Coding for Numerical Data Transmission*

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This paper considers the effectiveness of error-correcting codes for the transmission of numerical data. In such a situation, errors in the numerically most significant positions of a message are of greater consequence than are errors in the less significant positions. A measure of transmission fidelity based upon the average magnitude by which the numbers delivered to the destination differ from the transmitted numbers is developed and is referred to as the average numerical error (ANE). *Codes are compared by comparing the* ANE *that results from their use.*

Significant-bit codes are defined and the ANE *resulting from their use is determined. For constant-symbol-rate transmission, the relative effectiveness of various coding schemes is analyzed when the error probability in the channel is small. The* ANE *resulting from the use of certain specific codes is numerically evaluated and compared.*

I. INTRODUCTION

The usual approach to coding is to ignore the actual meaning of the transmitted symbols and to represent them in a purely statistical manner. As a result, all message errors are assumed to be equally costly and codes have been sought that simply reduce the probability that a message is received in error.

While this may be appropriate for the transmission of some types of data, there are situations in which other criteria of goodness are of greater merit. If, for example, one is interested in the transmission of the temperature of a satellite, the probability that a particular observation is transmitted incorrectly may have little direct relation to system performance whereas a measure of the average magnitude by which the received data differ from the data actually transmitted could prove useful.

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This paper develops a criterion of transmission fidelity for numerical data transmitted over a binary symmetric channel based upon the average numerical error which occurs. Significant-bit codes are defined and the average numerical error resulting from their use is determined for a binary symmetric channel with independent errors. For constant-symbol-rate transmission, the relative effectiveness of various coding schemes is analyzed when the probability that a symbol is received in error is small. In order to obtain a feeling for the utility of coding, the average numerical error resulting from certain specific codes is numerically evaluated.

II. PRELIMINARIES

Throughout this paper, the channel is taken to include all operations performed upon the symbols during transmission. A binary symmetric channel is defined to be a binary channel such that

(*i*) the channel always gives one of the binary symbols at its output,
(*ii*) the probability that any particular sequence of errors occurs is independent of the symbols transmitted.

In some sections, we shall consider a binary symmetric channel with independent errors. This is a binary symmetric channel for which the errors occur independently with probability p where $0 \leqq p \leqq \frac{1}{2}$ and $p = 1 - q$.

The elements of the Galois field of two elements are denoted by 0 and 1. Let the symbol $\oplus$ denote component by component modulo 2 addition of vectors (or n-tuples) whose components are field elements. The set of all such vectors forms a vector space Γ of dimension n over the field of two elements. Because a field element can be viewed as a vector with one component, $\oplus$ will also be used to denote the addition of field elements.

A binary group code V is a subset of Γ which forms a group. Over the field of two elements, any set of n-tuples that forms a group is indeed a vector space. Therefore, a binary group code V forms a subspace of Γ. The dimension of V is k.

The implementation of a binary group code can be viewed in the following manner. The encoder receives k binary information symbols (called a message) from the source and determines from the message $(n - k)$ binary parity check symbols (called an ending). The message and ending may be interleaved or transmitted sequentially to form a block of length n (called a code vector). The decoder operates upon the blocks of n binary symbols coming from the channel in an attempt to correct transmission errors and provides k binary symbols at its output. The notation (n,k) is used to denote such a code.

Consider the message $(m_k, m_{k-1}, \cdots, m_1)$. The code vector used to transmit this message will have $m_k, m_{k-1}, \cdots, m_1$ in the k information positions. The $(n - k)$ parity check positions that form the ending are denoted by $e_1, e_2, \cdots, e_{n-k}$. The order in which the information positions and the parity check positions are arranged for transmission is arbitrary.

Let H denote the parity check matrix for a binary group code. H is an $(n - k) \times n$ matrix whose entries are field elements. An n-tuple v is a code vector if and only if

$$v\tilde{H} = 0, \tag{1}$$

where $\tilde{H}$ denotes the transpose of H. H can be written in a form such that each column of H that corresponds to a parity check position in a code vector is a distinct weight* one $(n - k)$-tuple. When this is done, let $C_l(1 \leqq l \leqq k)$ denote the column in H that is in the position that corresponds to position m_l in a code vector.

For a binary symmetric channel, the order in which symbols are transmitted can affect code performance. For the binary symmetric channel with independent errors, the order in which symbols are transmitted does not affect performance. In the latter case, we can write H as

$$H = (C_k, C_{k-1}, \cdots, C_1 I_{n-k}), \tag{2}$$

where I_{n-k} denotes the $(n - k) \times (n - k)$ identity matrix.

III. FORMULATION OF A CRITERION OF CODING EFFECTIVENESS

A system for transmitting observations performed upon some physical process over a binary channel is shown in Fig. 1. So that the relationship between the observed numbers and the code will be clear, a general formulation will be presented.

If each quantization step is of uniform size, the quantizer output can be represented as $A + Bi$ where A and B are constants and the integer i indicates the quantization level. The "source scale-to-binary converter" receives $A + Bi$ from the quantizer and transmits i to the encoder. The "binary-to-source scale converter" receives some integer j from the decoder and delivers $A + Bj$ to the destination.

* The weight of a vector v is the number of nonzero components in v and is denoted by $w[v]$. The distance between two vectors u and v is $w[u \oplus v]$.

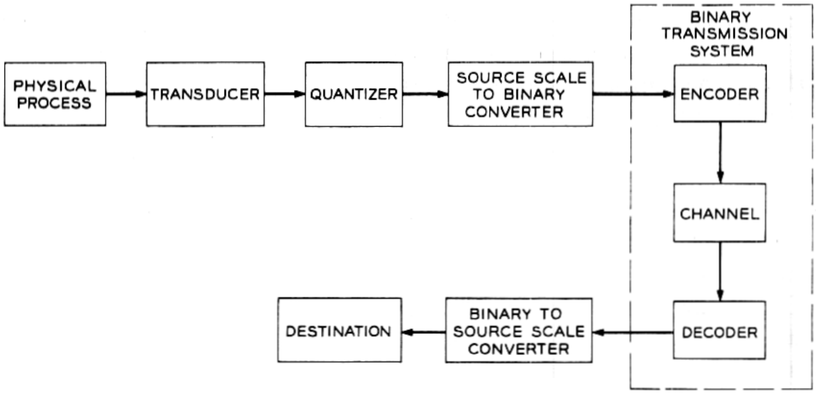

Fig. 1 — System model.

Let Pr $\{j \mid i\}$ be the probability of receiving j at the decoder output when i served as the encoder input and let Pr $\{i\}$ be the probability that i is sent. The average numerical error (ANE) that occurs is

$$\text{ANE} = \sum_i \sum_j \mid (A + Bj) - (A + Bi) \mid \Pr\{j \mid i\} \Pr\{i\}. \tag{3}$$

If all values of i are equally likely to be observed and if the range for i is $0 \leqq i \leqq 2^k - 1$, Pr $\{i\} = 2^{-k}$. The range for j is thus $0 \leqq j \leqq 2^k - 1$ and (3) becomes

$$\text{ANE} = \frac{B}{2^k} \sum_{i=0}^{2^k-1} \sum_{j=0}^{2^k-1} \mid j - i \mid \Pr\{j \mid i\}.$$

Because B is a constant not dependent upon the particular coding scheme implemented, B may be set equal to 1 when comparing the effectiveness of different codes. Accordingly, we shall consider the expression

$$\text{ANE} = \frac{1}{2^k} \sum_{i=0}^{2^k-1} \sum_{j=0}^{2^k-1} \mid j - i \mid \Pr\{j \mid i\}. \tag{4}$$

For a specified value of k, a given coding scheme is considered perferable to some other coding scheme if the ANE resulting from the implementation of the given code is less than the ANE resulting from the alternative code.

The code enters (4) through the terms Pr $\{j \mid i\}$. Thus, for a binary symmetric channel, the ANE will, in general, be dependent not only upon the error statistics of the channel but also upon the order in which the symbols are transmitted.

It is possible to simplify (4) to an expression that involves terms of the form Pr $\{j \mid 0\}$ exclusively. This reduces the number of terms by a factor of 2^k and demonstrates that knowledge of the error probabilities conditional upon zero being sent is sufficient to evaluate the ANE. However, it is necessary to develop some notation and to present two lemmas before proceeding to simplify (4). The proofs of the lemmas are omitted because the lemmas follow from the group property of the code.

When the integer i is to be sent, let us assume that the message ultilized is the k-bit binary representation of i (which is denoted by $B(i)$) such that

$$B(i) = (m_k , m_{k-1} , \cdots , m_1),$$

where

$$i = m_k \cdot 2^{k-1} + m_{k-1} \cdot 2^{k-2} + \cdots + m_1 .$$

The ending $E_i = (e_1 , e_2 , \cdots , e_{n-k})$ required to encode $B(i)$ is chosen so that the resulting code vector $C(i)$ satisfies (1).

Lemma 1: For any values of the integers i and j, $0 \leqq i \leqq 2^k - 1$ and $0 \leqq j \leqq 2^k - 1$, there exists an integer l such that Pr $\{j \mid i\} =$ Pr $\{l \mid 0\}$ where $B(l) = B(i) \oplus B(j)$ and $0 \leqq l \leqq 2^k - 1$.

Lemma 2: Let $B(l) = B(i) \oplus B(j)$ as in Lemma 1. For fixed $i(0 \leqq i \leqq 2^k - 1)$, as j successively takes on the values $0, 1, 2, \cdots , 2^k - 1$, l takes on each of the values in the range $0 \leqq l \leqq 2^k - 1$ once and only once.

Theorem 1: Let all messages be equally likely to be transmitted and let the channel be binary symmetric (but not necessarily with independent errors). For these conditions, the average numerical error is

$$\text{ANE} = \sum_{j=1}^{k} 2^{j-1} \sum_{i=2^{j-1}}^{2^j-1} \Pr \{i \mid 0\}. \tag{5}$$

Proof: By Lemmas 1 and 2, for each value of i and for a specified value of l, there will be a unique integer j_l such that Pr $\{j_l \mid i\} =$ Pr $\{l \mid 0\}$ where $B(l) = B(i) \oplus B(j_l)$. From (4),

$$\text{ANE} = \frac{1}{2^k} \sum_{l=1}^{2^k-1} \sum_{i=0}^{2^k-1} \mid j_l - i \mid \Pr \{l \mid 0\}, \tag{6}$$

where we have used the fact that $\mid j_l - i \mid = 0$ when $l = 0$.

For each value of l $(1 \leqq l \leqq 2^k - 1)$, we wish to determine $\sum_{i=0}^{2^k-1} \mid j_l - i \mid$. Let $\alpha(0 \leqq \alpha \leqq k - 1)$ be the largest integer such that

$2^\alpha \leqq l$. Define i' and j'_l as

$$B(j_l) = B(i') \oplus B(2^\alpha) \tag{7a}$$

$$B(i) = B(j'_l) \oplus B(2^\alpha) \quad \text{or} \quad B(j'_l) = B(i) \oplus B(2^\alpha). \tag{7b}$$

Then

$$B(i) \oplus B(j_l) = B(i') \oplus B(j'_l) = B(l). \tag{7c}$$

Because $l > 0$, $i \neq j_l$ and $i' \neq j'_l$. Suppose $i > j_l$. Then $j_l = i' - 2^\alpha$ and $j'_l = i - 2^\alpha$ by (7). It follows that $j_l - i = -2^\alpha - i + i'$ and $j'_l - i' = -2^\alpha + i - i'$. Conversely, if $i < j_l$, $j_l = i' + 2^\alpha$ and $j'_l = i + 2^\alpha$. Thus, $j_l - i = 2^\alpha - i + i'$ and $j'_l - i' = 2^\alpha + i - i'$.

Therefore,

$$| j_l - i | + | j'_l - i' | = | 2^\alpha + i - i' | + | 2^\alpha - i + i' |. \tag{8}$$

But $B(i) = B(2^\alpha) \oplus B(l) \oplus B(i')$ by (7). Thus, $| i - i' | < 2^\alpha$ and, from (8),

$$| j_l - i | + | j'_l - i' | = 2 \cdot 2^\alpha.$$

Because of the symmetries involved,

$$2 \sum_{i=0}^{2^k-1} | j_l - i | = \sum_{i=0}^{2^k-1} | j_l - i | + | j'_l - i' | = 2^k \cdot 2^{\alpha+1}.$$

Thus, (6) becomes

$$\text{ANE} = \sum_{l=1}^{2^k-1} 2^\alpha \Pr \{l \mid 0\}$$

or

$$\text{ANE} = \sum_{\alpha=0}^{k-1} 2^\alpha \sum_{l=2^\alpha}^{2^{\alpha+1}-1} \Pr \{l \mid 0\}. \qquad \text{QED}$$

In (5), notice that $\Pr \{0 \mid 0\}$ does not appear and that the terms $\Pr \{i \mid 0\}$ are not weighted linearly in i but that the weighting coefficients go in steps as powers of 2 with several conditional probabilities having the same weighting coefficient. Notice that the weighting coefficient for $\Pr \{i \mid 0\}$ is 2^{j-1} where $(j - 1)$ is the largest power of 2 in i. All errors with the same coefficient are of the same seriousness and a good code must reduce these sets of probabilities rather than simply minimize the probability that a few very large errors occur.

Because the set of messages $B(i)$ $(2^{j-1} \leqq i \leqq 2^j - 1)$ gives rise to the set of conditional probabilities whose weighting coefficient in the ANE expression is 2^{j-1}, we shall call these messages the j-level messages and the corresponding conditional probabilities, Pr $\{2^{j-1} \mid 0\}$ through Pr $\{2^j - 1 \mid 0\}$, the j-level conditional probabilities. The 0-level message is defined to be $B(0)$ and the 0-level conditional probability to be Pr $\{0 \mid 0\}$.

The j-level messages have the following interesting characteristics.

(*i*) Component m_j in each message is 1.

(*ii*) Components $m_i (j + 1 \leqq i \leqq k)$ in each message are 0.

(*iii*) Every possible $(j - 1)$-tuple occurs once and only once as components m_1 through m_{j-1} of some j-level message.

For a perfect error-correcting code used with a binary symmetric channel with independent errors, it is possible to compute the j-level conditional probabilities and thus the ANE from a knowledge of the weight distribution of the code vectors on each level (these weight distributions have been referred to as level weight structures.)[1] The problem of efficiently computing the level weight structures from knowledge of the parity check matrix has been discussed previously.[1]

IV. SIGNIFICANT-BIT CODES

In order to permit the error-correcting capabilities of a code to correspond somewhat to the significance of the information positions, it is possible to formulate a type of code which uses a subcode to protect the $(k - k_0)$ most significant positions of a message and simply transmits the remaining symbols unprotected. The name significant-bit code (SB code) is used for this type of code. An SB code is specified by the parity check matrix H_{SB} and the ANE resulting from the use of an SB code is ANE_{SB}.

The code utilized to protect the $(k - k_0)$ most significant information positions will be named the base code. Because it is confined to the $(k - k_0)$ most significant positions, we can abstract the base code and study it as a separate entity. Accordingly, the base code vectors are $(n - k_0)$-tuples of which the first $(k - k_0)$ positions are the base messages.

Although the concept of SB codes is applicable to any binary symmetric channel, we shall assume independent errors in the following analysis. Thus, from (2), the base code is specified by the base parity check matrix H_{B} where

$$H_{\text{B}} = (C'_{k-k_0}, C'_{k-k_0-1}, \cdots, C'_1 I_{n-k}).$$

In this case, the code vector $C(i) = B(i) \mid E_i$ where the symbol $\mid$ indicates that $C(i)$ can be partitioned into the k-tuple $B(i)$ and the

$(n - k)$-tuple E_i. Let $B(i)$ be partitioned so that $B(i) = B'(i') \mid B''(i'')$ where $B'(i')$ denotes the $(k - k_0)$ most significant positions of $B(i)$ and $B''(i'')$ denotes the k_0 least significant positions of $B(i)$. Then

$$C(i) = B'(i') \mid B''(i'') \mid E_i .$$

The range for i' is $0 \leqq i' \leqq 2^{k-k_0} - 1$ and for i'' is $0 \leqq i'' \leqq 2^{k_0} - 1$.

Let $\Pr_B \{i' \mid j'\}$ denote the probability of receiving i' when j' is sent using the base code. By Theorem 1, the ANE for the base code (ANE_B) is

$$\text{ANE}_B = \sum_{j=1}^{k-k_0} 2^{j-1} \sum_{i'=2^{j-1}}^{2^j-1} \Pr_B \{i' \mid 0\}. \tag{9}$$

Because the base code is used exclusively to protect the $(k - k_0)$ most significant information positions, H_{SB} must have the form

$$H_{SB} = (C'_{k-k_0}, C'_{k-k_0-1}, \cdots, C'_2, C'_1 \underbrace{0 \cdots 0}_{\substack{k_0 \\ \text{columns}}} I_{n-k})$$

where 0 is used to represent an all-zero column of H_{SB} and where the $C'_l (1 \leqq l \leqq k - k_0)$ are the columns of H_B. The coset leaders[2] in the standard array[2] for the SB code must be obtained from the coset leaders in the standard array for the base code by expanding the base coset leaders in length to n-tuples by inserting k_0 zeros in information positions 1 through k_0 of the expanded vectors. Because all vectors in column i of the standard array for the SB code will have $B''(i'')$ in information positions 1 through k_0,

$$\Pr \{i \mid 0\} = p^{w[B''(i'')]} q^{k_0 - w[B''(i'')]} \Pr_B \{i' \mid 0\}. \tag{10}$$

We shall now show that ANE_{SB} can be expressed in terms of the properties of the base code.

Theorem 2: Let the base code be defined as above. For a binary symmetric channel with independent errors and when all messages are equally likely to be transmitted,

$$\text{ANE}_{SB} = \Pr_B \{0 \mid 0\} \sum_{j=1}^{k_0} 2^{j-1} p q^{k_0 - j} + 2^{k_0} \text{ANE}_B . \tag{11}$$

Proof: Define

$$\text{ANE}' = \sum_{j=1}^{k_0} 2^{j-1} \sum_{i=2^{j-1}}^{2^j-1} \Pr \{i \mid 0\}$$

and

$$\text{ANE}'' = \sum_{j=k_0+1}^{k} 2^{j-1} \sum_{i=2^{j-1}}^{2^j-1} \Pr\{i \mid 0\}.$$

From Theorem 1, $\text{ANE}_{\text{SB}} = \text{ANE}' + \text{ANE}''$.

Let us first analyze ANE′. For $1 \leqq j \leqq k_0$, the sum of the j-level conditional probabilities is

$$\sum_{i=2^{j-1}}^{2^j-1} \Pr\{i \mid 0\} = \sum_{i''=2^{j-1}}^{2^j-1} p^{w[B''(i'')]} q^{k_0-w[B''(i'')]} \Pr_{\text{B}}\{0 \mid 0\}$$

where we have used (10) and realized that $i' = 0$ for all messages on this level. Because every $(j-1)$-tuple occurs as components m_1 through m_{j-1} of some j-level message and $m_j = 1$ in every j-level message, there are

$$\binom{j-1}{w[B''(i'')]-1}$$

messages of weight $w[B''(i'')]$ on the j-level. Thus,

$$\sum_{i=2^{j-1}}^{2^j-1} \Pr\{i \mid 0\} = \Pr_{\text{B}}\{0 \mid 0\} \sum_{l=1}^{j} \binom{j-1}{l-1} p^l q^{k_0-l}$$
$$= \Pr_{\text{B}}\{0 \mid 0\} p q^{k_0-j}$$

and

$$\text{ANE}' = \Pr_{\text{B}}\{0 \mid 0\} \sum_{j=1}^{k_0} 2^{j-1} p q^{k_0-j}.$$

Now consider ANE″. On level $k_0 + \xi$ $(1 \leqq \xi \leqq k - k_0)$, i has the range $2^{k_0+\xi-1} \leqq i \leqq 2^{k_0+\xi} - 1$. Divide this range into $2^{\xi-1}$ sets of consecutive integers each of size 2^{k_0}. Let the integer δ index these sets where $0 \leqq \delta \leqq 2^{\xi-1} - 1$. For a particular value of δ, as i increases from $2^{k_0+\xi-1} + \delta 2^{k_0}$ to $2^{k_0+\xi-1} + (\delta+1)2^{k_0} - 1$, $i' = 2^{\xi-1} + \delta$ and i'' runs through the range $0 \leqq i'' \leqq 2^{k_0} - 1$. Thus, using (10),

$$\sum_{i=2^{k_0+\xi-1}+\delta 2^{k_0}}^{2^{k_0+\xi-1}+(\delta+1)2^{k_0}-1} \Pr\{i \mid 0\}$$
$$= \sum_{i''=0}^{2^{k_0}-1} p^{w[B''(i'')]} q^{k_0-w[B''(i'')]} \Pr_{\text{B}}\{2^{\xi-1} + \delta \mid 0\}.$$

As i'' runs through the range $0 \leqq i'' \leqq 2^{k_0} - 1$, each possible k_0-tuple occurs once and only once. Therefore,

$$\sum_{i=2^{k_0+\xi-1}+\delta 2^{k_0}}^{2^{k_0+\xi-1}+(\delta+1)2^{k_0}-1} \Pr\{i \mid 0\} = \Pr_{\mathrm{B}}\{2^{\xi-1}+\delta \mid 0\} \sum_{l=0}^{k_0} \binom{k_0}{l} p^l q^{k_0-l} \tag{12}$$

$$= \Pr_{\mathrm{B}}\{2^{\xi-1}+\delta \mid 0\}.$$

Because of the manner in which the sets were chosen, ANE″ can be expanded as

$$\text{ANE}'' = \sum_{\xi=1}^{k-k_0} 2^{k_0+\xi-1} \sum_{\delta=0}^{2^{\xi-1}-1} \sum_{i=2^{k_0+\xi-1}+\delta 2^{k_0}}^{2^{k_0+\xi-1}+(\delta+1)2^{k_0}-1} \Pr\{i \mid 0\}. \tag{13}$$

Substituting (12) into (13), we obtain

$$\text{ANE}'' = 2^{k_0} \sum_{\xi=1}^{k-k_0} 2^{\xi-1} \sum_{i'=2^{\xi-1}}^{2^{\xi}-1} \Pr_{\mathrm{B}}\{i' \mid 0\}$$

which, from (9), is exactly 2^{k_0} ANE_{B}. QED

Notice that the situation $k = k_0$ can be included in this formulation if we define $\text{ANE}_{\mathrm{B}} = 0$ and $\Pr_{\mathrm{B}}\{0 \mid 0\} = 1$ when $k = k_0$. Thus, uncoded transmission can be regarded as an SB code in which $k = k_0$.

The interpretation of (11) is interesting. The quantity $\sum_{i=1}^{k_0} 2^{i-1}pq^{k_0-i}$ is the ANE that results from the uncoded transmission of k_0-tuples. Thus, ANE_{SB} is the ANE for uncoded transmission of k_0-tuples weighted by $\Pr_{\mathrm{B}}\{0 \mid 0\}$ plus 2^{k_0} times ANE_{B}.

(11) enables the computation of ANE_{SB} from the properties of the base code. Because the base code involves messages of length $(k - k_0)$, it is easier to analyze than the entire SB code.

V. CONSTANT-SYMBOL-RATE TRANSMISSION

Consider two error-correcting codes which are denoted as V_1 and V_2. Let V_1 be an (n_1, k) code and V_2 be an (n_2, k) code where n_1 may or may not be equal to n_2. Let ε_1 denote the minimum weight of the n_1-tuples that are not coset leaders in the standard array for V_1. Similarly, let ε_2 denote the minimum weight of the n_2-tuples that are not coset leaders in the standard array for V_2.

For a binary symmetric channel with independent errors, $\Pr\{i \mid 0\}$ for V_1 is

$$\Pr\{i \mid 0\} = \sum_{j=\varepsilon_1}^{n_1} \tau_{ij} p^j q^{n_1-j}$$

where τ_{ij} is the number of n_1-tuples of weight j in the column headed by $C(i)$ in the standard array for V_1. Thus, for V_1, the average numerical error (ANE_1) is

$$ANE_1 = \sum_{j=\varepsilon_1}^{n_1} \sigma_j p^j q^{n_1 - j},$$

where

$$\sigma_j = \sum_{l=1}^{k} 2^{l-1} \sum_{i=2^{l-1}}^{2^l - 1} \tau_{ij} .$$

Similarly, for V_2 ,

$$ANE_2 = \sum_{j=\varepsilon_2}^{n_2} \gamma_j p^j q^{n_2 - j},$$

where the γ_j are the appropriate constants.

However,

$$ANE_1 \to \sigma_{\varepsilon_1} p^{\varepsilon_1} q^{n_1 - \varepsilon_1} \quad \text{as} \quad p \to 0$$

and

$$ANE_2 \to \gamma_{\varepsilon_2} p^{\varepsilon_2} q^{n_2 - \varepsilon_2} \quad \text{as} \quad p \to 0.$$

Thus, for p sufficiently small, if $\varepsilon_1 > \varepsilon_2$, $ANE_1 < ANE_2$ and V_1 results in less ANE than V_2 .

The minimum weight of the vectors that are not coset leaders in an SB code is 1. Thus, consider two SB codes denoted by V_{SB1} and V_{SB2} where V_{SB1} is an (n_1, k) code and V_{SB2} is an (n_2, k) code. V_{SB1} protects the $(k - k_{01})$ most significant positions and V_{SB2} protects the $(k - k_{02})$ most significant positions of a message. By reasoning analogous to that above, for p small, if $k_{01} < k_{02}$ and if the base codes used in V_{SB1} and V_{SB2} correct all weight one errors, then V_{SB1} results in less ANE than V_{SB2} .

We thus have the following ranking of codes for p small. The ranking (in order of increasing effectiveness) assumes that the schemes are compared for the same value of k.

(*i*) Uncoded transmission.
(*ii*) An SB code protecting $(k - k_0)$ positions where $k \neq k_0$.
(*iii*) An SB code protecting $(k - k_0 + k')$ positions where $k' > 0$.
(*iv*) An e-error-correcting code where $e \geqq 1$.
(*v*) An $(e + e')$-error-correcting code where $e' > 0$.

To obtain a feeling for the utility of coding for numerical data transmission over a binary symmetric channel with independent errors, the ANE resulting from certain codes for $k = 26$ will be evaluated for constant-symbol-rate transmission. Ref. 3 contains similar information for $k = 1$, 4 and 11.

Let ANE_{UC} denote the ANE when no coding is used. Contrary to the concept of code equivalence that is obtained under the assumption that all errors are equally costly (i.e., when probability of message error is used as the measure of code performance), the ordering of the columns of the parity check matrix can affect code performance. Thus, for the (31, 26) perfect single error-correcting code (PSEC code), every ordering of the columns of the parity check matrix could yield a distinct ANE. Upper and lower bounds on the ANE for this code are obtained in Ref. 3 and are denoted herein as ANE_{UB} and ANE_{LB}, respectively.

By numerical computation, the ordering in (14) was found to result in as small an ANE as any other ordering tried. The number actually tried was by necessity a small fraction of all possible orderings of the 26 columns. However, notice that C_{12} through C_{26} each have a one in the same position thus assuring us that the number of weight three code vectors on levels 12 through 26 will be the theoretical minimum for this code (by Theorem 9 in Ref. 3). For values of p that are of primary interest (less than 10^{-1}), this assures us that it is not possible to find a different ordering that will result in a significantly better performance (although there are other orderings that in fact give equal performance). Let ANE_{P} denote the ANE that results from the code specified in (14).

$$H_{\text{P}} = \begin{bmatrix} 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & \\ 1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0 & \\ 1\,1\,1\,1\,0\,0\,0\,0\,1\,1\,1\,1\,0\,0\,0\,1\,1\,1\,1\,0\,0\,0\,1\,1\,1\,0 & I_5 \\ 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\,1\,1\,0\,0\,1\,1\,0\,1\,1\,0\,1 & \\ 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\,0\,1\,1\,0\,1\,1 & \end{bmatrix}. \tag{14}$$

If the columns of (14) are regarded as the 5-bit binary representations of integers, then the ordering from left to right corresponds to decreasing integer value (with powers of two omitted because they appear in I_5). Similar ordering was observed to be preferable for the (15, 11) PSEC code[3] and, by exhaustive search, actually found to be as good as any other ordering for the (7, 4) PSEC code[3].

Table I compares ANE_{LB}, ANE_{UB} and ANE_{P}. For convenience (and so that the values given will agree with the data plotted in Figs. 2, 3, and 4), the ANE has been normalized by dividing by $2^{26} - 1$ (i.e., the full-scale value).

The following SB codes are considered. For each, H_{B} and the notation used for the resulting ANE in Figs. 2, 3, and 4 is given. Theorem 2 permits the computation of the ANE for these codes from a knowledge of the base code.

TABLE I—VALUES OF ANE_{LB}, ANE_{UB} AND ANE_P DIVIDED BY $2^{26} - 1$

p	ANE_{LB}	ANE_P	ANE_{UB}
10^{-5}	$0.41992 \cdot 10^{-8}$	$0.41994 \cdot 10^{-8}$	$0.42993 \cdot 10^{-8}$
10^{-4}	$0.41920 \cdot 10^{-6}$	$0.41931 \cdot 10^{-6}$	$0.42932 \cdot 10^{-6}$
10^{-3}	$0.41212 \cdot 10^{-4}$	$0.41310 \cdot 10^{-4}$	$0.42329 \cdot 10^{-4}$
10^{-2}	$0.34850 \cdot 10^{-2}$	$0.35659 \cdot 10^{-2}$	$0.36894 \cdot 10^{-2}$
10^{-1}	$0.90222 \cdot 10^{-1}$	0.10446	0.12817

Base Code 1: (3, 1) PSEC code.

$$H_B = \begin{bmatrix} 1 & I_2 \\ 1 & \end{bmatrix}$$

The ANE is denoted as $ANE_{(3,1)}$.

Base Code 2: (5, 1) perfect double error-correcting code.

$$H_B = \begin{bmatrix} 1 & \\ 1 & I_4 \\ 1 & \\ 1 & \end{bmatrix}$$

The ANE is denoted as $ANE_{(5,1)}$.

Base Code 3: This base code uses independent (3, 1) PSEC codes to protect the two most significant information positions.

$$H_B = \begin{bmatrix} 1 & 0 & \\ 1 & 0 & I_4 \\ 0 & 1 & \\ 0 & 1 & \end{bmatrix}$$

Because the codes are used independently, the required conditional probabilities for the base code can be readily calculated. The ANE is denoted as $ANE_{(3,1),(3,1)}$.

Base Code 4: (7, 4) PSEC code.

$$H_B = \begin{bmatrix} 1 & 1 & 1 & 0 & \\ 1 & 1 & 0 & 1 & I_3 \\ 1 & 0 & 1 & 1 & \end{bmatrix}$$

The ANE is denoted as $ANE_{(7,4)}$.

Base Code 5: This base code uses a (3, 1) PSEC code to protect the most significant information position and a (7, 4) PSEC code to protect the next four most significant information positions.

$$H_{B} = \begin{bmatrix} 1\ 0\ 0\ 0\ 0 & \\ 1\ 0\ 0\ 0\ 0 & \\ 0\ 1\ 1\ 1\ 0 & I_5 \\ 0\ 1\ 1\ 0\ 1 & \\ 0\ 1\ 0\ 1\ 1 & \end{bmatrix}$$

The ANE is denoted as $\mathrm{ANE}_{(3,1),(7,4)}$.

Base Code 6: (15, 11) PSEC code.

$$H_{B} = \begin{bmatrix} 1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0 & \\ 1\,1\,1\,1\,0\,0\,0\,1\,1\,1\,0 & I_4 \\ 1\,1\,0\,0\,1\,1\,0\,1\,1\,0\,1 & \\ 1\,0\,1\,0\,1\,0\,1\,1\,0\,1\,1 & \end{bmatrix}$$

The ANE is denoted as $\mathrm{ANE}_{(15,11)}$.

Figs. 2, 3, and 4 present ANE_{UC}, ANE_{UB}, ANE_{LB}, ANE_{P}, and the ANE of the SB codes considered. In each case, the ANE has been normalized by dividing by $2^{26} - 1$. For clarity, logarithmic scales are used as p decreases from 10^{-1} until p becomes sufficiently small so that the results for small p apply.

The following observations can be made for constant-symbol-rate transmission.

(*i*) Improvements in transmission fidelity are obtainable by the utilization of codes. It should be noted that no one code is the most desirable for all p ($0 < p < \frac{1}{2}$) and in some cases the codes that are best for small p turn out to be less effective than uncoded transmission for the larger values of p.

(*ii*) For $k = 26$, it can be shown that the probability that a message is received in error when the PSEC code is used is less (for $0 < p < \frac{1}{2}$) than the probability that a message is received in error using any of the SB codes considered. Thus, under the criterion of minimizing the probability that a message is received in error, the PSEC code is preferable to any of the SB codes considered.

However, when the ANE is used as a measure of code effectiveness for numerical data transmission, we observe that the SB codes are preferable to the PSEC code for certain values of p. Thus, when comparing codes, the ranking obtained using probability of message error as the performance index may not correspond to the ranking obtained using ANE as an index. We can conclude that probability of message error and ANE are not equivalent measures of code performance and

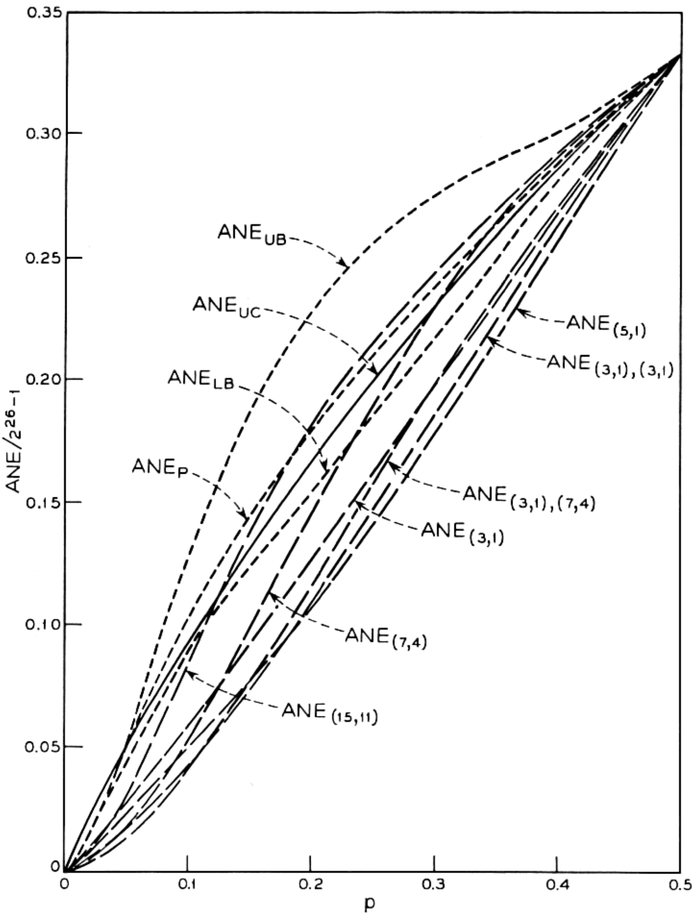

Fig. 2 — Constant-symbol-rate transmission; $k = 26$.

that, in some cases, the ANE can be reduced by using a code whose probability of message error is not minimal.

(*iii*) For $k = 26$, consider the relative performance of the PSEC code and the SB codes. When p is small, the PSEC code will be effective because it can correct all single errors (the only type that have much probability of occurring) whereas a single error in certain positions of an SB code will result in a message error. For larger values of p, there is an increasing chance that an error pattern will occur which the PSEC code cannot correct. The SB codes become effective in this situation. If multiple errors occur during transmission such that the errors occurring in the $(k - k_0)$ most significant information positions and the check positions form an error pattern correctable by the base code, this will be corrected leaving any errors in the k_0 least significant information positions uncorrected. Therefore, the most costly portion of a large number of error patterns can be corrected. As p increases, the number of positions in the base code must decrease so that uncorrectable error patterns in the positions covered by the base code have a sufficiently small probability of occurrence so that the base code can operate effectively. In other words, as p increases, more and more protection must be provided for the significant bits so that the most costly errors are prevented.

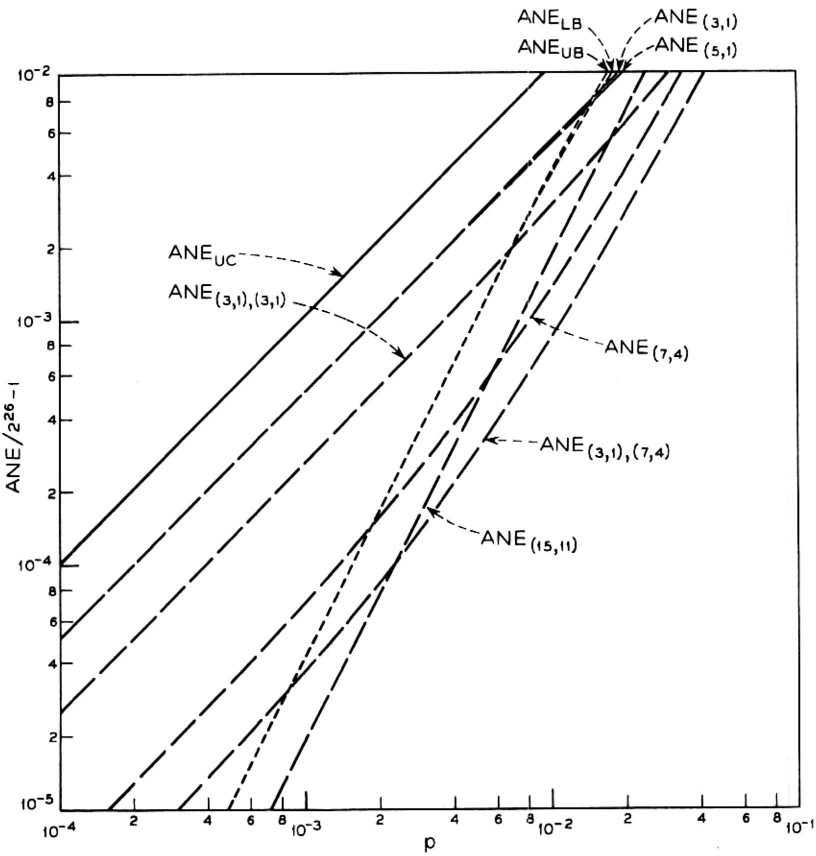

Fig. 3 — Constant-symbol-rate transmission; $k = 26$.

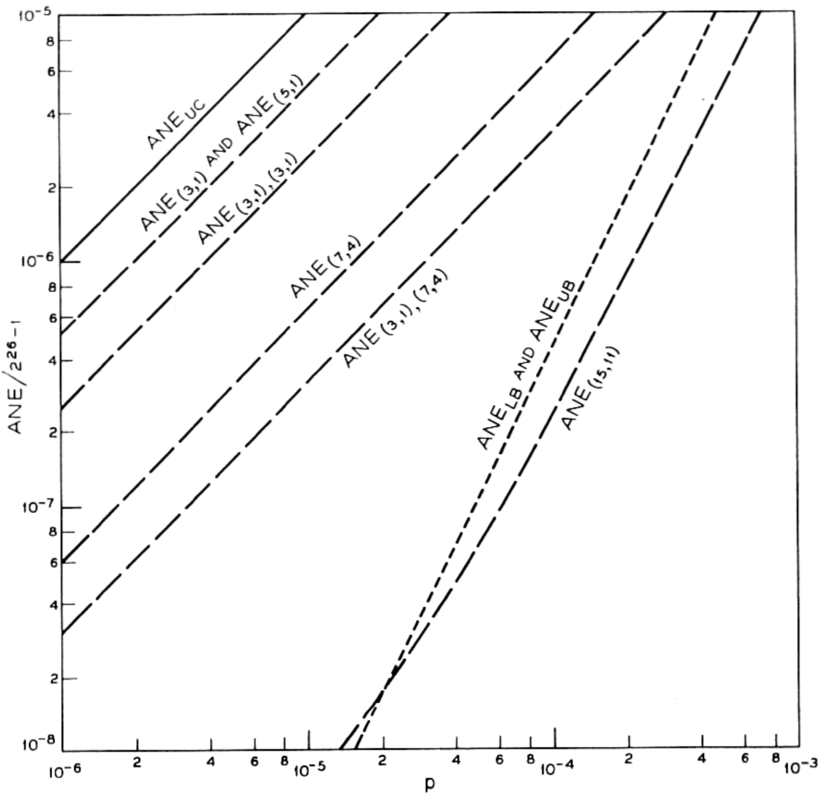

Fig. 4 — Constant-symbol-rate transmission; $k = 26$.

(*iv*) For p small, the ANE from uncoded transmission is approximately $(2^k - 1)p$. For small p, the ANE as a fraction of full scale for uncoded transmission is thus very nearly independent of k.

REFERENCES

1. Buchner, M. M., Jr., Computing the Spectrum of a Binary Group Code, B.S.T.J., *45*, March, 1966, pp. 441–449.
2. Slepian, D., A Class of Binary Signaling Alphabets, B.S.T.J., *35*, January, 1956, pp. 203–234.
3. Buchner, M. M., Jr., Coding for Numerical Data Transmission, Ph.D. Dissertation, The Johns Hopkins University, Baltimore, Maryland, 1965.