This is possible only if (11) holds. Note that the approximation $r\omega_0 \pm p \cong r\omega_0$ has been made in deriving (18).

IV. SLOW-VARYING FLUCTUATIONS

Suppose now that the input generator V_g is phase modulated and that its amplitude is kept constant. If the modulation frequency p is small enough, then the output phase deviation will be just N times that of the input generator. Furthermore, all amplitudes will remain constant. These properties are common to all multipliers and are well known.[5,6] Therefore, they will not be demonstrated here. As a consequence of these properties one has*

$$\lim_{p \to 0} T_\theta^{\rightarrow} = N \qquad \lim_{p \to 0} \rho_\theta^{\rightarrow} = 0.$$

By combining these two results with (11) one obtains

$$\begin{aligned} T_\theta^{\rightarrow} &= N & \rho_\theta^{\rightarrow} &= 0 \\ T_a^{\rightarrow} &= 1 & \rho_a^{\leftarrow} &= 0 \\ T_a^{\leftarrow *} &= \frac{1}{NT_\theta^{\leftarrow}} & \rho_a^{\rightarrow *} &= -\frac{\rho_\theta^{\leftarrow}}{NT_\theta^{\leftarrow}} \end{aligned} \tag{19}$$

which apply to the limiting case of slowly varying fluctuations. From (19) one can see that $T_a^{\rightarrow} = 1$. This has the meaning that if one amplitude modulates the input generator, then the output voltage will have an identical (percentage-wise) modulation.

* The multiplier is matched at the carrier frequency ω_0. Therefore, there is no reflection present at the input of the multiplier if modulation is absent. Consequently, input reflections will remain absent if the input phase is slowly modulated. Therefore, $\rho_\theta^{\rightarrow} = 0$ if $p \cong 0$.

Note that (19) leaves only two coefficients to be determined: two of $(\rho^{\rightarrow}_{a}{}^{*}, \rho^{\leftarrow}_{\theta}, T^{\leftarrow}_{\theta})$ or two of $(\rho^{\rightarrow}_{a}{}^{*}, \rho^{\leftarrow}_{\theta}, T^{\leftarrow}_{a}{}^{*})$. Both (11) and (19) represent quite general properties of ideal (i.e., lossless, tuned) multipliers. They do not depend on the actual circuit configuration of the multiplier, on the type of varactor characteristic, on the power level, etc.* Note that from (19) one has

$$(T^{\rightarrow}_{a}{}^{*}T^{\leftarrow}_{a}{}^{*})(T^{\rightarrow}_{\theta}T^{\leftarrow}_{\theta}) = 1.$$

This means that either the "PM round-trip transmission" $(T^{\rightarrow}_{\theta}T^{\leftarrow}_{\theta})$ or the "AM round-trip transmission" $(T^{\rightarrow}_{\theta}T^{\leftarrow}_{a})$ is larger than unity.† Therefore, reflections of the external circuit (the output load and the input generator) can produce instabilities.[1]

V. MULTIPLIER OF ORDER $N = 2^n$

In this section the following type of multiplier will be considered:

(*i*) the nonlinear capacitance has a square-law Q–V characteristic,
(*ii*) the order of multiplication is $N = 2^n$, and
(*iii*) it has the least number of idlers[7] $(2\omega_0, \cdots, 2^{n-1}\omega_0)$.

This type of multiplier is most important because it can be *exactly* treated with little difficulty and it is *realistic* at the same time. In fact, abrupt-junction varactors exactly satisfy (*i*); graded-junctions approximately satisfy (*i*); most practical designs are based on (*iii*) for reasons of simplicity. Condition (*ii*) excludes from this treatment two important cases: $N = 3$, $N = 5$. On the other hand, the (exact) results which will be obtained for $N = 2$, $N = 4$, $N = 8$, $N = 16$, $N = 32$, etc. may be qualitatively extended to the remaining cases $N = 3$, $N = 5$, etc.

In the Appendix, it is shown that, because of (*i*), (*ii*), and (*iii*), one has $T^{\leftarrow}_{\theta} = (-1)^n$, $\rho^{\rightarrow}_{a} = [N - (-1)^n]/3N$. Therefore, by combining these results with (19), one has

$$\hat{S}_{\theta} = \begin{vmatrix} \rho^{\rightarrow}_{\theta} & T^{\leftarrow}_{\theta} \\ T^{\rightarrow}_{\theta} & \rho^{\leftarrow}_{\theta} \end{vmatrix} = \begin{vmatrix} 0 & (-1)^n \\ N & \dfrac{1 - (-1)^n N}{3} \end{vmatrix} \tag{2a}$$

$$\hat{S}_{a} = \begin{vmatrix} \rho^{\rightarrow}_{a} & T^{\leftarrow}_{a} \\ T^{\rightarrow}_{a} & \rho^{\leftarrow}_{a} \end{vmatrix} = \begin{vmatrix} \dfrac{N - (-1)^n}{3N} & \dfrac{(-1)^n}{N} \\ 1 & 0 \end{vmatrix}. \tag{2b}$$

* They are valid both for a single stage and a multiplier consisting of many stages.
† Neglecting the possibility that they are both unity. See the next section.

Therefore, the multiplier scattering matrices are *uniquely determined as functions of only the order of multiplication* N.

The physical meaning of the scattering coefficients (2) may be illustrated by the following two examples.

5.1 *First example.*

Let a small signal $N\omega_0 + p$ be *injected* into R_N by means of a directional coupler connected as shown in Fig. 5(a). Then the voltage across R_N will be both amplitude and phase modulated, and the AM, PM indexes will have equal amplitudes according to (4). Let α be their amplitude. Next, reverse the connection of the directional coupler, as shown in Fig. 5(b). Now the following facts will be observed:

(*i*) *The voltage of R_N contains only PM*. That is, the AM component of $\omega + p$ *is absorbed without reflection by the output of the multiplier* (because $\rho^{\leftarrow}_{a} = 0$). The PM component, on the contrary, is reflected back into the load R_N. The PM index across R_N is $\rho^{\leftarrow}_{\theta}\alpha$ with $\rho^{\leftarrow}_{\theta}$ given by (2). Note that $|\rho^{\leftarrow}_{\theta}| \geqq 1$ for all values of N, and that $|\rho^{\leftarrow}_{\theta}| > 1$ for $N > 4$. Therefore, the PM component is reflected with amplification for $N > 4$. For instance, if $N = 8$ the gain is 3. All this indicates the circuit of Fig. 5(b) as a possible scheme for modulating the output phase of a multiplier without producing any AM (and with gain if $N > 4$).

(*ii*) The voltage across R_1 has both AM and PM. The PM index is $T^{\leftarrow}_{\theta}\alpha = (-1)^n\alpha$ and the AM index is $T^{\leftarrow}_{a}\alpha = (-1)^n/N \cdot \alpha$.

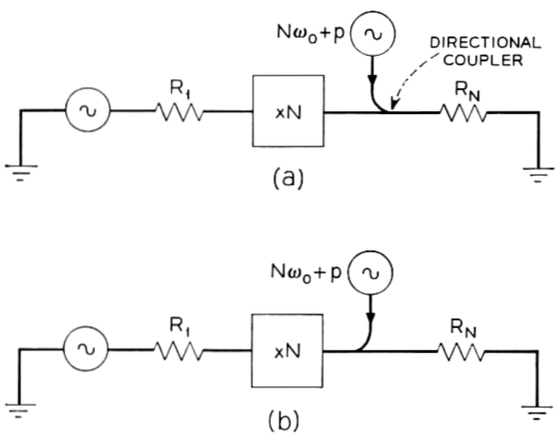


Fig. 5 — Injection of a tone at the output port of the multiplier.

5.2 *Second example.*

Now let the previous experiment be repeated at the input side of the multiplier, as shown by Fig. 6. Let α be the amplitude of the indexes generated across R_1 when the generator $\omega + p$ is connected as shown in Fig. 6(a). Then, after connecting $\omega + p$ as shown in Fig. 6(b), one has:

(*i*) The AM index across R_N is $T_a^{\rightarrow}\alpha = \alpha$ and the PM index is $T_\theta^{\rightarrow}\alpha = N\alpha$. These are general properties which have already been found and discussed in the more general case of the preceeding section [see (19)].

(*ii*) Across R_1 there is only AM and the AM index is $\rho_a^{\rightarrow}\alpha$ with $\rho_a^{\rightarrow}$ given by (2).

In practical applications it is important to bear in mind the following meaning of (2). If only phase modulation is present, then one can calculate the upper sidebands by simply replacing the multiplier with a linear and time-invariant amplifier whose scattering properties are given by (2a). Similarly, if only AM is present then one can calculate the upper sidebands by using (2b). All this is true provided the upper sidebands are normalized with respect to V_c, in the AM case, and with respect to jV_c, in the PM case.

Consider now the PM case. From (2a) one obtains that the round-trip PM transmission $T_\theta^{\rightarrow}T_\theta^{\leftarrow}$ is

$$| T_\theta^{\rightarrow}T_\theta^{\leftarrow} | = N. \tag{20}$$

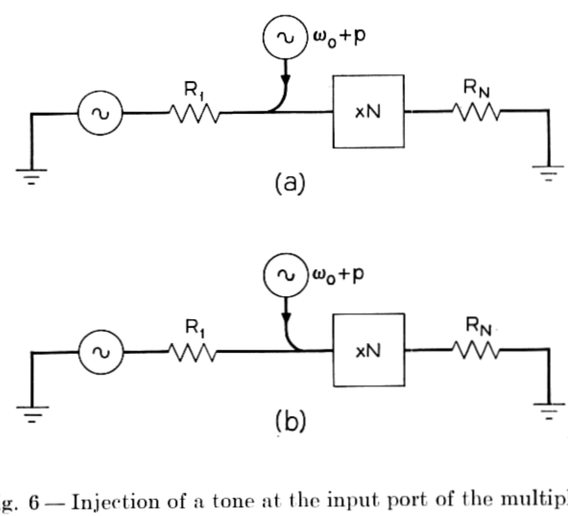


Fig. 6 — Injection of a tone at the input port of the multiplier.

Therefore, interactions between the multiplier and the other stages can cause phase instabilities.* Furthermore, from (20) one can see that the possibility of instabilities increases with the order of multiplication N. However, the presence of loss will decrease the round-trip transmission. More precisely, if η is the multiplier efficiency,† then the round-trip power transmission will become approximately‡

$$| T_{\theta}^{\rightarrow} T_{\theta}^{\leftarrow} |^2 = N^2 \cdot \eta^2 \tag{21}$$

and the possibility that interactions of the external circuit cause PM instability exists when approximately

$$\eta > \frac{1}{N}. \tag{22}$$

Note that amplitude instabilities cannot occur.§ In fact, if the output port of the multiplier is connected to an arbitrary passive stage, then the input AM reflection of the multiplier is always less than¶

$$| \rho_a^{\rightarrow} | + | T_a^{\rightarrow} T_a^{\leftarrow} | , \tag{23}$$

which is never greater than unity, as one may verify from (2b).

APPENDIX

Demonstrations for the multiplier of order 2^n which has the minimum number of idlers and uses a "square-law" varactor. Slowly varying modulations.

Let $S_t(t)$ be the total elastance of the varactor. It can be separated into the sum of a time-varying component $S(t)$ and an average component S_0

$$S_t(t) = S(t) + S_0 .$$

The "external circuit" connected to the varactor can be represented by an impedance Z_e in series with a voltage generator V_g, as it is shown in Fig. 7.

Let $Z(\omega)$ be the total impedance in series with $S(t)$. Then $Z = Z_e + S_0/j\omega$.

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* This is discussed in detail in the second paper.

† Output power divided by input power.

‡ η^2 is the round-trip transmission through an attenuator which has a forward (power) transmission η. Equation (21) has been obtained by representing the multiplier as the cascade connection of such an attenuator and a lossless multiplier.

§ It is important to note that it is assumed that the "bias circuit" is properly designed, so that low-frequency fluctuations of the average varactor capacitance C_0 are avoided.

¶ This can be obtained by using standard techniques.

The hypotheses are:

$$Z = 0 \quad \text{if} \quad \omega \cong 2^s\omega_0 \qquad (s = 1, \cdots, n-1) \tag{24a}$$

$$Z = R_1 \quad \text{if} \quad \omega \cong \omega_0 \tag{24b}$$

$$Z = R_N \quad \text{if} \quad \omega \cong N\omega_0 = 2^n\omega_0 \tag{24c}$$

$$Z = \infty \quad \text{if} \quad \omega \quad \text{is far from} \quad 2^s\omega_0 \qquad (s = 0, \cdots, n). \tag{24d}$$

Equation (24a) requires $Z = 0$ in the neighborhood of the idler frequencies $2\omega_0, \cdots, 2^{n-1}\omega_0$. Equations (24b), (24c) require Z to be equal to the input impedance R_1 for $\omega \cong \omega_0$, and to be equal to the output impedance R_N for $\omega \cong N\omega_0$. In (24d) it is required that current flow be limited to the frequencies of (24a), (24b), and (24c). Because

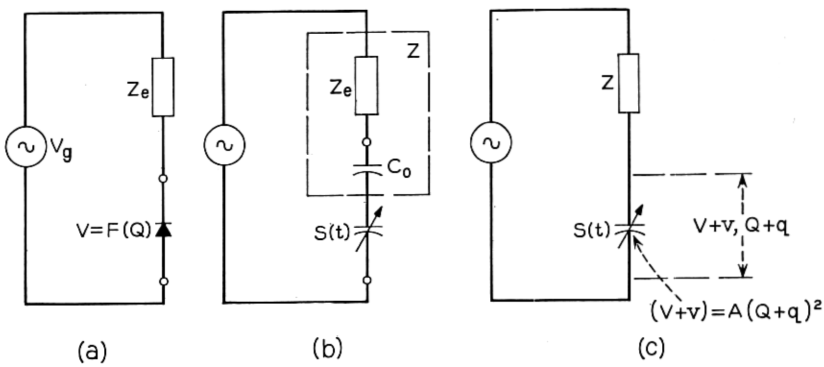

Fig. 7 — Equivalent circuit of the frequency multiplier.

of this last requirement, the variable component $Q(t)$ of the charge of the varactor (*when the sidebands are absent*) is of the type

$$Q(t) = \sum_{\substack{s=0,\cdots,n \\ r=\pm 2^s}} Q_r \exp(jr\omega_0 t). \tag{25}$$

Since the varactor has a square-law Q–V characteristic $F(Q)$, the total voltage* $V_t(t)$ across the varactor is related to $Q(t)$ through a relation of the type

$$F(Q(t)) = V_t(t) = AQ(t)^2 + Q(t)S_0 + V_0 . \tag{26}$$

Therefore, the time-varying part of V_t will consist of two components: one component $V(t)$ due to $AQ(t)^2$

* The sign convention in this appendix is that power is positive when flowing into the varactor.

$$V(t) = A \sum_{i \neq r} Q_i Q_r \exp[j(r + i)\omega_0 t] = \sum V_l \exp(jl\omega_0 t) \tag{27}$$
$$(i = \pm 2^s; \quad r = \pm 2^h; \quad s, h = 0, \cdots, n)$$

and another component $V' = Q(t)S_0$ which represents the contribution of a constant capacitance $C_0 = 1/S_0$. The total elastance is obtained by taking the derivative of (26) with respect to $Q(t)$:

$$S_t(t) = S(t) + S_0 = 2AQ(t) + \frac{1}{C_0} \tag{28}$$

and one finds a constant capacitance C_0 in series with a time-varying elastance component $S(t)$ originating from $AQ(t)$. Therefore, the equivalent circuit of the multiplier becomes that of Fig. 7(b) in which the nonlinear element has a Q–V characteristic of the type AQ^2, and $V(t)$, $Q(t)$, $S(t)$ represent its voltage, charge, and elastance, respectively, when the sidebands are absent. From (28), (25) one has

$$S(t) = 2AQ(t) = \sum_{\substack{s=0,\cdots,n \\ r=\pm 2^s}} S_r \exp(jr\omega_0 t) \tag{29}$$

with

$$S_r = 2AQ_r\,. \tag{30}$$

Let now

$$q(t) = \sum_{\substack{s=0,\cdots,n \\ r=\pm 2^s \\ i=\pm 1}} q_{r,i} \exp[j(r\omega_0 + ip)t] \tag{31}$$

be the sideband components of the charge of the varactor. Then the voltage sidebands across $S(t)$ are

$$v(t) = S(t)q(t) = \sum v_{r,i} \exp[j(r\omega_0 + ip)t]. \tag{32}$$

By substituting (29), (31), in (32) one obtains* $v_{r,i}$

$$v_{r,\pm 1} = \sum_{\substack{s=0 \\ u=\pm 2^s}}^{n} S_u q_{(r-u),\pm 1} \qquad (r = 1, \cdots, 2^n). \tag{33}$$

But from (24d) and from the fact that $Z(\omega + p) \cong Z(\omega)$, one has

$$q_{(r-u),\pm 1} = 0 \qquad \text{for} \qquad (r - u) \neq \pm 1, \cdots, \pm 2^n. \tag{34}$$

In (33), therefore, one has to consider only those values of u which satisfy (put: $r = 2^h$)

$$2^h - u = \pm 1, \cdots, \pm 2^n. \tag{35}$$

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* Note that $q_{r,i} = q^*_{r,-i}$, $v_{r,i} = v^*_{r,-i}$, (see footnote on p. 780). Therefore, consideration can be limited to the case $r > 0$.

Then, the only possible values of u are

$$u = 2^{h-1}, \qquad u = 2^{h+1}, \qquad u = -2^h, \tag{36}$$

because all other values of u (remember that u has to be of the type $u = \pm 2^s$) would cause $|r - u|$ to be either zero or odd. Therefore, remembering that $r = 2^h$, one finds that (33) consists of only three terms:

$$v_{r,\pm 1} = q_{r/2,\pm 1} S_{r/2} + q_{2r,\pm 1} S_{-r} + q^*_{r,\mp 1} S_{2r} \quad (r = 1, \cdots, 2^n). \qquad (37)^*$$

In conclusion, in order that $q(t)$, $v(t)$ represent sideband components produced by sources located only at the input and the output of the multiplier, it has to be (*necessary* and *sufficient* condition)

$$v_{r,\pm 1} = 0 \quad \text{for} \quad r = \pm 2, \cdots, \pm 2^{n-1} \tag{38a}$$

$$q_{r,\pm 1} \neq 0 \quad \text{only for} \quad r = \pm 1, \cdots, \pm 2^n \tag{38b}$$

$$v_{r,\pm 1} = q_{r/2,\pm 1} S_{r/2} + q_{2r,\pm 1} S_{-r} + q^*_{r,\mp 1} S_{2r} \quad \text{for} \quad r = 1, \cdots, 2^n. \tag{38c}$$

Equation (38a) follows from (24a); (38b) follows from (24d); (38c) is (37).

Note that (38a) and (38b), are the constraints given by the "external circuit" (which includes C_0), and (38c) is the constraint given by the elastance $S(t)$ of the varactor.

The "equilibrium equations" for the "carriers" Q_r, V_r are obtained in a similar way. They are

$$V_r = 0 \quad \text{for} \quad r = \pm 2, \cdots, \pm 2^{n-1} \tag{39a}$$

$$Q_r \neq 0 \quad \text{only for} \quad r = \pm 1, \cdots, \pm 2^n \tag{39b}$$

$$V_1 = j\omega_0 Q_1 R_1 \tag{39c}$$

$$V_N = -jN\omega_0 Q_n R_n \tag{39d}$$

$$V_r = A[Q_{r/2} Q_{r/2} + 2Q_{2r} Q_r^*] \quad \text{for} \quad r = 1, \cdots, 2^n. \tag{39e}$$

Equations (39a) and (39b) follow from (24a) and (24d), respectively. (39c) follows from (24b) and the hypothesis that the input generator is matched to the multiplier. Equation (39d) follows from (24c). Equation (39e) can be derived from (27) by using the *same* procedure used to derive (38c) from (32).

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* Note, if $r = 1$, the first term of the second member is zero; if $r = N$, only the first term of the second member is nonzero.

A.1 *Demonstration of* $T_a^{\leftarrow} = (-1)^n/N$

Theorem 1: If $\hat{q}$,$\hat{v}$ *satisfies* (38) *and if*

$$\tilde{q}_{r,\pm 1} = \hat{q}_{r,\pm 1} \exp\left[j\frac{\pi}{2}(-1)^s\right]$$
$$\tilde{v}_{r,\pm 1} = -\hat{v}_{r,\pm 1} \exp\left[j\frac{\pi}{2}(-1)^s\right] \qquad (r = 2^s;\, s = 0, \cdots, n) \tag{40}$$

then also $\tilde{q}$, $\tilde{v}$ *satisfles* (38), *i.e., it gives sidebands produced by sources at* $\omega_0 \pm p$, $N\omega_0 \pm p$.

Proof: $\tilde{q}$, $\tilde{v}$ clearly satisfies (38a) and (38b) because $\hat{q}$, $\hat{v}$ does. Therefore, it remains to be demonstrated that (38c) is satisfied.

By substituting (40) in (38c) one obtains

$$-\hat{v}_{r,\pm 1} \exp\left[j\frac{\pi}{2}(-1)^s\right] = \hat{q}_{r/2,\pm 1}S_{r/2} \exp\left[j\frac{\pi}{2}(-1)^{s-1}\right]$$
$$+ \hat{q}_{2r,\pm 1}S_{-r} \exp\left[j\frac{\pi}{2}(-1)^{s+1}\right] + \hat{q}^*_{r,\mp 1}S_{2r} \exp\left[-j\frac{\pi}{2}(-1)^s\right],$$

which is satisfied because $\hat{v}$, $\tilde{q}$ satisfies (38), and because

$$-\exp\left[j\frac{\pi}{2}(-1)^s\right] = \exp\left[j\frac{\pi}{2}(-1)^{s\mp 1}\right] - \exp\left[-j\frac{\pi}{2}(-1)^s\right].$$

Suppose now that $\hat{q}$, $\hat{v}$ is produced by a PM source located at the input of the multiplier. Therefore, remembering that $\rho_\theta^{\rightarrow} = 0$ because of (19), the input components of $\hat{q}$, $\hat{v}$ represent a PM forward wave, and also the output components constitute a PM forward wave. Consider now the input ($r = 1$) components of $\tilde{q}$, $\tilde{v}$. From (40) one has

$$\tilde{v}_{1,\pm 1} = -j\hat{v}_{1,\pm 1} \qquad \tilde{q}_{1,\pm 1} = j\hat{q}_{1,\pm 1} \tag{41}$$

from which one can see that, since $\hat{v}_{1,\pm 1}$, $\hat{q}_{1,\pm 1}$ is an FM "forward" wave, then $\tilde{v}_{1,\pm 1}$, $\tilde{q}_{1,\pm 1}$ is an AM "backward" wave. In a completely similar way one finds that also the output components $\tilde{v}_{N,\pm 1}$, $\tilde{q}_{N,\pm 1}$ represent an AM "backward wave". Finally, remembering (4) and the definitions of $T_\theta^{\rightarrow}$, $T_a^{\leftarrow}$, one has

$$T_\theta^{\rightarrow} = \left(\frac{\theta_N^{\rightarrow}}{\theta_1^{\rightarrow}}\right)_{\overleftarrow{\theta}_N = 0} = \frac{\hat{v}_{N,1}}{V_N}\frac{V_1}{\hat{v}_{1,1}}$$

$$T_a^{\leftarrow} = \left(\frac{a_1^{\leftarrow}}{a_N^{\leftarrow}}\right)_{\overrightarrow{a}_1 = 0} = \frac{\tilde{v}_{1,1}}{V_1}\frac{V_N}{\tilde{v}_{N,1}}.$$

By combining this result with (40) one has

$$\frac{\hat{v}_{N,1}}{\hat{v}_{1,1}} \frac{\tilde{v}_{1,1}}{\tilde{v}_{N,1}} = T_{\theta}^{\rightarrow} T_{a}^{\leftarrow} = (-1)^{N}.$$

But $T_{\theta}^{\rightarrow} = N$. Therefore, one obtains the desired result

$$T_{a}^{\leftarrow} = \frac{(-1)^{n}}{N}.$$

A.2 *Demonstration of* $\rho_{a}^{\rightarrow} = [N - (-1)^{n}]/N \cdot 3$

$$\rho_{a}^{\rightarrow} = \frac{N - (-1)^{n}}{N \cdot 3}. \tag{42}$$

First it will be shown that the charge $\mathring{q}(t)$ defined by

$$\mathring{q}_{r,1} = \mathring{q}_{r,-1} = (1 - \rho(s))Q_{r} \qquad (r = 2^{s}, s = 0, \cdots, n) \tag{43}$$

with

$$\rho(s) = \frac{2^{n-s} - (-1)^{n-s}}{2^{n-s} \cdot 3} \tag{44}$$

gives the charge sidebands produced by an AM wave of amplitude $a_{1}^{\rightarrow} = 1$ arriving at the input of the multiplier.

Let first the fact that $\mathring{q}(t)$ satisfies (38) be demonstrated. Substitute (43) in (37) and take into account (30). One obtains $\mathring{v}$, the voltage associated with $\mathring{q}$,

$$\begin{aligned}\mathring{v}_{r,\pm 1} = 2A\{&Q_{r/2}Q_{r/2}(1 - \rho(s - 1)) \\ &+ Q_{2r}Q_{-r}[(1 - \rho(s)) + (1 - \rho(s + 1))]\} \\ &\qquad (r = 2^{s}; s = 1, \cdots, n - 1).\end{aligned} \tag{45}$$

From (44) one has

$$2 - \rho(s) - \rho(s + 1) = 2(1 - \rho(s - 1)).$$

Therefore, by using (39e), (45) gives

$$\mathring{v}_{r,\pm 1} = 4(1 - \rho(s - 1))V_{r} = 0 \quad \text{for} \quad r = 2, \cdots, 2^{n-1}$$

because of (39a). Therefore, $\mathring{q}$, $\mathring{v}$ satisfies (38). Furthermore, it represents AM because the sidebands $\mathring{q}_{r,\pm 1}$ are "in-phase" with the carriers Q_{r}. This is shown by (43).

Now let the output indexes $a_{N}^{\rightarrow}$, $a_{N}^{\leftarrow}$ of $\mathring{q}$, $\mathring{v}$ be calculated. Consider the following expressions: (38c) with $r = N$; (30) with $r = N/2$; (43)

with $r = N/2$; (39e) with $r = N$. From them one obtains

$$\mathring{v}_{N,1} = V_N .$$

From this first relation, by using (39d) and (43) with $r = N$, one obtains

$$\mathring{v}_{N,1} = -R_N(jN\omega_0\mathring{q}_{N,1}).$$

This second relation gives $a_N^{\leftarrow} = 0$, because $jN\omega_0\mathring{q}_{N,1}$ is the current at $N\omega_0 + p$. Then the first relation gives at once $a_N^{\rightarrow} = 1$.

From the fact that $a_N^{\leftarrow} = 0$, $a_N^{\rightarrow} = 1$ and from the fact that (19) gives $T_a^{\rightarrow} = 1$, the input indexes must be

$$a_1^{\rightarrow} = 1, \qquad a_1^{\leftarrow} = \rho_a^{\rightarrow} .$$

Therefore, the input voltage sidebands must be

$$\mathring{v}_{1,\pm1} = (1 + \rho_a^{\rightarrow})V_1 . \tag{46}$$

But, by substituting (43) in (37) with $r = 1$, one has

$$\mathring{v}_{1,1} = (1 - \rho(1))Q_2S_{-1} + (1 - \rho(0))Q_1^*S_2$$

from which, remembering that $Q_2S_{-1} = Q_1^*S_2 = V_1$ because of (39e) and (30), one has

$$\mathring{v}_{1,1} = (2 - \rho(1) - \rho(0))V_1 = (1 + \rho(0))V_1 \tag{47}$$

in which use has been made of (44). Finally, by combining (47) with (46), one obtains

$$\rho_a^{\rightarrow} = \rho(0) = \frac{N - (-1)^n}{3N}$$

which is the desired result.

A.3 *Demonstration of the fact that the multiplier does not produce AM $\leftrightarrows$ PM conversion, if p is small enough.*

An input PM source does not cause AM if p is small enough. In fact, one may verify that the sidebands produced by an input PM forward wave of amplitude $\theta_1^{\rightarrow} = \theta$ are

$$q_{r,1} = jrQ_r\theta, \qquad q_{r,-1} = jrQ_r\theta^* \tag{48}$$

which are in "quadrature" with respect to the carriers Q_r. To demonstrate that there is no PM $\rightarrow$ AM conversion it therefore remains to be shown that an output PM source does not produce AM. This can be done by applying transformation (40) to the AM sidebands given by (43). In fact, in this way one obtains PM sidebands. They are produced by PM sources located both at the input and the output of the multiplier. Therefore, one concludes that there is no PM $\rightarrow$ AM, if $p \cong 0$.

By using the same procedure one can show that there is no AM $\rightarrow$ PM conversion. In fact, the discussion following (43) shows that an input AM source does not produce PM. Then, by applying (40) to (48), one finds that an output AM source does not produce PM. This concludes the demonstration.

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