Phase and Amplitude Modulation in High-Efficiency Varactor Frequency Multipliers of Order $N = 2^n$ – Stability and Noise

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A general analysis of the stability conditions of frequency multipliers of order $N = 2^n$ is presented. The frequency multipliers to be considered are harmonic generators which use varactor diodes as nonlinear elements. The type of instability investigated is that which causes spurious tones to appear at the output of a multiplier in the vicinity of the desired harmonic. It arises because an efficient multiplier is potentially unstable with respect to the quadrature components of its sidebands.

This paper shows how instability arises and how it can be avoided. One of the main results is that, to obtain stability in practical cases, it is sufficient that the bandwidths of the various resonant circuits satisfy some simple conditions.

I. SUMMARY OF RESULTS

A general analysis of the stability conditions of frequency multipliers of order $N = 2^n$ is presented. The frequency multipliers to be considered are harmonic generators which use varactor diodes as nonlinear elements.

That stability is one of the most serious problems in high-efficiency multipliers is a widely known experimental fact.[1,2]

At the present time little is known of the restrictions placed by the condition of stability on the available circuit configurations. Consequently, present design procedures leave the problem of stability to be solved experimentally, and this is often done at the expense of efficiency. Furthermore, multipliers which are individually stable may become unstable when connected together to form a chain. As a result, isolators are often needed. The isolators will guarantee stability but will lower the overall efficiency.

The practical importance of the problem to be analyzed is illustrated by the following example.

Suppose that one has designed a stable and very efficient octupler, and that one wants to reduce the output noise by using a bandpass filter consisting of a high Q resonant circuit connected in series with the load. In general, one will not obtain the desired result. In fact, the output noise will, in general, increase rather than decrease. Furthermore, if the bandwidth of the filter is too narrow, then the multiplier will become unstable.*

This example illustrates the important fact that, as the circuit approaches an unstable condition, the output noise level increases indefinitely. Then, when the multiplier is on the point of becoming unstable, the output noise becomes very large at some frequencies. Furthermore, when instability arises, spurious tones appear in the vicinity of the carriers. Therefore, not only is it important that a multiplier be stable, but it is also important that it be far from instability, if one wants a low-noise multiplier.

This paper shows how instability arises, and how to avoid it. It also shows how to derive the output noise from a knowledge of the various noise sources. Some of the results which have been obtained are summarized by the following statements.

1.1 *Doubler*

Consider the doubler first.

(*i*) The stability of the doubler will not, in general, depend on the impedance presented by the input circuit (to the varactor) in the "vicinity" of the input carrier ω_0. More precisely, in the design of the input circuit, consideration can be limited to those frequencies whose distance from the carrier ω_0 is larger than half the bandwidth of the output circuit [roughly; see (44), (45), (46)]. At those input frequencies which are not in the "vicinity" of the input carrier,

(*ii*) the impedance presented by the input circuit to the varactor should be large rather than small, as compared to R_1 [see (13), Theorem 2, Theorem 3], where R_1 is the impedance presented by the input circuit at ω_0. Furthermore:

(*iii*) The output bandwidth should be large compared to the input bandwidth [see (35), etc.]. Furthermore:

(*iv*) An efficient chain of more than two doublers which are individually stable will, in general, be stable if each doubler has been designed according to (*iii*) and if, in addition, each doubler is sufficiently broadband with respect to the preceding one [see (58)]. Furthermore:

* This example is derived at the end of this paper. See (51), (52), etc.

(*v*) Those circuit configurations should be preferred which produce PM ⇄ AM conversion [see (55)]. Therefore, low-pass circuit configurations at the input, and high-pass circuit configurations at the output, are, in general, preferable to bandpass circuit configurations.

1.2 *Multiplier of Order $N = 2^n$*

Consider a multiplier of order $N = 2^n > 2$ which has the minimum number of idlers. This memorandum shows that such a multiplier is equivalent to a chain of doublers.* Therefore, results obtained for the doubler can be extended to this type of multiplier. For instance (*iv*) gives:

An efficient multiplier of order 2^n which has the minimum number of idlers should have

$$B_1 \ll B_2 \ll \cdots \ll B_{n-1} \ll B_n ,$$

where B_1 is the bandwidth of the input circuit, B_2 is the "equivalent" bandwidth of the first idler, etc.

Note that the above results† apply to the case of a lossless multiplier. The presence of losses reduces the restrictions placed by the condition of stability. If $\eta < 1/N$, then the question of stability does not arise any more, in general. This has been shown by Ref. 3.

II. PRELIMINARY CONSIDERATIONS

2.1 *Method of approach of the mathematical description of the multiplier and its signals*

This paper is concerned with the presence of amplitude and phase fluctuations in the multiplier signals. Suppose, for the moment, that these two types of fluctuations are independent from each other. That is, suppose that the multiplier does not produce AM-to-PM conversion, and vice-versa. Suppose, furthermore, that only PM is present. Then each signal will consist of a carrier and of a pair of sidebands in quadrature with respect to the carrier. Since either sideband can be obtained from a knowledge of the other, then one may consider only one of the two sidebands and ignore the other one. Let then, for instance, the

* See Theorem 1 of the second section. Note that the equivalence is exact only if the varactor has a square-law Q–V characteristic.

† These results implied that the impedance presented by the circuit (to the variable capacitance) is "large" at the frequencies which are far from the various carriers. Therefore, subharmonic oscillations and bias instabilities are not included in this analysis.

upper sideband be chosen as the variable, and let it be described in terms of propagating waves.

Then the input variables of the multiplier will be the two waves* $v^{\rightarrow}_{\omega_0+p}$, $v^{\leftarrow}_{\omega_0+p}$ which constitute the upper sideband of the input carrier ω_0. Similarly, the output waves $v^{\rightarrow}_{N\omega_0+p}$, $v^{\leftarrow}_{N\omega_0+p}$ represent the output variables. ω_0 is the input "carrier" frequency of the multiplier, and p is the frequency of the fluctuations. At this point the scattering formalism furnishes a convenient way of describing the properties of the multiplier. More precisely, one may define the PM scattering parameters of the multiplier as the reflection and transmission coefficients which relate the "scattered" waves $v^{\leftarrow}_{\omega_0+p}$, $v^{\rightarrow}_{N\omega_0+p}$ to the "incident" ones $v^{\rightarrow}_{\omega_0+p}$, $v^{\leftarrow}_{N\omega_0+p}$. One obtains in this way the PM scattering matrix $\hat{S}_\theta$ of the multiplier.

Through the analysis $v^{\rightarrow}_{\omega_0+p}$, $v^{\leftarrow}_{\omega_0+p}$, $v^{\rightarrow}_{N\omega_0+p}$, $v^{\leftarrow}_{N\omega_0+p}$, have been normalized with respect to the carriers. In this way the variables become the dimensionless coefficients

$$j\theta_1^{\rightarrow} = \frac{v^{\rightarrow}_{\omega_0+p}}{V_{\omega_0}}, \qquad j\theta_1^{\leftarrow} = \frac{v^{\leftarrow}_{\omega_0+p}}{V_{\omega_0}}, \qquad j\theta_N^{\rightarrow} = \frac{v^{\rightarrow}_{N\omega_0+p}}{V_{N\omega_0}}, \qquad j\theta_N^{\leftarrow} = \frac{v^{\leftarrow}_{N\omega_0+p}}{V_{N\omega_0}},$$

which represent "modulation indexes". Note that V_{ω_0} is the Fourier coefficient of the input carrier ω_0, and $V_{N\omega_0}$ is the Fourier coefficient of the output carrier $N\omega_0$.

In a completely similar way the AM case is treated. By normalizing the AM upper sidebands with respect to V_{ω_0}, $V_{N\omega_0}$, one obtains four modulation indexes $a_1^{\rightarrow}$, $a_1^{\leftarrow}$, $a_N^{\rightarrow}$, $a_N^{\leftarrow}$ which represent the AM variables. Furthermore, by considering how these waves are scattered by the multiplier, one finds the AM scattering matrix $\hat{S}_a$ which describes the multiplier AM properties.

2.2 *General Scattering Properties of an "Ideal" Multiplier*

Consider a multiplier of order $N = 2^n$ which has the following properties: it is lossless, it has the minimum number of idlers $(n - 1)$, and it uses a varactor having a square-law Q–V characteristic. The first paper (See Ref. 3) has shown that, if the fluctuations are slow enough ($p \cong 0$), then such a multiplier does not produce AM $\leftrightarrows$ PM conversion† and it has the following scattering matrices:

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* $v^{\leftarrow}_{\omega}$, $v^{\rightarrow}_{\omega}$ designate the Fourier coefficients of the voltage components of frequency ω. They propagate in the directions indicated by the arrows. For more details on the mathematical description of the multiplier and its signals see Ref. 3.

† It is important to note that it has been assumed that the "bias circuit" of the multiplier is properly designed, so that low-frequency fluctuations of the average capacitance C_0 are avoided. See Ref. 4.

$$\hat{S}_a = \begin{vmatrix} \rho_a^{\rightarrow} & T_a^{\leftarrow} \\ T_a^{\rightarrow} & \rho_a^{\leftarrow} \end{vmatrix} = \begin{vmatrix} \dfrac{N-(-1)^n}{3N} & \dfrac{(-1)^n}{N} \\ 1 & 0 \end{vmatrix} \tag{1}$$

$$\hat{S}_\theta = \begin{vmatrix} \rho_\theta^{\rightarrow} & T_\theta^{\leftarrow} \\ T_\theta^{\rightarrow} & \rho_\theta^{\leftarrow} \end{vmatrix} = \begin{vmatrix} 0 & (-1)^n \\ N & \dfrac{1-(-1)^n N}{3} \end{vmatrix} . \tag{2}$$

A multiplier for which (1), (2) apply will be called "ideal". According to the preceding discussion, one has that, if only PM is present and if a multiplier is "ideal", then

$$\begin{aligned} \frac{\overleftarrow{v}_{\omega_0+p}}{V_{\omega_0}} &= \rho_\theta^{\rightarrow} \frac{\overrightarrow{v}_{\omega_0+p}}{V_{\omega_0}} + T_\theta^{\leftarrow} \frac{\overleftarrow{v}_{N\omega_0+p}}{V_{N\omega_0}} \\ \frac{\overrightarrow{v}_{N\omega_0+p}}{V_{N\omega_0}} &= T_\theta^{\rightarrow} \frac{\overrightarrow{v}_{\omega_0+p}}{V_{\omega_0}} + \rho_\theta^{\leftarrow} \frac{\overleftarrow{v}_{N\omega_0+p}}{V_{N\omega_0}} . \end{aligned} \tag{3}$$

If only AM is present, the relations existing among the upper sidebands are similar to (3) (Replace $\rho_\theta^{\rightarrow}$, $T_\theta^{\leftarrow}$, etc. with $\rho_a^{\rightarrow}$, $T_a^{\leftarrow}$, etc.).

In conclusion, one can calculate the sidebands present in an "ideal" multiplier in the following way. First, separate the PM components from the AM components. Next, solve separately the two cases AM, PM. The PM case is solved by means of (3). That is, one replaces the multiplier with a linear amplifier which has

(*i*) a forward transmission coefficient $T_\theta^{\rightarrow}$,
(*ii*) a reverse transmission coefficient $T_\theta^{\leftarrow}$,
(*iii*) an input reflection coefficient $\rho_\theta^{\rightarrow}$, and
(*iv*) an output reflection coefficient $\rho_\theta^{\leftarrow}$.

Then one supposes that the variables consist of the PM upper sidebands only, as it is illustrated by (3), and one readily calculates them. In a completely similar way one calculates the AM upper sidebands.

2.3 *General Considerations on Stability*

The frequency response of an unstable circuit which is on the point of becoming stable is infinite.* Hence, if the parameters† of a circuit are continuously varied, a circuit can go from a stable to an unstable situation if and only if its frequency response becomes infinite. Therefore, a certain situation A is stable if‡

* See Ref. 5, p. 316, or Ref. 6, p. 112.
† These parameters are discussed in the next sections.
‡ Note that there are stable circuits for which (4) is not satisfied (conditionally

one can cause the circuit to pass from situation A to a stable situation B by continuously varying its parameters, *without causing its frequency response to become infinite.* (4)

For instance, consider a multiplier which is individually stable and which is connected to a generator and a load which have reflections. Suppose, furthermore, that one is concerned about the possibility that these reflections cause instability. Then, according to (4) one may find out whether or not the circuit is unstable (or conditionally stable) by examining whether or not the circuit response becomes infinite as the reflections are decreased in amplitude. This can be done by inserting attenuators between the multiplier and the other stages (the load and the generator). In this example, situation B occurs when all reflections become zero. The importance of (4) and of this example will be better understood in the next sections.

Consider now the case of a multiplier which does not produce AM $\rightleftarrows$ PM conversion. Then only phase instabilities can occur, as has been shown by Ref. 3. Therefore, one is concerned about its behavior with respect to PM only.

In general, when the multiplier is on the point of becoming unstable, all four PM scattering coefficients ($T^{\rightarrow}, T^{\leftarrow}, \rho^{\rightarrow}, \rho^{\leftarrow}$) become infinite simultaneously.* Therefore, one may choose any one of them to study the stability of the multiplier. However, since the forward transmission $T^{\rightarrow}$ is of special interest because it gives the output response to input PM signals, it will be selected as the multiplier frequency response to be analyzed.

The stability of a multiplier depends on the values of certain parameters which will be discussed in the following sections. Those values of these parameters for which the circuit is stable constitute the so called "region of stability". At the boundary of the region of stability the PM forward transmission becomes infinite at one or more frequencies, as has already been pointed out. Therefore, in order to find the stability region, one must find those "critical" values of the parameters for which $T^{\rightarrow} = \infty$. Then, once the boundary is found, one must identify which one of the two regions separated by the boundary is stable. This can be done by applying the stability test (4) or any of the usual stability

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stable circuits. See Ref. 6, p. 162). However, for practical reasons, one is interested in designing a multiplier which does not become unstable if the losses are increased. (i.e., which is unconditionally stable.) For such a multiplier (4) is necessary and sufficient.

* See Ref. 6, p. 164. Since the discussion will be mostly confined to the PM case, the subscripts a, θ used in (1), (2), and (3) to distinguish the AM case from the PM case will be omitted unless necessary.

tests (such as the "Nyquist plot", etc.) to one point of either one of the two regions.

As will be seen, a multiplier can be represented by means of a chain of stages which are individually stable. The analysis of such a chain may be carried on as follows. Consider the cascade connection of the two stages (*b*), (*c*), shown in Fig. 1. If the two stages are individually stable the overall forward transmission becomes infinite if and only if

$$\overleftarrow{\rho}_b \overrightarrow{\rho}_c = 1, \tag{5}$$

where $\overleftarrow{\rho}_b$ is the output reflection of the first stage* and $\overrightarrow{\rho}_c$ is the input reflection of the second stage. Equation (5) may be obtained in the

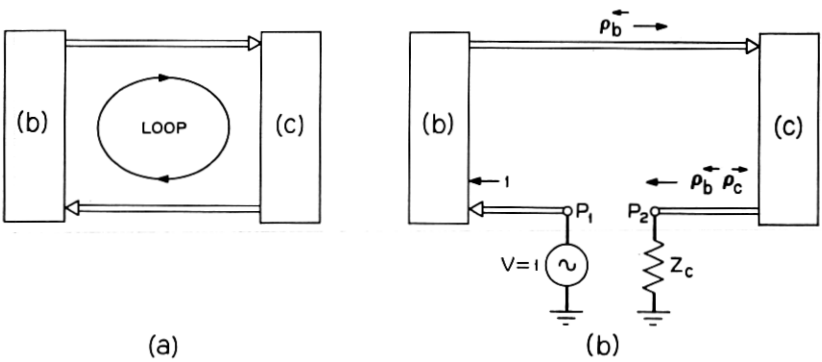

Fig. 1 — "Loop transmission" at the interconnection between two stages.

following way. Consider the interconnection between the two stages (*b*), (*c*), and suppose that the reverse path is separated from the forward path, as illustrated in Fig. 1(a). Next, break the reverse-path connection as it is shown in Fig. 1(b). In addition, terminate P_2 in the characteristic impedance Z_c of the interconnection, and apply a unit voltage to P_1, as illustrated in Fig. 1(b). Then the voltage which appears at P_2 is $\overleftarrow{\rho}_b \overrightarrow{\rho}_c$. Therefore, $\overleftarrow{\rho}_b \overrightarrow{\rho}_c$ represents the "return voltage" (i.e., the familiar loop transmission $\mu\beta$)† of the loop indicated in Fig. 1(a), and (5) is demonstrated.

In some cases, one will be interested in knowing the reflection $\overleftarrow{\rho}'$ presented by the output of a stage when its input port is connected to a generator which has an output reflection $\overleftarrow{\rho}_g \neq 0$ (see Fig. 2). The output reflection $\overleftarrow{\rho}'$ is given by

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* Which occurs when the other port of the stage is terminated in its characteristic impedance.

† See Ref. 6, p. 44.

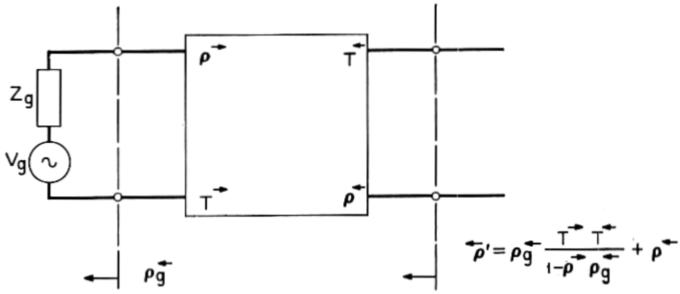

Fig. 2 — Effect of $\rho_g^{\leftarrow}$ on the output reflection of a stage.

$$\rho^{\leftarrow\prime} = \rho_g^{\leftarrow} \frac{T^{\rightarrow}T^{\leftarrow}}{1 - \rho^{\rightarrow}\rho_g^{\leftarrow}} + \rho^{\leftarrow}, \tag{6}$$

where $\rho^{\rightarrow}$, $\rho^{\leftarrow}$, $T^{\rightarrow}$, $T^{\leftarrow}$ designate the scattering coefficients of the stage when its ports are terminated in the respective characteristic impedances. Equation (6) may be obtained by using standard techniques.[7]

III. EFFECT OF INTERACTIONS BETWEEN A MULTIPLIER AND TWO PASSIVE STAGES

Consider a multiplier connected between two stages as shown in Fig. 3(a). Such a problem is encountered in two practical cases. One is the design of the input and output networks of a multiplier, as will be seen in the next section. The other case occurs when the load and the generator have reflections and one is concerned about the possibility that these reflections may cause instability. This is the case considered here and it is illustrated in Fig. 3(b). It should be clear, however, that results obtained for either case can be extended to the other one.

In this section it is supposed that the multiplier is lossless. Furthermore, it is assumed that interaction only occurs in the vicinity of the carriers* so that one can approximate the multiplier properties by means of (2) (which is valid if losses are absent and if $p \cong 0$). It is assumed, in addition, that the generator and the load do not produce AM ⇄ PM conversion. This is the worst case, as is explained in the next section.

Since there is no AM ⇄ PM conversion, one is only concerned about phase instabilities and, therefore, it will be assumed that there is only

* This will be justified by the results obtained in the next section. In fact, the next section shows that, if the multiplier is "properly" designed, then its PM transmissions ($T^{\rightarrow}$, $T^{\leftarrow}$) are maximum for $p \cong 0$ and decrease monotonically with p. Therefore, interaction is in general more likely to occur for $p \cong 0$ (i.e., for those values of p for which (2) holds), in practical applications.

PM. Therefore, consideration can be limited to the upper sidebands only as it is indicated in Fig. 3 where $Z_g(\omega_0 + p)$, $Z_L(N\omega_0 + p)$ are the impedances of the generator and the load at the upper-sidebands, respectively. Let ρ_g, ρ_L* be the reflection coefficients of the generator and the load, respectively. Then

$$\rho_g = \frac{Z_g(\omega_0 + p) - R_1}{Z_g(\omega_0 + p) + R_1} \qquad \rho_L = \frac{Z_L(N\omega_0 + p) - R_N}{Z_L(N\omega_0 + p) + R_N}, \tag{7}$$

where R_1, R_N are the values of Z_G, Z_L for $p = 0$.† Now let ρ' indicate* the reflection presented by the multiplier to the load when the input

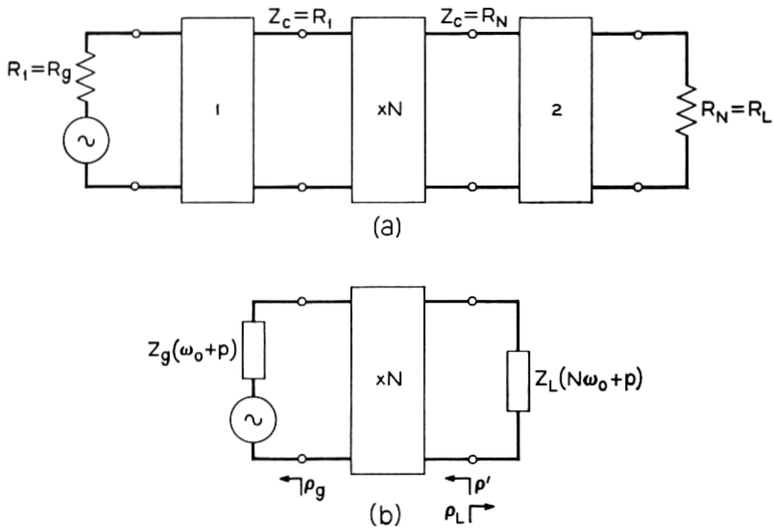

Fig. 3 — Multiplier connected between two stages.

port is connected to the generator. From (6), taking into account the fact that now $\rho^{\rightarrow} = 0$ [see (2)], one has

$$\rho' = \rho^{\leftarrow} + \rho_g T^{\rightarrow} T^{\leftarrow}, \tag{8}$$

where $\rho^{\leftarrow}$, $\rho^{\rightarrow}$, $T^{\rightarrow}$, $T^{\leftarrow}$ designate the multiplier PM scattering coefficients. From (2) one has

$$\rho^{\leftarrow} = (-1)^n \frac{N - (-1)^n}{3}, \qquad T^{\rightarrow}T^{\leftarrow} = (-1)^n N. \tag{9}$$

* The arrows will be omitted, unless necessary.

† R_1 is the input impedance of the multiplier at ω_0 (the multiplier is assumed to be matched: $Z_g(\omega_0) = R_1$).

According to (5), infinite transmission occurs if (and only if)

$$\rho_L \rho' = 1. \tag{10}$$

Equation (10) is obtained from (5) by considering the cascade connection of the generator and the multiplier as stage (*b*), and by considering the load as stage (*c*). By substituting (8) in (10), and by taking into account (9), one obtains

$$\rho_L \rho' = (-1)^n \rho_L \left(N \rho_g - \frac{N - (-1)^n}{3} \right) = 1. \tag{11}$$

According to the stability test given by (4), one has that the circuit is unstable depending on whether or not at some frequencies*

$$\begin{aligned} (\mathrm{Re})(\rho_L \rho' - 1) &\geqq 0 \\ (\mathrm{IM})[\rho_L \rho'] &= 0. \end{aligned} \tag{12}$$

This can be shown by applying (4) as illustrated by the example of the preceding section. In fact, consider a circuit for which (12) occurs at some frequencies. Next, insert an attenuator† between the load and the multiplier. Then, as the attenuator is varied from zero to infinite attenuation, the forward transmission becomes infinite, because of (12), (11). Therefore, (12) guarantees instability. Next, consider a circuit for which (12) never occurs. In this case, as the attenuator is increased, the forward transmission never becomes infinite. Therefore, one concludes that the circuit is stable if and only if (12) never occurs.

3.1 *Discussion of (11)*

Let the effect of ρ_g on ρ' be considered first. Since both N and $[N - (-1)^n]/3$ are always positive, one has from (11) that the magnitude of ρ' is maximum when $\rho_g = -1$. Therefore,

The output reflection (ρ') is maximum when

$$| Z_g | \ll R_1 . \tag{13}$$

Since $| \rho_g | \leqq 1$, the first relation of (12) gives that instability can occur only if

$$| \rho_L | \geqq \frac{3}{4N - (-1)^n} \equiv \rho_{LO} . \tag{14}$$

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* (RE) means "Real part of". (IM) means "Imaginary part of".

† The attenuator is supposed to be ideal, i.e., without phase delay. Note that the circuit is stable when the attenuator provides infinite attenuation. In fact, in this case $| \rho_L | = 0$ and therefore (12) is never satisfied.

Therefore,

In order to guarantee stability it is sufficient to require $|\rho_L| < |\rho_{Lo}|$. Note that, in a practical case, it will suffice to require (14) in the vicinity of the output carrier only.* That is, consideration can be limited to those frequencies which fall within the pass-band of the output circuit. (15)

Now put (11) into the form

$$(-1)^n\left(N\rho_g - \frac{N-(-1)^n}{3}\right) - \frac{\rho_L^*}{|\rho_L|^2} = 0. \tag{16}$$

By considering the imaginary part of (16) one finds that (11) [and therefore (12), also] is satisfied only if either

$$\rho_{LI} = \rho_{gI} = 0 \tag{17}$$

or

$$(-1)^{n+1}\rho_{LI}\rho_{gI} > 0, \tag{18}$$

where ρ_{LI}, ρ_{gI} are the imaginary components of ρ_L, ρ_g. According to (17), (18) one can say

Instability can occur only if at some frequencies p either one of the following two situations occur: (19)

(*i*) both Z_g, Z_L are real.
(*ii*) $X_L X_g(-1)^{n+1} > 0$ $(N = 2^n)$.

Consider now the imaginary part of (11). After some manipulations one obtains

$$\rho_{LR}\left[N\rho_{gR} - \frac{N-(-1)^n}{3}\right] = \frac{-N\rho_{LR}^2}{\rho_{LI}^2}(\rho_{gI}\rho_{LI}). \tag{20}$$

Therefore, if ρ_{gI}, $\rho_{LI} \neq 0$, then (20), (18) give

$$(-1)^n\rho_{LR}\left[N\rho_{gR} - \frac{N-(-1)^n}{3}\right] > 0. \tag{21}$$

Note that, if $\rho_{gI} = \rho_{LI} = 0$, then (21) follows directly from the first relation of (12). According to (21) one can say

Instability can occur only if, at some frequencies, (21) is satisfied. (22)

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* See footnote on p. 806.

It is recalled that $\rho_{LR} > 0$ or $\rho_{LR} < 0$ depending on whether or not $|Z_L| > R_N$ (and similarly for ρ_{gR}, $|Z_g|$, R_1). The importance of (19), (22) follows from the fact that they allow the question of stability to be answered in many cases simply by looking at the signs of X_g, X_L and at the magnitudes of $|Z_g|$, $|Z_L|$ (more precisely, of $|Z_g|/R_1$, $|Z_L|/R_N$). This will be illustrated by the examples of the next section. Note that the presence of $(-1)^n$ in (11), (18), (21), follows from the fact that there is a reversal in the sign of both $\rho^{\leftarrow}$, $T^{\rightarrow}T^{\leftarrow}$ each time N is increased by a factor of two. This is explained by Theorem 1 of the next section which shows that a multiplier of order 2^n is equivalent to a chain of doublers. Since the round-trip transmission of a doubler is negative, each doubler gives a contribution of 180° to the phase of the overall round-trip transmission (from which $(-1)^n$ follows).

Consider now some special cases. If $N = 2, 4$, then (11) becomes

$$\rho_L \rho' = (-1)^n \rho_L (N\rho_g - 1) = 1.$$

Therefore,

if $N = 2$ or 4, instabilities can occur only if both the generator and the load are interacting (i.e., both ρ_L, $\rho_g \neq 0$). (23)

If $N > 4$, on the other hand, then $|\rho'| > 1$ even if $|\rho_g| = 0$. Therefore,

if $N > 4$, then instability can arise even if the generator does not interact (i.e., $\rho_g = 0$). (24)

This discussion of interactions will be concluded by emphasizing that

if $\rho_L = 0$ there is no interaction at all. Therefore, one may say that the load reflection ρ_L is the primary cause of instabilities. (25)

IV. ANALYSIS OF THE FREQUENCY DEPENDENCE OF THE PROPERTIES OF A MULTIPLIER OF THE TYPE (26) AND DISCUSSION OF THE STABILITY CONDITIONS

Consider a multiplier which has the following characteristics:

The order of multiplication is $N = 2^n$. It has the minimum number $(n - 1)$ of idlers. It is tuned. (26)

The analysis of the frequency dependence of the properties of such a multiplier is based on the following theorem which is demonstrated in the Appendix.

*Theorem 1:** *A multiplier of the type* (26) *is equivalent to the chain of stages shown in Fig. 4. The doublers of Fig. 4 are "ideal". More precisely, they are frequency independent and lossless, and they do not produce AM ⇄ PM conversion. Their scattering matrices are given by* (2) *(with* $N = 2$*). That is,*

$$\hat{S} = \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} \qquad \text{(If only PM is present)}$$
$$\hat{S} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{vmatrix} \qquad \text{(If only AM is present).} \tag{27}$$

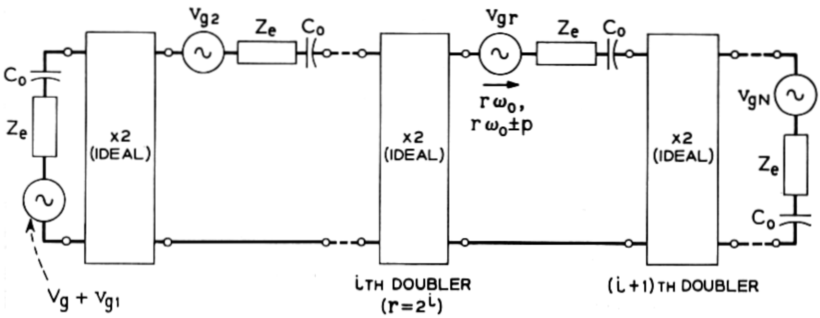

Fig. 4 — Chain of doublers equivalent to the multiplier of order $N = 2^n$ shown in Fig. 5.

Note that this theorem applies to the general case of a multiplier which is lossy and produces AM ⇄ PM conversion. Fig. 5 shows the equivalent circuit of the actual multiplier from which the chain is derived. Z_e is the impedance presented by the external circuit to the varactor. R_s is the series resistance of the varactor. C_0 is the varactor average capacitance. The generator V_G consists of a carrier voltage V_g of frequency ω_0, and of noise terms $v_{g1}, v_{g2}, \cdots, v_{gN}$ which correspond to the various sidebands $\omega_0 \pm p$, $2\omega_0 \pm p$, etc.

The impedance Z_e resonates with C_0 at the "desired" carriers ω_0, $2\omega_0, \cdots, N\omega_0$. Furthermore, Z is so large for ω far from $\omega_0, 2\omega_0, \cdots, N\omega_0$ that current flows through the varactor only at the desired carriers and their sidebands.

Let now the chain of Fig. 4 be examined. Consider the impedance

* In deriving (27) the approximation $\omega_0 \pm p \cong \omega_0$ has been made. For more details, see the considerations made in the Appendix on the frequency dependence of the properties of the "ideal" doublers of Fig. 4.

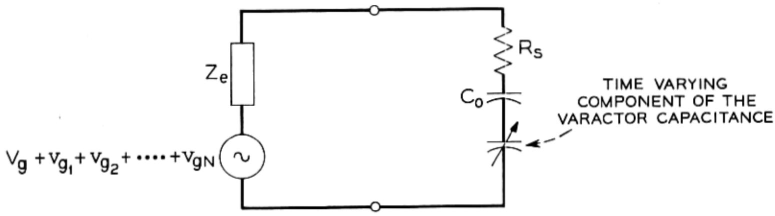

Fig. 5 — Multiplier equivalent circuit.

connecting the output of the ith doubler to the input of the $(i + 1)$th doubler $(i < n)$. Current is flowing only at the frequencies $r\omega_0$, $r\omega_0 \pm p$ $(r = 2^i)$, through this impedance. Furthermore, at $\omega \cong r\omega_0$ (i.e., $\omega = r\omega_0$, $r\omega_0 \pm p$) Z_e gives the impedance presented by the $r\omega_0$-idler circuit to the varactor. Therefore, the impedance connected to the output of the ith doubler corresponds to the ith idler and can be represented by means of a series resonant circuit (resonant at $r\omega_0$) as illustrated in Fig. 6. The resistance R_r shown in Fig. 6 represents the losses of the idler circuit and it is equal to R_s, the series resistance of the varactor, if the varactor is the only lossy element. The chain of Fig. 4 can be represented by the diagrammatic circuit of Fig. 7 in which, according to the preceding considerations, S_{i+1} represents the ith idler circuit and can be approximated as illustrated in Fig. 6. The impedance $Z_{\text{in}}^{(i+1)}$ is the characteristic impedance of the input port of the $(i + 1)$th doubler. It is equal to the impedance presented by the input port of the $(i + 1)$th doubler at the carrier frequency $r\omega_0$. $Z_{\text{out}}^{(i)}$ is the characteristic impedance of the output port of the ith doubler and it is related to $Z_{\text{in}}^{(i)}$ through the formula $Z_{\text{out}}^{(i+1)} = Z_{\text{in}}^{(i)} + R_r$ (with $r = 2^i$). $Z_{\text{in}}^{(i)}$ is calculated in the Appendix.

At $\omega \cong N\omega_0$, the impedance Z_e is the impedance presented to the

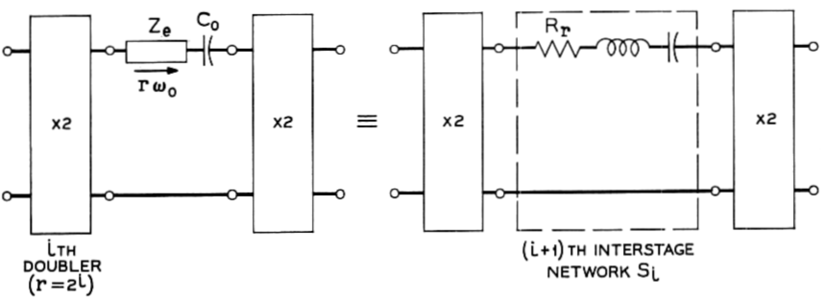

Fig. 6 — $(i + 1)$th interstage network S_{i+1}.

varactor by the output circuit of the multiplier. Accordingly, the impedance connected to the output of the last doubler of Fig. 4 can be represented as shown in Fig. 7, where S_{n+1} corresponds to the output circuit of the multiplier. The input circuit of the multiplier is represented in Fig. 7 by means of a network S_1 connected between the generator and the first doubler. Note that S_1, S_{n+1} include R_s and C_0. Furthermore, they are tuned at the carrier frequencies and therefore they provide unity transmission at the carriers if losses are absent.

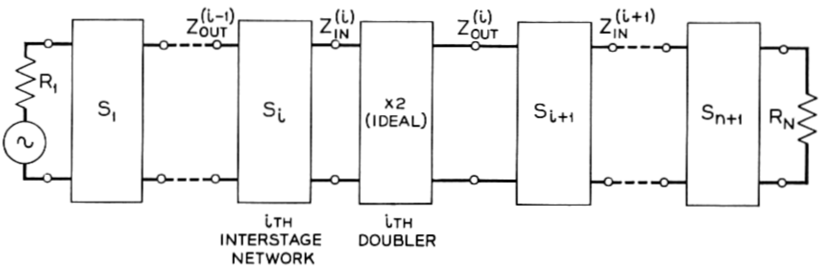

Fig. 7 — Schematic representation of the chain of doublers equivalent to the multiplier of order $N = 2^n$.

4.1 *Analysis of the Worst Case (No Losses, No AM ⇄ PM)*

Assume that the multiplier does not produce AM ⇄ PM conversion, and that it is lossless. (That is, S_1, S_2, $\cdots$, S_{n+1} are lossless and do not produce AM ⇄ PM conversion.)

Let the doubler be examined in detail. It is the most important multiplier because it represents the elementary constituent of any multiplier or chain of multipliers.

4.2 *Discussion of the Doubler*

According to the preceding discussion, the equivalent circuit of a doubler consists of three stages: an input stage S_1, an "ideal" doubler, and an output stage S_2. Therefore, the general considerations of the preceding section on the interactions between a multiplier and two other stages are applicable to the analysis of a doubler. Suppose that the scattering coefficients of the input and output stages are labeled 1 and 2, respectively. Then the overall forward PM transmission is given by*

---

* (28) can be obtained by means of standard techniques. See for instance Ref. 7.

$$T_i^{\rightarrow} = \frac{2T_1^{\rightarrow}T_2^{\rightarrow}}{1 - \rho_2^{\rightarrow} + 2\rho_1^{\leftarrow}\rho_2^{\rightarrow}}. \tag{28}$$

Since in this section it is assumed that S_i does not produce AM $\rightleftarrows$ PM conversion and is lossless, its PM scattering parameters are simply given by its reflection and transmission coefficients at the frequency $\omega + p$ ($\omega = \omega_0$ or $2\omega_0$ depending on whether $i = 1$ or $i = 2$). The characteristic impedances of the two ports of S_i are equal because S_i is lossless; they will be denoted $Z_c^{(i)}$. If, for instance, S_i is either one of the two simple circuits of Fig. 8, then

$$\hat{S}_i = \begin{vmatrix} \rho_i & T_i \\ T_i & \rho_i \end{vmatrix}$$

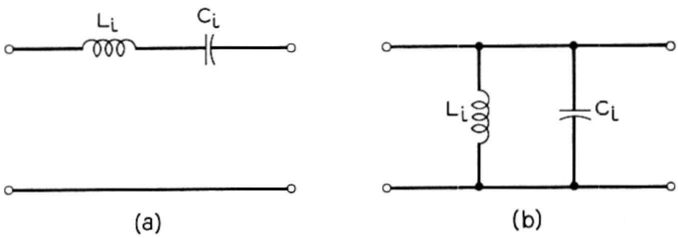

Fig. 8 — Single-tuned resonant circuits.

with†

$$\begin{aligned} T_i &= T_i^{\rightarrow} = T_i^{\leftarrow} = \frac{B_i}{B_i + jp} \\ \rho_i &= \rho_i^{\rightarrow} = \rho_i^{\leftarrow} = \pm(1 - T_i) \\ B_i &= \frac{Z_c^{(i)}}{L_i}, \end{aligned} \tag{29}$$

where the negative sign applies to the case of Fig. 8(b). Note that B_i is the 3-dB bandwidth of T_i. Note, furthermore, that the only difference between the two circuits is that they have opposite reflections. If $p \neq 0$, ∞, the signs of the real and imaginary components of the reflections are

$$\begin{aligned} &\text{circuit (a)} \qquad \rho_{iR} > 0, \qquad \rho_{iI} > 0 \\ &\text{circuit (b)} \qquad \rho_{iR} < 0, \qquad \rho_{iI} < 0, \end{aligned} \tag{30}$$

† The approximation $\omega \cong \omega + p$ has been made in deriving (29). That is, the resonant circuit has been supposed to be very narrow band.

where ρ_{iR} is the real component of ρ_i, and ρ_{iI} is the imaginary component.

Suppose now that both S_1, S_2 can be approximated by either one of the two circuits (a), (b) of Fig. 8. From (18), (21) one has that instability may occur only if

$$\rho_{1I}\rho_{2I} > 0, \qquad \rho_{2R}(1 - 2\rho_{1R}) > 0. \tag{31}$$

From the first of (31) and from (30) it follows that the circuit is unstable only if $\rho_{1R}\rho_{2R} > 0$. Therefore, by combining this result with the second inequality of (31), one has that instability requires

$$\rho_{2R} > 0, \qquad \rho_{1R} > 0. \tag{32}$$

Therefore, the only case in which instability may occur, is that in which both S_1, S_2 are of the type (a). Let now this case be examined in detail (see Fig. 9).

4.3 *Discussion of the Case in Which Both S_1 and S_2 can be Approximated by Means of a Single-Tuned Series Resonant Circuit*

This case is most important for two main reasons. A first reason is that the equivalent circuit of a varactor includes the series connection of C_0 and the inductance of the diode mount. Therefore, in most practical cases it will be possible to represent both the input and output circuit of a multiplier by means of a series resonant circuit, by first approximation. The second reason is that, in the case of an idler circuit, it has already been pointed out that the equivalent circuit is a series resonant circuit, to a first approximation.

Let the frequency p be normalized with respect to the output bandwidth (that is, $B_2 = 2$) and let $\gamma = B_1/B_2$. From (28) and (29) one has

$$T_t^{\rightarrow} = \frac{2\gamma}{(\gamma - 2p^2) + jp}. \tag{33}$$

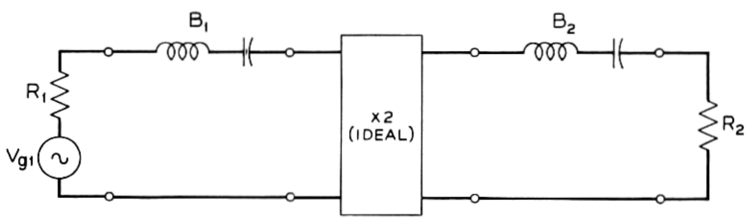

Fig. 9 — Case in which both the input and output circuit of the doubler can be approximated by means of single-tuned series resonant circuits.

By analyzing (33) one finds that the behavior of $T_t^{\rightarrow}$ is different depending on whether $\gamma < \frac{1}{4}$ or $\gamma > \frac{1}{4}$. More precisely,

(*i*) if $\gamma < \frac{1}{4}$, then maximum transmission ($T_M^{\rightarrow}$) occurs at $p = 0$; therefore, $T_M^{\rightarrow} = 2$; and

(*ii*) if $\gamma > \frac{1}{4}$, then maximum transmission occurs at $p = \sqrt{(4\gamma - 1)/8}$ and it is given by

$$T_M^{\rightarrow} = \frac{2\gamma}{1/4 + j\sqrt{(4\gamma - 1)/8}}.$$

All this is illustrated in Fig. 10, where $20 \log_{10} |T_t^{\rightarrow}|$ is plotted as a function of p for different values of $\gamma (= B_1/B_2)$. One can see from Fig. 10 that if the output bandwidth is enough greater than the input bandwidth (i.e., $\gamma < \frac{1}{4}$), then the transmission curve decreases monotonically with p. On the other hand, if $\gamma > \frac{1}{4}$, then

a peak appears in the transmission curve, and the maximum transmission increases indefinitely with γ. (34)

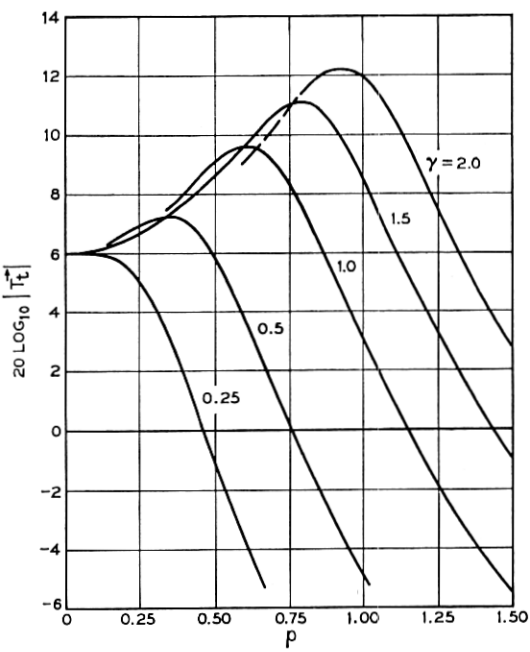

Fig. 10 — Effect of γ ($=B_1/B_2$) on the forward PM transmission of the doubler of Fig. 9.

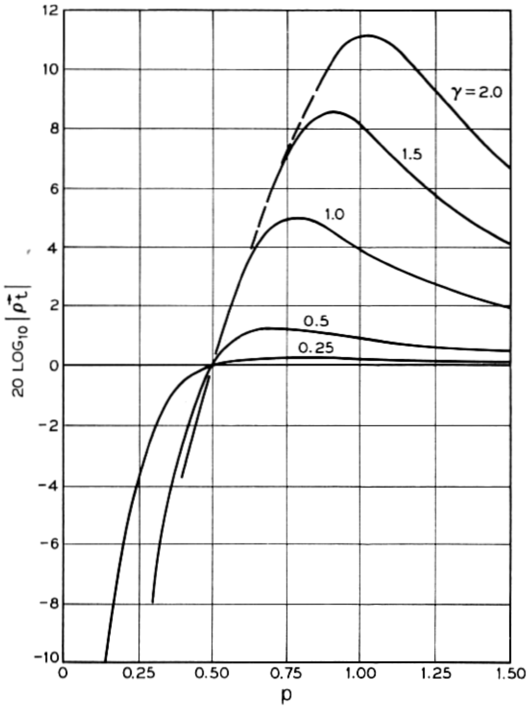

Fig. 11 — Effect of γ (=B_1/B_2) on the input PM reflection of the doubler of Fig. 9.

This is in accordance with (2). One concludes from (34) that small values of B_1/B_2 are desirable if large values of $| T_t^{\rightarrow} |$ are to be avoided. Small values of B_1/B_2 are desirable also for the following stability considerations. If B_1/B_2 is large, then the input PM reflection $\rho_t^{\rightarrow}$ becomes larger than unity, as shown in Fig. 11. In addition, large values of B_1/B_2 enhance the reverse overall transmission $T_t^{\leftarrow}$, as shown in Fig. 12. Consequently, one may conclude

> Large values of $\gamma(B_1/B_2)$ deteriorate the stability of the doubler and enhance the PM forward transmission (i.e., the output noise) of the multiplier in the vicinity of the carriers, if both S_1, S_2 are of the type shown in Fig. 8(a). (35)

It is important to note that the circuit becomes unstable only in the limiting case $\gamma = \infty$. Remember that it has been pointed out in the preliminary considerations of the first section that all scattering coeffi-

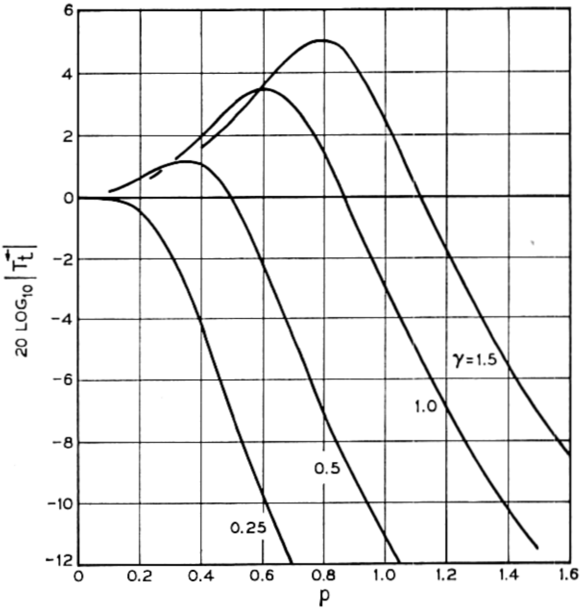

Fig. 12 — Effect of γ $(=B_1/B_2)$ on the reverse PM transmission of the doubler of Fig. 9.

cients become infinite when the multiplier is on the point of becoming unstable.

4.3.1 *Remaining cases*

In the remaining cases, [in which both S_1, S_2 are of the type (a), (b) and at least one of S_1, S_2 is of the type (b)], the overall transmission $T_t^{\rightarrow}$ decreases monotonically with p for all values of B_1/B_2. This is in accordance with (32) and it can be directly verified by using (28), (29).

4.4 *Discussion of the General Case*

In most practical cases the output and input circuits of a multiplier are far more complicated than the simple cases considered in the preceding discussion. For instance, the impedance of the external circuit $Z_e + 1/j\omega C_0$ may have spurious resonances at the sidebands, etc. It is therefore important to consider the case in which the input and output circuits of a multiplier have more complicated configurations. Let first the input circuit be considered in detail.

4.5 *Discussion of the Case in which only the Output Circuit can be Approximated by Means of a Single-Tuned Series Resonant Circuit*

According to the terminology of the first section, let Z_g be the impedance presented by the input circuit (i.e., by the cascade connection of S_1 and R_1) to the "ideal" doubler of Fig. 13. Similarly, let Z_L be the impedance presented by the output circuit to the output port of the "ideal" doubler.

Consider now the two cases illustrated in Fig. 14. The parameters B_{1a} , B_{1b} , B_2 represent the bandwidths of the various resonant elements (when they are considered individually). In order to investigate the

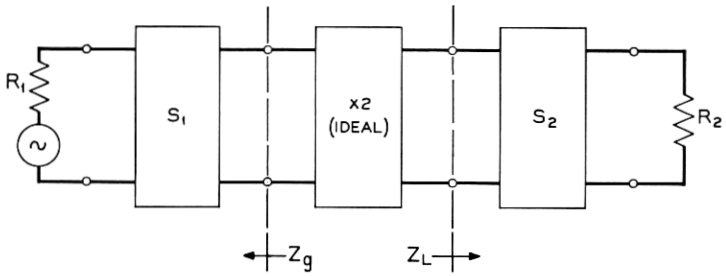

Fig. 13 — Impedances Z_g and Z_L presented by the input and output circuits to the "ideal" doubler.

possibility of instabilities, consider the reflections ρ_L , ρ_g , defined in the previous section ($\rho_g = \overleftarrow{\rho_1}$, $\rho_L = \overrightarrow{\rho_2}$). Then

$$\rho_L = \frac{jp}{B_2 + jp}. \tag{36}$$

ρ_g is different in the two cases (a), (b). In case (a), ρ_g is given by

$$\rho_g = \frac{-p^2(B_{1a} + B_{1b}) - jp(B_{1a}^2 - B_{1b} + 2p^2)}{[B_{1a}^2 B_{1b} - p^2(3B_{1a} + B_{1b}) + jp(2B_{1a} + B_{1a}^2 - 2p^2)]}. \tag{37}$$

Case (b) can be obtained simply by changing the sign of (37) and by interchanging B_{1a} , B_{1b} .

Consider now the infinite gain condition*

$$1 - \rho_L + 2\rho_L\rho_g = 0 \tag{38}$$

which is obtained by letting the denominator of (28) equal zero. Those values of B_{1a} , B_{1b} , B_2 for which (38) is satisfied at one or more fre-

* Note that (38) is equal to (11) ($n = 1$, $N = 2$).

quencies furnish the "boundary" of the region of stability (see the general considerations on stability of the first section).

Consider now the circuit of Fig. 14(*a*). By substituting (36), (37), in (38) one finds that the circuit becomes unstable under certain conditions. These conditions are illustrated in Fig. 15. One can see from Fig. 15 that, if both B_{1b}/B_{1a}, B_2/B_{1a} are too small, then the circuit is unstable. If $B_{1b} > B_{1a}$, then the input circuit does not have spurious resonances,

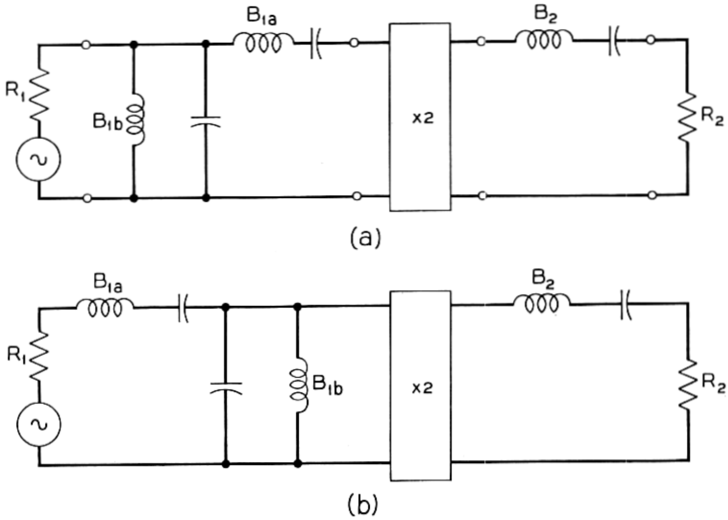

Fig. 14 — Two typical examples in which only the output circuit can be approximated by means of a single-tuned series resonant circuit.

as one may verify from (37).* But Fig. 15 shows that the circuit may still be unstable. Therefore, one can say

instability may occur even if the circuit does not have spurious resonances. (That is, even if the circuit impedance is real only at the desired carriers ω_0, $2\omega_0$.) (39)

Fig. 15 shows that if $B_{1b} \gg B_{1a}$, then stability is secured. This can be explained by noticing that for $B_{1b} \gg B_{1a}$ the circuit reduces to that of Fig. 9, which has already been found to be stable.

Fig. 15 shows also that stability is secured if $B_2 > 1.2B_{1a}$ (approximately). This can be explained as follows. Consider first the following theorem.

* If $B_{1b} > B_{1a}$, then $\rho_{gI} \neq 0$ for $0 < p < \infty$.

Theorem 2: If at all frequencies p either one of the following two conditions

$$|\rho_L| < |\rho_{LO}| \tag{40}$$

$$|Z_g| \gg R_1 \tag{41}$$

is satisfied, then the doubler is stable. $|\rho_{LO}|$ *is* $\frac{1}{3}$ *if losses are absent, and it is greater than* $\frac{1}{3}$ *if losses are present.*

Proof: The preceding section has shown* that the first condition guarantees $T_t^{\rightarrow} \neq \infty$. Consider therefore, the second condition. Note

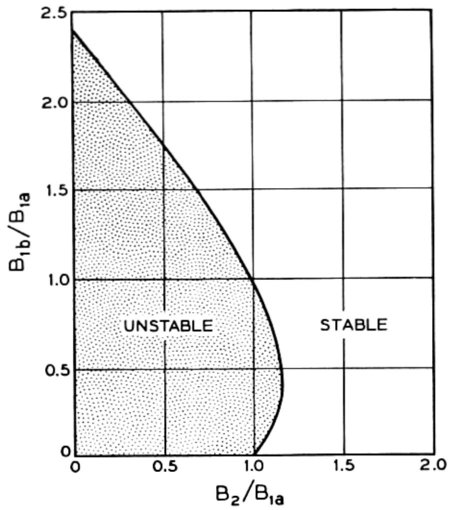

Fig. 15 — Stability region of the doubler of Fig. 14(a).

that $|Z_g| \gg R_1$ is equivalent to $\rho_g \cong 1$. Therefore, by substituting $\rho_g = 1$ in (38) and neglecting the exceptional case $\rho_L = -1$, one finds that (41) also guarantees $T_t^{\rightarrow} \neq \infty$. One concludes therefore, that either one of (40), (41) guarantees $T_t^{\rightarrow} \neq \infty$. Consider now a circuit which satisfies (40), (41). Next, decrease the amplitude of $|\rho_L|$. Since, as $|\rho_L|$ is decreased, the forward transmission never becomes infinite one concludes that the circuit is stable, according to (4).

Consider now the circuit of Fig. 14(a) and suppose $B_{1a} \ll B_2$. Note that condition (40) is satisfied for

$$p < \frac{1}{2\sqrt{2}} B_2 , \tag{42}$$

* See (14). Note that $|\rho_L| < \frac{1}{3}$ corresponds to the worst case (no losses; $\rho_g = 1$).

as one can verify from (36). Furthermore, since $B_{1a} \ll B_2$, one has that condition (41) is satisfied at the frequencies p

$$p > \frac{1}{2\sqrt{2}} B_2 \gg B_{1a} . \tag{43}$$

Therefore, the circuit is stable, as shown by Fig. 15.

The preceding demonstration used the fact that the condition of stability places restrictions on only those values of ρ_g (i.e., of Z_g) which occur at those frequencies p for which $|\rho_L| < |\rho_{LO}|$. This property is most important and will be further emphasized by stating

> the condition of stability places restrictions only at those frequencies p which are rejected by the output filter. Therefore, at those frequencies which are passed by the output filter, the impedance Z_g produced by the output filter can be quite arbitrary (all this applies to the case of a doubler). (44)

In this statement, the frequencies which are "rejected" by the output filter are given by the complement of (42)

$$p > \frac{1}{2\sqrt{2}} B_2 \tag{45}$$

if losses are absent. The frequencies which "pass through" the output filter are given by (42)

$$p < \frac{1}{2\sqrt{2}} B_2 . \tag{46}$$

Consider now the circuit of Fig. 14(*b*). Consider first the following theorem.

Theorem 3: *If $R_g > R_1$, then infinite transmission never occurs.*

Proof: Let $Z_g = R_g + jX_g$, and assume

$$R_g > R_1 . \tag{47}$$

Then, after some manipulations one finds

$$|1 - 2\rho_g| = \left| \frac{(R_g + R_1) - 2(R_g - R_1) - jX_g}{(R_g + R_1) + jX_g} \right| \leqq 1.$$

Therefore, (38) is never satisfied for all values of ρ_L. This demonstrates Theorem 3.

Consider now the circuit of Fig. 14(b). Since $X_L > 0$, from (19) one has that infinite gain occurs only if

$$X_g > 0. \tag{48}$$

But one can directly verify that the circuit of Fig. 14(b) has the following properties

$$R_g > R_1 \text{ when } X_g > 0.$$

Therefore, infinite transmission never occurs because of Theorem 3. This demonstrates that the circuit is always stable. Theorem 3 has a very important consequence.

> Spurious resonances which occur at the input sidebands do not produce instability, provided they produce high impedances ($R_g > R_1$) at the varactor terminals (this is valid for a doubler). (49)

Note that (49) is valid for any arbitrary output circuit.

4.6 *Case in which the Input Circuit can be Approximated by Means of a Single-Tuned Series Resonant Circuit*

Suppose now that the input network consists of a series resonant circuit and that the output circuit is quite arbitrary (see Fig. 16). Then one has

$$\rho' = 1 - 2\rho_g = \frac{B_{1a} - jp}{B_{1a} + jp}.$$

Therefore, $|\rho'| = 1$. Consequently instability may arise in the limiting case $|\rho_L| = 1$. If one neglects this limiting case, one can say

> If the input circuit of the doubler can be approximated by a single-tuned resonant circuit in series with the generator, then stability is secured for any arbitrary output circuit configuration (in the case of a doubler). (50)

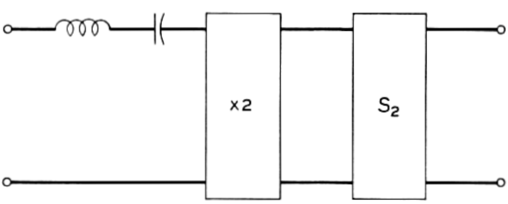

Fig. 16 — Case in which only the input circuit can be approximated by means of a single-tuned series resonant circuit.

Note that (50) is in agreement with (13), (41) which show that it is in general desirable that $|Z_g| > R_1$.

4.7 *Discussion of a Frequency Multiplier of Order $N > 2$ (both a Single Stage and a Chain of Stages)*

The following discussion is concerned with the problem of designing a stable chain of doublers. This problem arises both in the design of a single stage multiplier of order $2^n > 2$, and in the design of a multiplier of order 2^n consisting of many stages. That the chain may be unstable even if each doubler is individually stable, is a widely-known experimental fact. It will be shown here that if $N > 4$, then unstability may arise even if: each doubler is individually stable and it is, in addition, of the type shown in Fig. 9 (which is the simplest case which may occur).

Consider an octupler and suppose that both its input and output circuits consist of single tuned resonant circuits connected in series to R_1 and R_L. Then, according to Fig. 6, its equivalent circuit is that shown in Fig. 17 and the bandwidths B_1, B_2, B_3, B_4 are the parameters on which the frequency dependence of $T_t^{\rightarrow}$ (and therefore, also the stability of the circuit) depends. To simplify the analysis assume the following conditions:

$$\frac{B_3}{B_4} = \frac{B_2}{B_3} = \frac{B_1}{B_2} = \gamma, \tag{51}$$

and let the frequency p be normalized with respect to $B_4/2$ (i.e., $B_4 = 2$). In this way $T_t^{\rightarrow}$ depends on the parameter γ, only. $T_t^{\rightarrow}$ has been calculated for different values of γ and the results are shown in Fig. 18. One can see from Fig. 18 that the multiplier is unstable if

$$\gamma = 1.425 \text{ (approximately)}. \tag{52}$$

This result is in agreement with (35), which pointed out that small values of γ are desirable. The conclusion from (52) is that

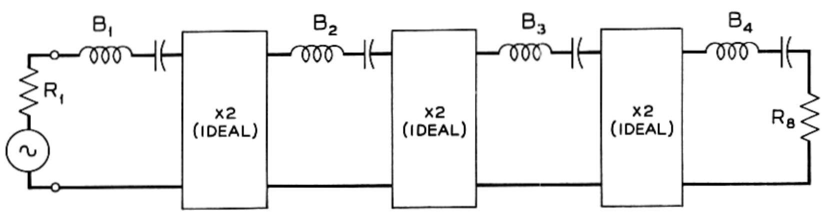

Fig. 17 — Equivalent circuit of an octupler in which both the input and output circuits consists of single-tuned series resonant circuits.

if three or more doublers of the type shown in Fig. 9 are connected in cascade, the chain will be unstable unless each stage is broadband enough with respect to the preceding one. (53)

Consider now a doubler which is individually stable, and suppose that the input circuit is made more narrowband than the output circuit. Then, if a stage is connected to the input port of the doubler, instability will not occur. In fact, according to (44), instability may arise only from interactions occurring at the frequencies which are rejected by

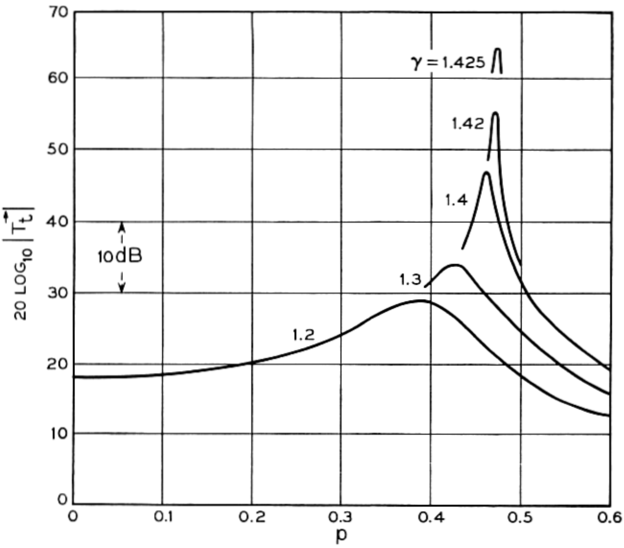

Fig. 18 — Effect of γ $(=B_i/B_{i+1})$ on the forward PM transmission of the octupler of Fig. 17.

the output circuit. But these frequencies are also rejected by the input circuit. Therefore, the input filter prevents these "dangerous" interactions. Note that if a broadband* stage is connected to the output of the doubler, then instability does not occur because the output circuit of the doubler will reject those frequencies at which the broadband stage presents reflections. The conclusion can be stated as follows.

Consider a chain consisting of doublers which are individually stable and which have $B_1 \ll B_2$ (where B_1, B_2 are the input and output bandwidths of the general doubler, respectively). The chain will be stable, provided the input circuit of each (54)

* Broadband with respect to the output circuit of the doubler.

doubler is broadband with respect to the output circuit of the preceding doubler.

This is also in accordance with (52) and with the conclusions derived from it.

4.8 *Discussion of the Effect of AM ⇄ PM conversion*

Consider a doubler and suppose, for instance that the output circuit S_2 produces AM ⇄ PM conversion. Then, if a unit PM wave is incident to the input port of S_2, two waves will be reflected back,

$$\rho_{(\mathrm{PM},\mathrm{PM})}\ ,\ \rho_{(\mathrm{PM},\mathrm{AM})}$$

where the subscript (,) indicates the type of incident wave and the type of reflected wave (in this order). Conservation of energy requires

$$|\ \rho_{(\mathrm{PM},\mathrm{PM})}\ | \leqq 1 - |\ \rho_{L(\mathrm{PM},\mathrm{AM})}\ |$$

which shows that the amplitude of the reflected PM wave will in general be decreased by the presence of PM → AM conversion. *This is a desirable effect.* Note that the AM wave generated by S_2 may be reflected back and converted again to PM by S_2. However, for this to happen, the AM wave must travel through the "ideal" multiplier in the reverse direction, be reflected back by S_1, and travel through the multiplier in the forward direction. Through this path the AM wave is attenuated because the AM round-trip (power) transmission of the "ideal" doubler is $\frac{1}{4}$, (see (1)). The conclusion is that, when the AM wave is converted back to PM by S_2, it has small effect in most practical cases. Similar arguments can be applyed to the case in which S_1 has AM ⇄ PM conversion. The conclusion is

input and output circuits which produce AM ⇄ PM conversion are in general preferable to those which do not produce AM ⇄ PM conversion. (55)

Because of the qualitative nature of the discussion leading to (55) a more convincing argumentation may be desirable. A rigorous demonstration requires considerably more space and is beyond the scope of this paper.

V. CONCLUSIONS ON THE DESIGN OF A STABLE MULTIPLIER

Consider first a doubler. From (35), (44), and (54), one has

the output bandwidth should be broadband with respect to the input bandwidth. That is, (56)

$$B_2 \gg B_1\ .$$

The input circuit can have any arbitrary frequency dependence at those frequencies p which correspond to the passband of the output filter [see (44)]. In fact, this will not cause any instability.

Instability may arise *only if* both the input circuit and the output circuit satisfy simultaneously certain conditions. For instance it must be

$$X_g X_L \geqq 0, \qquad R_g < R_1 , \qquad | \rho_L | > \tfrac{1}{3} \tag{57}$$

for some frequencies in order that instability may occur. From (57) one can see that spurious resonances of the input circuit do not produce instability if they cause

$$R_g > R_1 .$$

Furthermore, in the design of the input and output circuits, those circuit configurations should be preferred which produce AM $\rightleftarrows$ PM conversion. Therefore, low-pass circuit configurations at the input, and high-pass circuit configurations at the output, are preferable to bandpass circuit configurations.

Finally, under certain circuit conditions, instability may arise even in the absence of spurious resonances. However, instability can be avoided by making the output circuit broadband enough with respect to the input circuit. This last part of the statement follows from (44).

The need for condition (56) is further emphasized by considering the stability conditions for a chain of doublers (or for a multiplier of order $N = 2^n$, which has been shown to be equivalent to a chain of doublers).

A chain of doublers will be stable if the following conditions are satisfied: each doubler is individually stable and satisfies (56); furthermore, each doubler is broadband enough with respect to the preceding one [see (54)]. (58)

If (58) is not satisfied, (53) shows that instability may arise even in the simplest case in which the input and output circuits of each doubler consist of (single-tuned) series resonant circuits.

Note that (56), (57) etc. apply also to the design of a multiplier of order $N = 2^n$. In this case, one should design the input circuit by considering the doubler which consists of the cascade connection of: the input circuit, an "ideal" doubler, the first idler (connected in series to the output of the "ideal" doubler). The output circuit should satisfy the stability conditions of the doubler which consists of the last idler connected in series to the input of the "ideal doubler", and of the output circuit connected to the output of the "ideal" doubler. Note that requirement (56) gives

$$B_1 \ll B_2 \ll \cdots \ll B_n \ll B_{n+1} , \tag{59}$$

where B_1 is the input circuit bandwidth, B_2 is the first idler bandwidth [see (29) and the Appendix], B_n is the last idler bandwidth, and B_{n+1} is the output circuit bandwidth.

It is important to emphasize that all these requirements have been derived in the case of no losses. The presence of losses reduces the limitations [such as (56) and (59)] placed by the condition of stability. If the losses are so large that the multiplier efficiency is less than $1/N$ [see (24) of Ref. 3] then the multiplier will be in general stable.

Note that it has always been implied that the impedance presented to the varactor terminals by the external circuit* is "large" at frequencies very far from the carriers. If this is not true, other types of instabilities may arise, such as subharmonic generation, etc.

APPENDIX

Equivalence of a Multiplier of Order $N = 2^n$ to a Chain of Doublers

Consider a multiplier of the type defined by (26). Its equivalent circuit is shown in Fig. 19. V_g is the input voltage generator of frequency

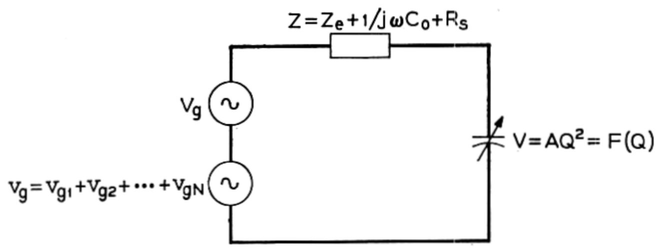

Fig. 19 — Multiplier equivalent circuit.

ω_0 . v_{g1} , v_{g2} , $\cdots$, v_{gN} are the noise terms present at the various sidebands $\omega_0 \pm p$, $2\omega_0 \pm p$, $\cdots$, $N\omega_0 \pm p$. The series resistance R_s and the average capacitance C_0 of the varactor are included in the impedance $Z(\omega)$. Therefore, the nonlinear capacitance shown in Fig. 19 represents the nonlinear part of the capacitance of the varactor and it has a Q–V

* The external circuit includes C_0 , the average capacitance of the varactor, the inductance of the varactor, etc.

characteristic of the type: $V = F(Q) = AQ^2$ (see the Appendix of Ref. 3 for more details).

The hypothesis are

$$Z(r\omega_0) = R_r \quad (r = 2^s = 1, 2, \cdots, N) \tag{60a}$$

$$Z(\omega) = \infty \quad \text{for } \omega \text{ far from } r\omega_0 \quad (r = 2^s = 1, 2, \cdots, N). \tag{60b}$$

In (60a) it is assumed that the circuit is resonant at the idler frequencies and at the input and output frequency. R_1 is the impedance of the input generator; $R_2, \cdots, R_{N/2}$ represent the losses of the idlers; R_N is the impedance of the load at $N\omega_0$. In (60b) it is required that current flow be limited to the frequencies $\omega_0, 2\omega_0, \cdots, N\omega_0$ and their sidebands. In this appendix both positive and negative frequencies will be considered, as illustrated by the Fourier Series (61), (62), etc. In (60) consideration is confined to $\omega > 0$ because the case $\omega < 0$ is given by $Z(\omega) = Z(-\omega)^*$, where $(\ )^*$ indicates the complex conjugate.

When the sidebands are absent (i.e., when $v_{g1} = v_{g2} = \cdots = v_{gN} = 0$) the charge of the nonlinear capacitance is of the type

$$Q(t) = \sum_{\substack{r=\pm 2^s \\ s=0,\cdots,n}} Q_r \exp(jr\omega_0 t). \tag{61}$$

Notice that $Q_r = Q_{-r}^*$ because $Q(t)$ is real. The elastance $S(t)$ is given by

$$S(t) = 2AQ(t) = \sum_{\substack{r=\pm 2^s \\ s=0,\cdots,n}} S_r \exp(jr\omega_0 t) \tag{62}$$

with

$$S_r = 2AQ_r . \tag{63}$$

Consider now the voltage $V(t)$ across the nonlinear capacitance. From $V = AQ^2$ and (61) one has

$$V(t) = A \sum_{i \neq r} Q_i Q_r \exp[j(r+i)\omega_0 t] = \sum V_l \exp[jl\omega_0 t]$$

$$(i = \pm 2^h; \quad r = \pm 2^s; \quad s, h = 0, \cdots, n)$$

from which one obtains the Fourier coefficient of $V(t)$ relative to $\omega = r\omega_0$

$$V_r = A[Q_{r/2}Q_{r/2} + 2Q_{2r}Q_{-r}] \qquad (r = \pm 2^s; s = 0, \cdots, n). \tag{64}$$

The linear circuit connected to the nonlinear capacitance gives an additional expression for V_r

$$V_r = -jr\omega_0 Q_r Z(r\omega_0) + V_{gr} \qquad (r = \pm 2^s; s = 0, \cdots, n), \tag{65}$$

where V_{gr} is the Fourier coefficient of V_g relative to $\omega = r\omega_0$. Therefore, $V_{gr} = 0$ for $|r| \neq 1$. From (65), (64) one obtains

$$V_{gr} - jr\omega_0 Q_r Z(r\omega_0) = A(Q_{r/2}Q_{r/2} + 2Q_{2r}Q_{-r}) \qquad (r = \pm 2^s;\, s = 0, \cdots, n). \tag{66}$$

Equation (66) gives the "equilibrium equations" of the circuit of Fig. 19 at the carriers $\pm\omega_0$, $\pm 2\omega_0$, $\cdots$, $\pm N\omega_0$.

Consider now the sidebands. Let

$$q(t) = \sum_{\substack{s=0,\cdots,n \\ r=\pm 2^s \\ i=\pm 1}} q_{r,i} \exp\,[j(r\omega_0 + ip)t] \tag{67}$$

be the sidebands of the charge of the nonlinear capacitance. Then the voltage sidebands are given by

$$v(t) = S(t)q(t). \tag{68}$$

By substituting (67), (62) in (68) one obtains $v_{r,i}$, the Fourier coefficient of $v(t)$ relative to $\omega = r\omega_0 + ip$

$$v_{r,i} = q_{r/2,i}S_{r/2} + q_{2r,i}S_{-r} + q_{-r,i}S_{2r} \qquad (i = \pm 1). \tag{69}$$

For more details on the derivation of (69) see the Appendix of Ref. 3.

The linear circuit gives a second expression for $v_{r,i}$

$$v_{r,i} = (v_g)_{r,i} - j(r\omega_0 + ip)q_{r,i}Z(r\omega_0 + ip), \tag{70}$$

where $(v_g)_{r,i}$ indicates the Fourier coefficient of $v_g = v_{g1} + \cdots + v_{gN}$ relative to $\omega = r\omega_0 + ip$. By combining (69) with (70) one obtains the "equilibrium equation" of the circuit of Fig. 19 at the sidefrequency $r\omega_0 + ip$

$$(v_g)_{r,i} = j(r\omega_0 + ip)q_{r,i}Z(r\omega_0 + ip) + q_{r/2,i}S_{r/2} + q_{2r,i}S_{-r} + q_{-r,i}S_{2r}. \tag{71}$$

Consider now the chain illustrated in Fig. 20. Let the Q–V characteristic of the nonlinear capacitances of Fig. 20 be equal to that of Fig. 19. Furthermore, let $Z_r(\omega)$ be equal to $Z(\omega)$ for $\omega \cong r\omega_0$ and be infinite for ω far from $r\omega_0$. That is,

$$\begin{aligned} Z_r(\omega) &= Z(\omega) \quad \text{for} \quad \omega = r\omega_0,\, r\omega_0 \pm p \\ Z_r(\omega) &= \infty \quad \text{for} \quad \omega \quad \text{far from} \quad r\omega_0 \qquad (r = 2^s = 1, 2, \cdots, N), \end{aligned} \tag{72}$$

where only positive frequencies are considered because $Z_r(\omega) = Z_r^*(-\omega)$. Note that because of (72), the spectrum of the charge of the $(s + 1)$th

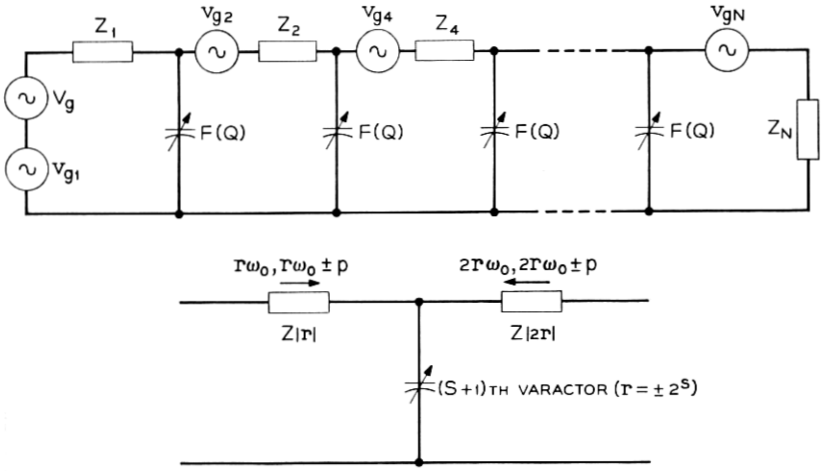

Fig. 20 — Chain of doublers equivalent to the multiplier of Fig. 19.

nonlinear capacitance of Fig. 20 is restricted to the carrier frequencies $r\omega_0$, $2r\omega_0$ ($r = \pm 2^s$) and to their sidebands $r\omega_0 \pm p$, $2r\omega_0 \pm p$. Let the symbol $(\ )'$ be used to distinguish the variables of the circuit of Fig. 19 from those of the circuit of Fig. 20. Then the charge components of the $(s + 1)$th capacitance of Fig. 20 are Q_r', Q_{2r}', $q_{r,i}'$, $q_{2r,i}'$, with $r = \pm 2^s$ and $i = \pm 1$.

It will be shown that, under these hypothesis, the chain of Fig. 20 is equivalent to the multiplier of Fig. 19. More precisely, it will be shown that the charges and voltages of the two circuits are equal.

Demonstration: Let first the fact that $Q_1, \cdots, Q_N$ are equal to Q_1', $\cdots$, Q_N' be demonstrated. Consider therefore, the "equilibrium equation" of the chain of Fig. 20 for $\omega = r\omega_0$ ($r = \pm 2^s$). With reference to Fig. 21, it is obtained by applying Kirchoff's law to the $(s + 1)$th loop.

Across the first capacitance of Fig. 21, the voltage component of frequency $r\omega_0$ is

$$AQ_{r/2}'Q_{r/2}' \qquad (r = \pm 2^s, \mid r \mid \neq 1, N). \tag{73}$$

The voltage component of frequency $r\omega_0$ produced by the second capacitance is

$$2AQ_{-r}'Q_{2r}' \qquad (r = \pm 2^s, \mid r \mid \neq 1, N). \tag{74}$$

Since only the current $jr\omega_0 Q_r'$ is flowing through $Z_{|r|}$, the voltage

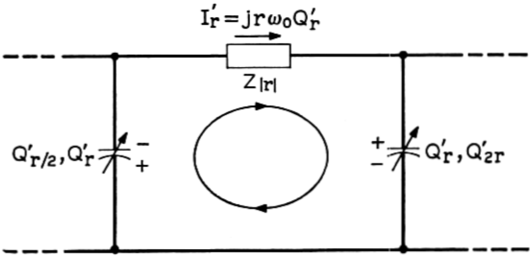

Fig. 21 — Sth loop ($r = 2^s$) of the chain of Fig. 20.

across $Z_{|r|}$ is

$$jr\omega_0 Q_r' Z_{|r|}(r\omega_0) = jr\omega_0 Q_r' Z(r\omega_0) \qquad (r = \pm 2^s, |r| \neq 1, N). \tag{75}$$

Equilibrium requires that the sum of these three voltages be zero. Therefore, one obtains

$$0 = jr\omega_0 Q_r' Z(r\omega_0) + A[Q_{r/2}' Q_{r/2}' + 2Q_{2r}' Q_{-r}'] \tag{76}$$

which is identical to the "equilibrium equation" of the circuit of Fig. 19 for $\omega = r\omega_0$ (with $r = \pm 2^s$, $|r| \neq 1, N$).

In a completely similar way, one finds that also for $|r| = 1$ and $|r| = N$ the "equilibrium equations" of the two circuits are identical. If $r = 1$, for instance, the loop consists of the voltage generator V_g in series with Z_1 and the first nonlinear capacitance of Fig. 20. One obtains

$$V_{g1} - j\omega_0 Q_1' Z(\omega_0) = 2AQ_2' Q_{-1}'$$

which is identical to (66) for $r = 1$. The case $r = N$ is obtained by considering the last loop and one finds

$$-jN\omega_0 Q_N' Z(N\omega_0) = AQ_{N/2}' Q_{N/2}'$$

which is identical to (66) for $r = N$. All this demonstrates that $Q_1, \cdots, Q_N$ are identical to $Q_1', \cdots, Q_N'$. Therefore, the two circuits are equivalent at the carrier frequencies.

Consider now the sidebands. Notice that the elastance components of the $(s + 1)$th nonlinear capacitance of Fig. 20 are S_r, S_{2r} ($r = \pm 2^s$). Therefore, the equilibrium of the $(s + 1)$th loop for $\omega = r\omega_0 + ip$ requires

$$\begin{aligned}(v_g)_{r,i} &= j(r\omega_0 + ip)q_{r,i}' Z(r\omega_0 + ip) \\ &+ q_{r/2,i}' S_{r/2} + S_{2r} q_{-r,i}' + S_{-r} q_{2r,i}' \qquad (r = \pm 2^s, i = \pm 1).\end{aligned} \tag{77}$$

The first term of the second member represents the voltage component across $Z_{|r|}$, the second term corresponds to the voltage across the first capacitance, and the last two terms correspond to the voltage across the second capacitance. Equation (77) is equivalent to (71). Therefore, the two sets of charge components $(q_{r,i})$, $(q'_{r,i})$ ($|r| = 1, 2, \cdots, N$; $i = \pm 1$) are identical. This concludes the demonstration of the equivalence of the two circuits of Figs. 19 and 20. Note that the voltage across $Z_{|r|}$ is equal to the voltage components of Z due to the frequencies $\pm r\omega_0$ and their sidebands.

A.1 *Discussion of the Circuit of Fig. 20*

The circuit of Fig. 20 can be represented by the chain of doublers illustrated in Fig. 22. The equivalent circuit of the $(s + 1)$th doubler of Fig. 22 is shown in Fig. 23. It consists of an "ideal" input filter $F_{\text{in}}^{(s+1)}$ connected in series to the input, an "ideal" output filter $F_{\text{out}}^{(s+1)}$ connected in series to the output, and an "ideal" lossless varactor which has a Q–V characteristic of the type $V = AQ^2$. The filter $F_{\text{in}}^{(s+1)}$ has zero impedance at the input frequencies $r\omega_0$, $r\omega_0 \pm p$ and it has infinite impedance for ω far from $r\omega_0$ ($r = \pm 2^s$). Similarly, $F_{\text{out}}^{(s+1)}$ limits the output current to the frequencies $2r\omega_0$, $2r\omega_0 \pm p$. Consequently, the series connection of $F_{\text{out}}^{(s)}$, Z, $F_{\text{in}}^{(s+1)}$ is equivalent to $Z_{|r|}$, according to (72). Therefore, the two circuits of Figs. 22 and 20 are equivalent. By first approximation, the properties of the "ideal" doublers of Fig. 22 are independent of the modulation frequency p. More precisely, the frequency dependence of the "ideal" doublers can be neglected with respect to the frequency dependence of the impedances Z_1, Z_2, etc. This is more precisely explained by the following considerations.

Consider (77). Let $i_{r,i}$ indicate the Fourier coefficient of the current flowing through $Z_{|r|}$ for $\omega = r\omega_0 + ip$. Then

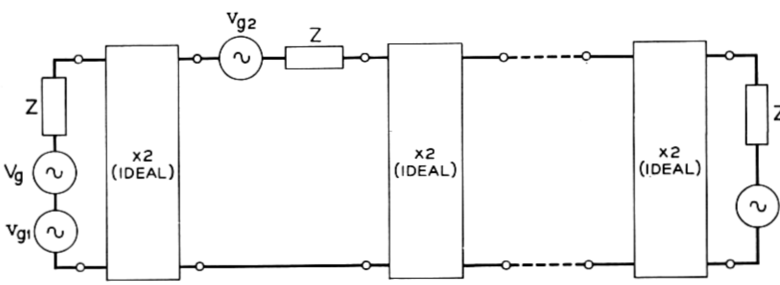

Fig. 22 — Chain of doublers equivalent to the circuit of Fig. 20.

$$i_{r,i} = j(r\omega_0 + ip)q_{r,i} \qquad (i = \pm 1). \tag{78}$$

If one approximates (78) by means of $i_{r,i} = jr\omega_0 q_{r,i}$, one obtains from (77)

$$Z(r\omega_0 + ip)i_{r,i} - \frac{2j}{r} S_{r/2} i_{r/2,i} + \frac{j}{r} S_{2r} i_{-r,i} - \frac{j}{2r} S_{-r} i_{2r,i} = (v_g)_{r,i} . \tag{79}$$

Equation (79) gives the approximate version of (77) which is obtained by neglecting the frequency (p) dependence of the behavior of the nonlinear capacitances of Fig. 20. In fact, one can see that the only term in (79) which depends on p is the first term, and this term is caused by the impedance $Z_{|r|}$ of Fig. 20. In this analysis the approximate expression (79) is valid because it is assumed $Z(\omega) \cong \infty$ for ω far from $r\omega_0$. Therefore, at those frequencies p for which the approximation $i_{r,i} \cong r\omega_0 q_{r,i}$ becomes invalid, the only important term in (79) is the first, which does not contain any approximation. However, it is important to notice that (79) is exact only in the limiting case of a narrow-band multiplier. That is, when $Z_{|r|}(\omega)$ varies so rapidly with p that it is infinite outside a very narrow band around $\omega = r\omega_0$.

According to the preceding considerations, the properties of the "ideal" doublers of Fig. 22 can be assumed independent of the modulation frequency p, by first approximation.

In the following part of this Appendix, it will be shown that the impedance presented to the output of each "ideal" doubler by the following part of the chain is real at the carrier frequency. Furthermore, each doubler is lossless. Therefore, according to the results derived in Ref. 3, one concludes that each doubler does not produce AM ⇄ PM

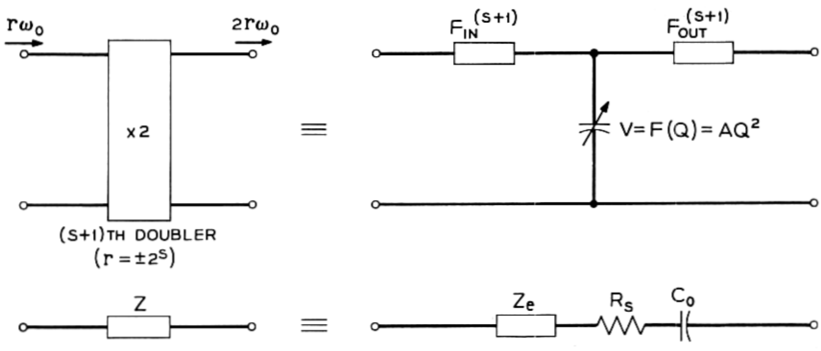

Fig. 23 — Constituents of the chain of Fig. 22.

and that it is characterized by the two scattering matrices given by (27). The characteristic impedances of the input and output ports of the various doublers are derived in the following part.

A.2 *Calculation of the Output and Input Impedances of the Doublers of Fig. 22*

At the carrier frequency $r\omega_0$, the input impedance of the $(s+1)$th doubler $(r = 2^s)$ is given by

$$Z_{\text{in}}^{(s+1)} = \frac{V_{\text{in}}^{(s+1)}}{jr\omega_0 Q_r} = \frac{2AQ_{-r}Q_{2r}}{jr\omega_0 Q_r} \qquad (r = \pm 2^s), \tag{80}$$

where $V_{\text{in}}^{(s+1)}$, the input voltage of the $(s+1)$th doubler, has been obtained from (74).

The output impedance of the $(s+1)$th doubler is given by

$$Z_{\text{out}}^{(s+1)} = Z(2r\omega_0) + Z_{\text{in}}^{(s+2)} \qquad (r = \pm 2^s). \tag{81}$$

But one also has

$$Z_{\text{out}}^{(s+1)} = \frac{-V_{\text{out}}^{(s+1)}}{j2r\omega_0 Q_{2r}} = \frac{-AQ_rQ_r}{j2r\omega_0 Q_{2r}} \qquad (r = \pm 2^s), \tag{82}$$

where $V_{\text{out}}^{(s+1)}$ is the output voltage of the $(s+1)$th doubler and it is given by (73) with $r/2$ replaced with r. Therefore, from (80) and (82) one obtains

$$Z_{\text{out}}^{(s+1)} Z_{\text{in}}^{(s+1)} = \frac{A^2 \mid Q_r \mid^2}{r^2\omega_0^2} \qquad (r = \pm 2^s). \tag{83}$$

By combining (81) and (83), one has

$$Z_{\text{in}}^{(s+1)}[Z_{\text{in}}^{(s+2)} + Z(2r\omega_0)] = \frac{A^2 \mid Q_r \mid^2}{r^2\omega_0^2} \qquad (r = \pm 2^s). \tag{84}$$

The output impedance of the last doubler $(s = n)$ is

$$Z_{\text{out}}^{(n)} = Z(N\omega_0) = R_N . \tag{85}$$

Equation (85) and the recurrent formulas (81) and (84) allow the output and input impedances of the various doublers to be readily calculated (R_N, A, $\mid Q_r \mid$ are known).

Note that Z is real at the frequencies ω_0, $2\omega_0$, $\cdots$, $N\omega_0$, because the circuit is resonant at these frequencies. $Z(2\omega_0)$, $\cdots$, $Z(N/2\omega_0)$ represent the idler losses. If, for instance, the varactor is the only lossy element of the circuit, then

$$Z(2\omega_0) = Z(4\omega_0) = \cdots = R_s , \tag{86}$$

where R_s is the series resistance of the varactor. Therefore, since (85), (84), and (81) are real, one concludes that $Z_{in}^{(s)}$, $Z_{out}^{(s)}$ are real.

Finally, according to the convention of the first paper, the input and output characteristic impedances of the sth doubler are given by $Z_{in}^{(s)}$, $Z_{out}^{(s)}$, respectively.

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