

# Transistor Distortion Analysis Using Volterra Series Representation

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*Intermodulation distortion due to nonlinear elements in transistors is analyzed using Volterra series representation. It is shown that this technique is well suited for the analysis of transistor distortion where the nonlinearities are small but frequency dependent. An ac transistor model incorporating four nonlinearities is briefly described. The nonlinear nodal equations of the model are successively solved by expressing nodal voltages in terms of the Volterra series expansion of the input voltage. Based on this analysis, a digital computer program has been developed which computes the second and the third harmonic distortion for a given set of input frequencies and transistor parameters. The results compare favorably with measured values. This method also enables the derivation of closed form ac expressions for a simplified model; these expressions show the dependence of distortion on frequency, load and source impedances, bias currents and voltages, and the parameters of the transistor. The technique is also extended to cascaded transistors, and simplified expressions for the overall distortion in terms of the distortion and gain of individual transistors are derived. Finally, a few pertinent practical applications are discussed.*

## I. INTRODUCTION

Solid-state long-haul analog communication systems are being designed for higher frequencies to meet the growth in demand. One of the more critical and significant problems facing the system designer is intermodulation noise arising from transistor nonlinearities. Thus, an analysis of transistor distortion at higher frequencies is a practical problem; this paper investigates the transistor distortion using the Volterra series as an analysis tool.

Transistor distortion has been investigated in some detail previously. Many authors have considered the exponential nonlinear relation between emitter current and emitter-to-base voltage which is important

at low currents.<sup>1,2,3,4,5</sup> The effect of frequency on this nonlinear source alone has been reported.<sup>5</sup> Three nonlinearities (exponential, avalanche, and  $h_{FE}$  at dc) have been examined by Riva, Beneteau and Dallavolta.<sup>6</sup> For currents up to 20 mA and frequencies up to 100 kHz, Meyer<sup>7,8,9</sup> has developed a more accurate and complex model obtaining the nonlinearities from  $h$ -parameters. However, he takes into account the frequency dependence by assuming that the  $h$ -parameters can be written as  $h' + j\omega h''$ . Moreover, he does not take into account avalanche distortion, nor has he extended the model to higher currents (100 mA) and frequencies (20 MHz). The model described here considers four nonlinearities; they are, exponential, avalanche,  $h_{FE}$ , and collector capacitance nonlinearities. These nonlinearities are superimposed on a linear ac equivalent circuit.<sup>10,11</sup> Much of the initial development of the model with three nonlinearities was done by Thomas.<sup>10</sup>

The transistor model is analyzed using a Volterra series representation; this series is a generalization of the power series. In a now classic report, Wiener applied this analysis technique<sup>12</sup> to find the response of a nonlinear device to noise.<sup>13</sup> Bose has carried the theory further.<sup>14</sup> Following a series of lectures by Wiener,<sup>15</sup> the theoretical framework, higher-dimensional transforms, and optimization with Gaussian inputs were considered by Brilliant,<sup>16</sup> George,<sup>17</sup> and Chesler,<sup>18</sup> respectively. Barrett<sup>19</sup> has treated statistical inputs. The synthesis problem has been examined by Van Trees,<sup>20</sup> who also applied the method to phase-locked loops.<sup>21</sup> The technique has been extended to discrete systems,<sup>22,23,24</sup> and a class of time-variant systems.<sup>24,25</sup> More recently the theory of the convergence of the series has been treated.<sup>26</sup> This work relies more on George's work on the higher-dimensional transform theory.<sup>17</sup>

Even though much work has been done in this area, the Volterra series has not found a wide application in solving nonlinear system problems due to several reasons; if the rate of convergence is not rapid, the higher-degree terms, which are cumbersome to handle, cannot be neglected; hence, it cannot conveniently represent gross nonlinearities. It is not simple to invert the multidimensional transforms to the time domain, and it is not a useful technique to determine the stability of a nonlinear differential equation.

The Volterra series method does, however, offer certain distinct advantages in analyzing transistor distortion. Since transistor distortion is frequency dependent, the power series is inadequate to characterize it; the Volterra series does indeed represent frequency dependent systems. The nonlinearities in the transistors under consideration are extremely small so that the second- and third-degree terms suffice to

characterize them. Since the output corresponding to sinusoidal input signals is of interest, there is no need to find the inverse of the higher-dimensional transforms; the output can be expressed in terms of the transform of the kernel. The higher-dimensional transforms of the kernel are complex numbers when  $s_i = j\omega_i$ , where  $s_i$  is the complex variable in the transform domain; hence, these kernels can be numerically evaluated using the computer (see Section IV). Moreover, for a slightly simpler model closed form ac expressions can be derived. Since the kernels retain phase information, this approach will be useful for the AM-to-PM conversion problem at IF frequencies. Finally, in an amplifier two or more transistors are cascaded; the nonlinear behavior of such cascaded transistors is a significant problem. The Volterra series approach can be easily extended to study such cascaded transistors.

## II. AN INTRODUCTION TO VOLTERRA SERIES REPRESENTATION

A brief exposition of Volterra series with pertinent reference to the problem under consideration is presented below. For further details the reader is referred to the references cited.

Consider a simple memoryless nonlinear system described by the following power series; let  $y(t)$  be the output and  $x(t)$  the input; the system is represented by

$$y(t) = c_1 x(t) + c_2 [x(t)]^2 + c_3 [x(t)]^3, \quad (1)$$

where  $c_1, c_2, c_3$  are constants. For a time-invariant system with memory (capacitors and inductors in an electrical network), the linear term  $\{c_1 x(t)\}$  is replaced by the convolution integral ( $x(t) = 0; t < 0$ )

$$y_1(t) = \int_0^t c_1(t - \tau) x(\tau) d\tau. \quad (2)$$

In the transform domain, (2) may be written

$$Y_1(s) = C_1(s)X(s). \quad (3)$$

This transform domain representation of the system  $[C_1(s)]$  has been an invaluable aid to the communication engineers since it brings into focus the frequency behavior of the system.

A generalization of the second-degree term,  $c_2[x(t)]^2$ , is the double convolution integral

$$y_2(t) = \int_0^t \int_0^t c_2(t - \tau_1, t - \tau_2) \prod_{i=1}^2 x(\tau_i) d\tau_i. \quad (4)$$

The output depends on the past values of the input; the above expression involves a product of the input with itself, thus representing a quadratic system.  $c_2(t - \tau_1, t - \tau_2)$  is known as the second-degree Volterra kernel.

A two-dimensional Laplace transform can be defined for (4) after introducing dummy variables  $t_1$  and  $t_2$ . As shown in Appendix A, (4) becomes

$$Y_2(s_1, s_2) = C_2(s_1, s_2) \prod_{i=1}^2 X(s_i). \quad (5)$$

When two sinusoidal signals at frequencies  $f_a$  and  $f_b$  are applied (Appendix A), the output at the harmonic frequency  $f_a \pm f_b$  is given by  $[|C_2(f_a \pm f_b)| \cos(2\pi(f_a \pm f_b)t + \phi_{a \pm b})]$ . Since in general  $C_2(f_a, f_b)$  will not be equal to  $C_2(f_a, -f_b)$ , different values of distortion at different harmonic frequencies are directly reflected in the kernel. Moreover, as in the power series case, the  $2f$  product is less by a factor of two.

Likewise, the third-degree term  $[c_3(x(\tau))^3]$  can be generalized to a triple convolution integral;

$$y_3(t) = \int_0^t \int_0^t \int_0^t C_3(t - \tau_1, t - \tau_2, t - \tau_3) \prod_{i=1}^3 x(\tau_i) d\tau_i. \quad (6)$$

In the transform domain (6) may be written

$$Y_3(s_1, s_2, s_3) = C_3(s_1, s_2, s_3) \prod_{i=1}^3 X(s_i). \quad (7)$$

The magnitude of the signal at the harmonic frequency  $f_a + f_b - f_c$  due to the three fundamental signals at  $f_a$ ,  $f_b$  and  $f_c$  is given by  $|C_3(f_a, f_b, -f_c)|$ . The constants like  $1/4$  for a ' $3f_a$ ' product are the same as obtained from the power series approach.

Later in the paper (in Section IV) the cascade relations in the transform domain are frequently used; their physical significance is discussed in detail in Section VI. (See also Fig. 1.) The cascade formulae and the procedure for deriving them are given in Appendix A.

The second and third harmonic distortion are defined as the second and third harmonic power in dBm, respectively, when the fundamental power at the output of the transistor is at zero dBm (one milliwatt). In the analysis of the model in Section IV, the output voltage is expressed in terms of a Volterra series of the input voltage. Thus, the kernels  $C_1(s_1)$ ,  $C_2(s_1, s_2)$ , and  $C_3(s_1, s_2, s_3)$  are the voltage transfer ratios; for a given load  $R_L$ , the second and the third harmonic distortion in dBm are given by the following expressions:



$$M_{2_{a \pm b}} = 20 \log \frac{1}{2} \frac{|C_2(f_a \pm f_b)| \sqrt{10^{-3} R_L}}{|C_1(f_a)| |C_1(\pm f_b)|} \quad (8)^*$$

$$M_{3_{a \pm b \pm c}} = 20 \log \frac{1}{4} \frac{|C_3(f_a \pm f_b \pm f_c)| 10^{-3} R_L}{|C_1(f_a)| |C_1(\pm f_b)| |C_1(\pm f_c)|} \quad (9)^*$$

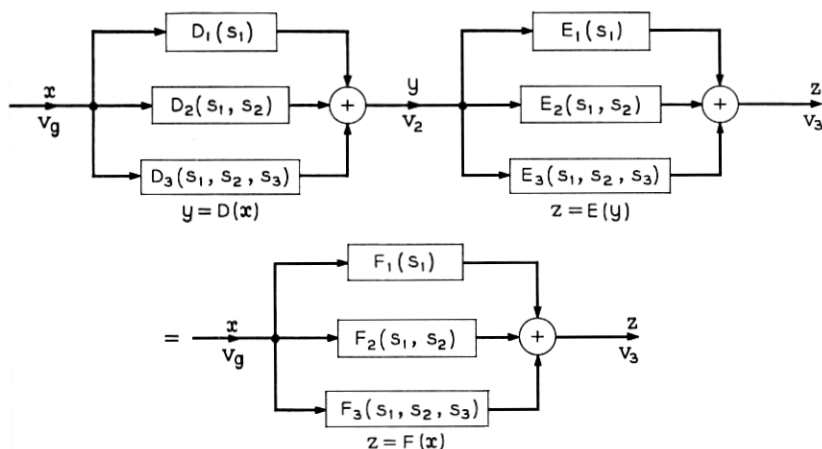


Fig. 1 — Two cascaded systems.

### III. THE JUNCTION TRANSISTOR NONLINEAR MODEL

A model is a simple but realistic representation of a physical phenomenon in terms of measurable parameters such that the phenomenon can be analyzed, and controlled if possible. The linear equivalent circuit of a transistor is one such example. In reality, several elements of the transistor equivalent circuit are not linear but are linearized versions of nonlinear functions; they are the first-degree terms of the Taylor's series expansion of the nonlinear functions. Hence, a logical way to develop the nonlinear model is to consider the second- and third-degree terms of the Taylor's series expansion; thus, the emitter resistance (exponential nonlinearity), current gain ( $h_{FE}$  and avalanche nonlinearity), and the collector capacitance (collector capacitance nonlinearity) have been represented by nonlinear voltage dependent current generators whose parameters are higher-degree Taylor's series terms. This approach has another advantage in that it is difficult to measure the nonlinearities since they are small; but, it is not too difficult to measure the overall functions and to curve fit with the known theoretical

\* The factors  $\frac{1}{2}$  and  $\frac{1}{4}$  normalize the distortion to  $2f$  and  $3f$  products.

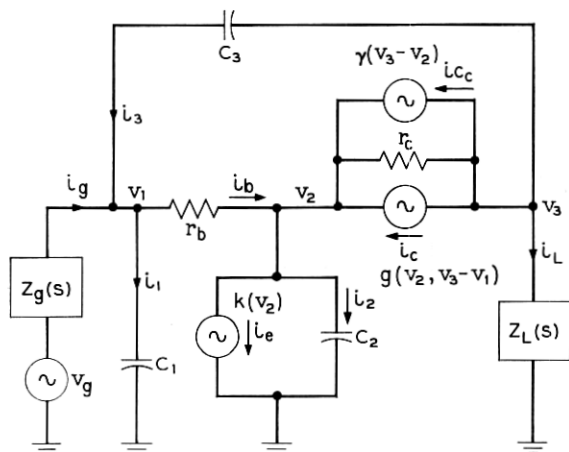


Fig. 2 — Common-emitter nonlinear equivalent circuit.

or empirical relations. These nonlinearities are superimposed on the linear equivalent circuit (Fig. 2). The nonlinearities are described next.

### 3.1 Exponential Nonlinearity

The emitter current,  $I_E$ , is related to the emitter voltage,  $V_2$ , by the exponential relation

$$I_E = A \left[ \exp \left( q \frac{V_2}{KT} \right) - 1 \right] + B, \quad (10)$$

where  $K$  = Boltzmann's constant,

$q$  = electron charge in coulombs,

$T$  = Temperature in degrees Kelvin,

and  $A$  and  $B$  are constants which depend on the transistor parameters (Ref. 27; p. 181, p. 249). An experimental curve of the emitter current  $I_E$  and the emitter-to-base voltage  $V_{be}$  is shown in Fig. 3. This nonlinearity is expressed as a voltage-dependent current generator by a Taylor's series expansion of (10) as follows:

$$i_e = K(v_2) = K_1 v_2 + K_2 v_2^2 + K_3 v_2^3, \quad (11)$$

where the Taylor's series coefficients are derived in terms of known parameters, the emitter resistance  $r_e$ , and the emitter bias current  $I_E$ ; i.e.,

$$K_1 = \frac{1}{r_{e1}}; \quad K_2 = \frac{1}{2(r_{e1})^2} \frac{1}{I_E}; \quad K_3 = \frac{1}{6} \frac{1}{(I_E)^2} \frac{1}{(r_{e1})^3}. \quad (12)$$

### 3.2 Avalanche and $h_{FE}$ Nonlinearity

The collector current is a nonlinear function of the emitter current at higher values of current ( $h_{FE}$  nonlinearity) and of the collector-to-base voltage at higher values of voltage (avalanche nonlinearity).<sup>27</sup>  $h_{FE}$ , the ratio of  $I_C$  to  $I_B$ , is plotted as a function of collector current  $I_C$  in Fig. 4. It is seen that the following empirical relation<sup>6</sup> matches the experimental result (Fig. 4):

$$h_{FE} = \frac{h_{FE \max}}{1 + a \log^2 \frac{I_C}{I_{C \max}}}, \quad (13)$$

where  $h_{FE \max}$  is the maximum value of  $h_{FE}$ ,  $I_{C \max}$  is the value of  $I_C$  at which  $h_{FE \max}$  occurs, and  $a$  is a constant.

The avalanche nonlinearity is due to avalanche multiplication which occurs at higher collector-to-base voltage. It is determined from the collector characteristic which is a plot of collector current ( $I_C$ ) and collector-to-emitter voltage ( $V_{CE}$ ), (Fig. 5). The empirical Miller's avalanche multiplication factor is given by

$$\frac{1}{1 - \left( \frac{V_{CE}}{V_{CEO}} \right)^n}, \quad (14)$$

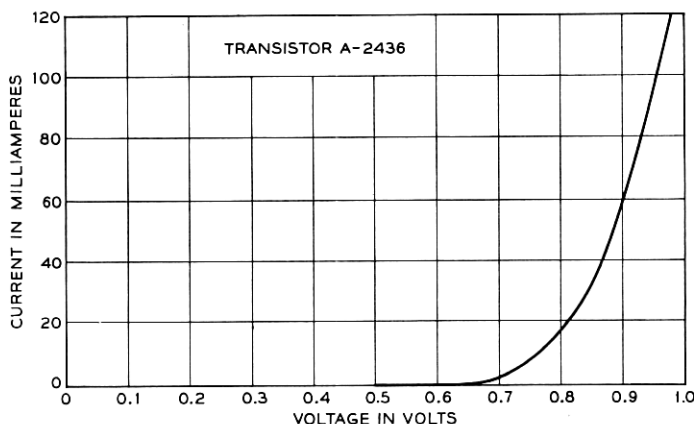
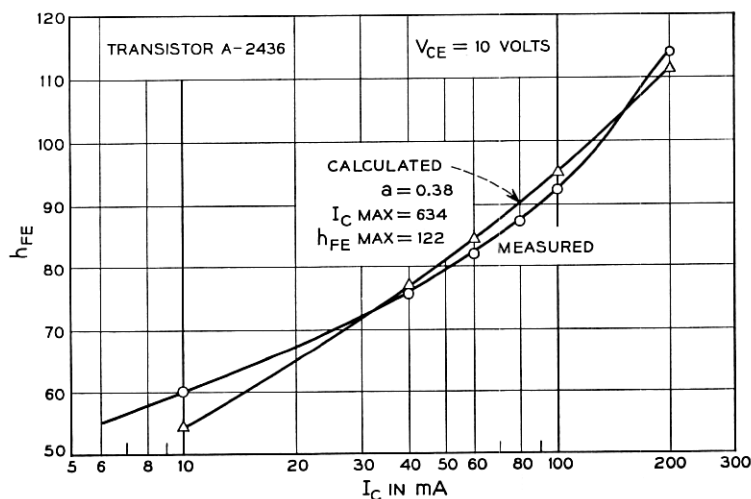


Fig. 3 — Exponential nonlinearity — measured curve.

Fig. 4— $h_{FE}$  nonlinearity — calculated and measured curves.

where  $V_{CE0}$  is the sustained voltage, and the exponent  $n$  is determined by experiment. From expressions (13) and (14), the ratio  $I_C v_S I_E$  is given by

$$\frac{I_C}{I_E} = \frac{h_{FE \text{ max}}}{1 + h_{FE \text{ max}} + a \log^2 \left( \frac{I_C}{I_{C \text{ max}}} \right)} \cdot \frac{1}{1 - \left( \frac{V_{CB}}{V_{CB0}} \right)^n}, \quad (15)$$

where  $V_{CB0} = V_{CE0}/n \sqrt{1 - \alpha}$  and  $V_{CB} \cong V_{CE}$ .

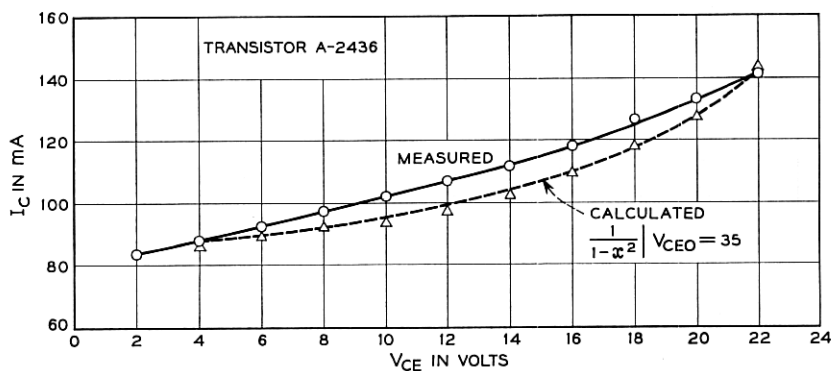


Fig. 5—Avalanche nonlinearity — calculated and measured curves.

The ac  $i_e$  can be expressed in terms of  $i_e$  and  $v_{cb}[v_3 - v_1]$  by a Taylor's series expansion of (15). Since  $i_e$  is a function of emitter voltage  $v_2$ ,  $i_e$  is represented by a current generator  $g(v_2, v_3 - v_1)$ ; for convenience in notation it is separated into a linear term  $g_1(v_2, v_3 - v_1)$ , a second-degree term  $g_2(v_2, v_3 - v_1)$  and a third-degree term  $g_3(v_2, v_3 - v_1)$ . The linear term equals  $\hat{M}_0(\alpha_1 K_1)v_2 + \hat{M}_1(v_3 - v_1)$ . The second-degree term is given by  $\alpha_2 \hat{M}_0 K_1^2 (v_2)^2 + m_2(v_3 - v_1)^2 + (\alpha_1 \hat{M}_1) K_1 v_2 (v_3 - v_1)$ . The coefficients  $\alpha_1$ ,  $\alpha_2$ ,  $\hat{M}_1$ ,  $m_2$ , etc., and the third-degree term are given in Appendix B.

### 3.3 Collector Capacitance Nonlinearity

The collector capacitance is a nonlinear function of collector-to-base voltage ( $V_{CB}$ ) since the depletion layer width is a function of  $V_{CB}$ . The exact functional relationship is determined by plotting the common-base imaginary part of  $h_{22}$  as a function of collector-to-base voltage ( $V_{CB}$ ) as shown in Fig. 6.<sup>9</sup> It is evident from the figure that  $C_e$  follows the 1/3 voltage law (Ref. 19; Equation 5-96);

$$C_e = k(V_{CB})^{-1/3}. \quad (16)$$

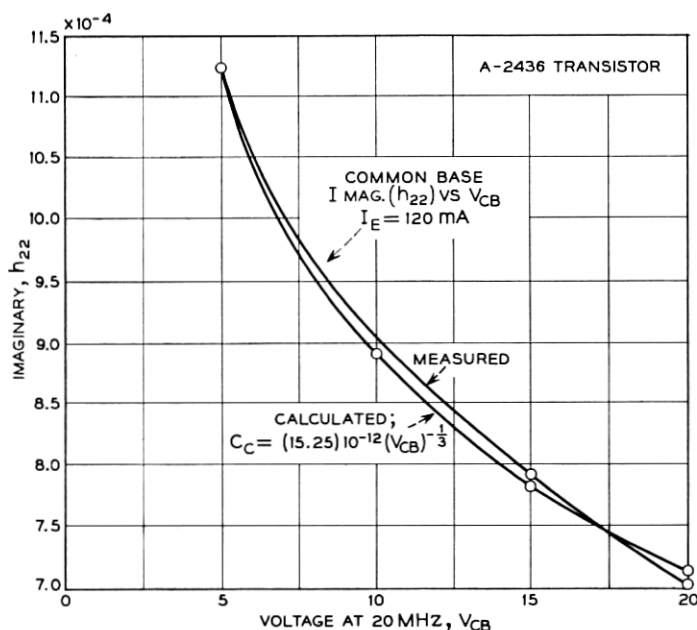


Fig. 6—Collector capacitance nonlinearity — calculated and measured curves.

This nonlinearity is represented as a frequency (differentiation) and voltage-dependent current generator as follows:

$$\begin{aligned} i_{e_c} &= \gamma(v_3 - v_1) \\ &= \gamma_1 \frac{d}{dt}(v_3 - v_1) + \gamma_2 \frac{d}{dt}(v_3 - v_1)^2 + \gamma_3 \frac{d}{dt}(v_3 - v_1)^3, \end{aligned} \quad (17)$$

where  $\gamma_1 = C_c$ , and where  $\gamma_2$  and  $\gamma_3$  are known from (16).

The above nonlinear current generators are incorporated in the linear equivalent circuit as shown in Fig. 2. The linear equivalent circuit parameters are obtained from the equivalent circuit characterization. They can, for example, be computed from the  $h$ -parameters at different frequencies. In general, the distortion is not a critical function of the linear parameters. (Figs. 14 to 17).

All the nonlinear coefficients ( $K_2$ ,  $\alpha_2$ ,  $m_2$ , etc.) are easily obtained from a simple computer program. The parameters to be specified along with typical values for transistor type A-2436 are listed in Appendix C.

#### IV. THE VOLTERRA KERNELS FOR THE NONLINEAR MODEL

The Volterra series method is applied to the model to compute the second and the third harmonic distortion. The voltage at each node is a nonlinear frequency-dependent function of the input voltage. Each nodal voltage is expressed by a Volterra series expansion of the generator voltage; since the nonlinearities are small only three terms are considered. The kernels at each node are determined from Kirchhoff's current equations.

##### 4.1 Nodal Equations

The Kirchhoff's current law is applied at each node; the currents are next expressed in terms of the generator voltage  $v_g$ , the three nodal voltages  $v_1$ ,  $v_2$ , and  $v_3$ , and the known linear and nonlinear parameters. The impedances are represented by their transforms and  $o$  denotes that it operates on the voltage across it. The nodal equations are given below.

$$\frac{1}{Z_g(s)} o(v_g - v_1) + (sC_3) o(v_3 - v_1) = (sC_1) o v_1 + \frac{1}{r_b} o(v_1 - v_2), \quad (18)$$

$$\begin{aligned} \frac{1}{r_b} o(v_1 - v_2) &= K(v_2) + (sC_2) o v_2 + \left(\frac{1}{r_c}\right) o(v_2 - v_3) - \gamma(v_3 - v_2) \\ &\quad - g(v_2, v_3 - v_1), \end{aligned} \quad (19)$$

$$-\gamma(v_3 - v_2) + \frac{1}{r_c}(v_2 - v_3) - g(v_2, v_3 - v_1) = (sC_3) o(v_3 - v_1) + \left(\frac{1}{Z_L(s)}\right) o v_3, \quad (20)$$

where  $K(v_2)$ ,  $\gamma(v_2 - v_3)$  and  $g(v_2, v_3 - v_1)$  are the nonlinear current generators.

#### 4.2 Solution Using Volterra Series

Since each nodal voltage is to be expressed in terms of three Volterra kernels, there are nine unknown Volterra kernels to be determined from the three equations. The problem of solving for nine unknowns from three equations is resolved by noting that the polynomials  $x$ ,  $x^2$  and  $x^3$  are linearly independent; hence, each degree term is separately and successively solved. The linear kernels are first determined; then the second-degree kernels are determined in terms of the linear kernels; lastly, the third-degree kernels are evaluated in terms of the first- and second-degree kernels.

Let  $A_1(s)$ ,  $B_1(s)$ ,  $C_1(s)$  denote the transforms of the linear kernels at nodes one, two and three, respectively. From the nodal equations (18) to (20), the following vector matrix equation is derived.

$$\begin{Bmatrix} \frac{1}{Z_v(s)} \\ 0 \\ 0 \end{Bmatrix} = P_E(s) \begin{Bmatrix} A_1(s) \\ B_1(s) \\ C_1(s) \end{Bmatrix}, \quad (21)$$

where

$$P_E(s) = \begin{bmatrix} \frac{1}{Z_v(s)} + s(C_3 + C_1) + \frac{1}{r_b} & -\frac{1}{r_b} & -sC_3 \\ -\frac{1}{r_b} + m_1 & \frac{1}{r_b} + sC_2 + \frac{1}{r_c} + K_1(1 - \alpha) + s\gamma_1 & -\frac{1}{r_c} - m_1 - s\gamma_1 \\ -sC_3 - m_1 & -\frac{1}{r_c} + \alpha K_1 - s\gamma_1 & \frac{1}{r_c} + sC_3 + \frac{1}{Z_L(s)} + m_1 + s\gamma_1 \end{bmatrix}. \quad (22)$$

Equation (21) is solved by inverting matrix  $P_E(s)$  and post-multiplying by the vector

$$\begin{Bmatrix} \frac{1}{Z_v(s)} \\ 0 \\ 0 \end{Bmatrix}.$$

For a given frequency  $s = j\omega$ , the computation is done numerically.

The second-degree terms are equated next in (18) to (20). There are two types of second-degree terms; those arising from the unknown second-degree kernels [for example,  $(s_1 + s_2)C_1A_2(s_1, s_2)$ ] and those arising from the known nonlinear coefficients and the known linear kernels [for example,  $K_2 \prod_{i=1}^2 B_1(s_i)$ ]. The terms associated with the unknown second-degree kernels are the same as were associated with the unknown linear kernels in (21), but at the harmonic frequency  $(s_1 + s_2)$ . The following vector matrix equation is obtained for the second-degree kernels:

$$\begin{Bmatrix} 0 \\ [\hat{g}_2(B_1, C_1 - A_1) \\ + \gamma_2(C_1 - B_1) - K_2(B_1)] \\ [-\hat{g}_2(B_1, C_1 - A_1) \\ - \hat{\gamma}_2(C_1 - B_1)] \end{Bmatrix} = P_E(s_1 + s_2) \begin{Bmatrix} A_2(s_1, s_2) \\ B_2(s_1, s_2) \\ C_2(s_1, s_2) \end{Bmatrix}, \quad (23)$$

where  $\hat{g}_2$  and  $\hat{\gamma}_2$  represent the second harmonic contribution due to  $g_2(v_2, v_3 - v_1)$  and  $\gamma_2(v_3 - v_2)$ ; hence,

$$\begin{aligned} \hat{g}_2(B_1, C_1 - A_1) &= [\alpha_1 K_2 + \alpha_2 K_1^2] \prod_{i=1}^2 B_1(s_i) \\ &+ \frac{\alpha_1 M_1 K_1}{2} [B_1(s_1)[C_1(s_2) - A_1(s_2)] \\ &+ B_1(s_2)[C_1(s_1) - A_1(s_1)]] \\ &+ m_2 \prod_{i=1}^2 [C_1(s_i) - A_1(s_i)] \end{aligned} \quad (24)$$

$$\hat{K}_2(B_1) = K_2 \prod_{i=1}^2 B_1(s_i) \quad (25)$$

$$\hat{\gamma}_2(C_1 - B_1) = K_2(s_1 + s_2) \gamma_2 \prod_{i=1}^2 [C_1(s_i) - B_1(s_i)]. \quad (26)$$

$P_E(s_1 + s_2)$  is the matrix  $P_E(s)$  with  $s$  replaced by  $(s_1 + s_2)$ .

The vector on the left side of (23) is known. Thus, the unknown kernels are determined by inverting the matrix  $P_E(s_1 + s_2)$  and post-multiplying by the vector on the left-hand side of (23). When  $s_1 = j\omega_b$ , the inversion of the matrix and the post multiplication by the vector can be done numerically.



The procedure for obtaining the third-degree kernels is almost the same; the significant difference is that the vector on the left side not only contains terms arising from the third-degree nonlinear parameters but also includes second-degree coefficients which give rise to third-degree terms by the interaction of the first- and the second-degree kernels. These interaction terms are denoted by  $\hat{K}_{23}$ ,  $\hat{g}_{23}$ ,  $\hat{\gamma}_{23}$ , respectively. For example,  $K_3(B_1) = K_3 \Pi_{i=1}^3 B_1(s_i)$ , whereas  $K_{23} = 2k_2 B_1(s_1) B_2(s_2, s_3)$  which shows the interaction of the first- and the second-degree kernels. The third-degree kernels are derived from the following equations:

$$\left\{ \begin{array}{c} 0 \\ [\hat{g}_3(B_1, C_1 - A_1) + \hat{\gamma}_3(C_1 - B_1) \\ - \hat{K}_3(B_1) + \hat{g}_{23} + \hat{\gamma}_{23} + \hat{K}_{23}] \\ [-\hat{g}_3(B_1, C_1 - A_1) - \hat{\gamma}_3(C_1 - B_1) \\ - \hat{g}_{23} - \hat{\gamma}_{23}] \end{array} \right\} = P_E(s_1 + s_2 + s_3) \left\{ \begin{array}{c} A_3(s_1, s_2, s_3) \\ B_3(s_1, s_2, s_3) \\ C_3(s_1, s_2, s_3) \end{array} \right\}, \quad (27)$$

where  $\hat{g}_3$ ,  $\hat{g}_{23}$  are given in Appendix B.

A computer program has been developed which calculates the kernels and the second and the third harmonic distortion. It uses existing programs to invert the matrix  $P_E(s)$ . The nonlinear coefficients are computed from the known and measured parameters. Computed and measured results at different currents are given in Fig. 7. The program has been extended to common-base and common-collector configurations.

#### V. SIMPLIFIED DISTORTION EXPRESSIONS, THEIR PHYSICAL SIGNIFICANCE AND COMPARISON WITH EXPERIMENTAL RESULTS

Another advantage of the Volterra series method is that it permits derivation of closed-form expressions for second and third harmonic distortion. These equations show the interaction between the various nonlinear parameters and the effect of frequency.

The model includes the base resistance ( $r_b$ ), the emitter resistance ( $r_{e1}$ ), the diffusion capacitance ( $C_2$ ), the load ( $R_L$ ) and the source impedances  $Zg(s)$ , and three nonlinearities, namely, exponential, avalanche, and  $h_{FE}$  nonlinearities. In the computer program  $C_{bc}$ ,  $C_{be}$ ,

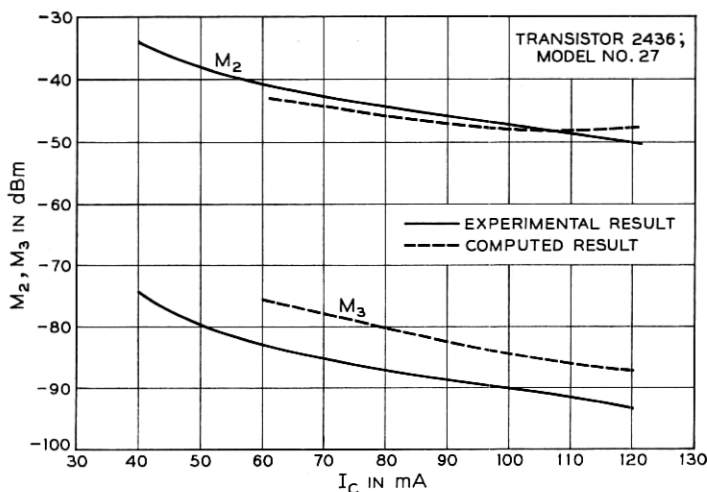


Fig. 7—Comparison of experimental and computed results.

$r_e$ ,  $C_e$ ,  $m_1$  and collector capacitance nonlinearity have been taken into account. The expressions given below are for the common-emitter configuration.

### 5.1 The Second Harmonic Distortion Term

The second harmonic distortion in dBm (8) is given by

$$M_{2_{a \pm b}} \approx 20 \log \frac{1}{2} \sqrt{\frac{10^{-3}}{R_L}} \left| \left[ \frac{(r_b + Z_o(s)) \cdot (K_1 + sC_2) + 1}{(r_b + Z_o(s)) \cdot [K_1(1 - \alpha_1) + sC_2] + 1} \right] \cdot \left[ \frac{\alpha_2}{(\alpha_1)^2} - \hat{M}_1 \left( R_L + \frac{sC_2}{2} \frac{r_b}{\alpha_1 K_1} \right) + m_2 \prod_{i=1}^2 \left( R_L + \frac{s_i C_2 r_b}{\alpha_1 K_1} \right) + \frac{K_2}{\alpha_1 K_1^2} \left( \frac{(Z_o(s) + r_b) \cdot sC_2 + 1}{(Z_o(s) + r_b) \cdot (K_1 + sC_2) + 1} \right) \right] \right| \quad (28)$$

where  $s_1 = j\omega_a$ ,  $s_2 = \pm j\omega_b$  and  $s = j\omega_a \pm j\omega_b$ .

### 5.2 The Third Harmonic Distortion Term

In the third harmonic distortion term given below, the interaction terms due to the first- and the second-degree kernels have not been included mainly to reduce the complexity; in certain cases, they may be significant.

$$M_{3_{a+b+c}} \approx 20 \log \frac{1}{4} \frac{10^{-3}}{R_L} \left| \left[ \frac{(r_b + Z_o(s)) \cdot (sC_2 + K_1) + 1}{(r_b + Z_o(s)) \cdot (K_1(1 - \alpha_1) + sC_2) + 1} \right] \right|$$

$$\begin{aligned}
& \cdot \left[ \frac{\alpha_3}{(\alpha_1)^3} + \hat{M}_1 \left( R_L^2 + \frac{2}{3} \frac{R_L \underline{s} C_2 r_b}{\alpha_1 K_1} + \frac{1}{3} \frac{\overline{s_i s_j} C_2^2 r_b^2}{(\alpha_1 K_1)^2} \right) - m_3 \prod_{i=1}^3 \left( R_L + \frac{s_i C_2 r_b}{\alpha_1 K_1} \right) \right. \\
& + \frac{K_3}{(\alpha_1 K_1)^3} \frac{\alpha_1 [(r_b + Z_g(\underline{s}))(\underline{s} C_2) + 1]}{(r_b + Z_g(\underline{s}))(\underline{s} C_2 + K_1) + 1} + \frac{2\alpha_2 K_2}{(\alpha_1)^3 K_1^2} \\
& \left. - \frac{\alpha_2 \hat{M}_1}{(\alpha_1)^3} \left[ \alpha_1 R_L + \frac{\underline{s}}{3} \frac{C_2 r_b}{(K_1)^3} \right] \right] \Bigg|, \quad (29)
\end{aligned}$$

where  $s_1 = j\omega_a$ ,  $s_2 = j\omega_b$ ,  $s_3 = \pm j\omega_c$  and  $\underline{s} = s_1 + s_2 + s_3$  and  $\overline{s_i s_j} = s_1 s_2 + s_2 s_3 + s_3 s_1$ .

### 5.3 Physical Interpretation of the Distortion Terms

The interaction of different nonlinearities and their dependence on load impedance, source impedance, bias currents, bias voltage and frequency is indeed somewhat complex. However, the closed form expressions derived above give a general qualitative picture which will be discussed now.

#### 5.3.1 Effect of Frequency

It is important to know the effect of frequency on distortion. The distortion depends not only on the frequencies of the fundamental tones but also on the harmonic frequency of interest. As shown in Fig. 8,

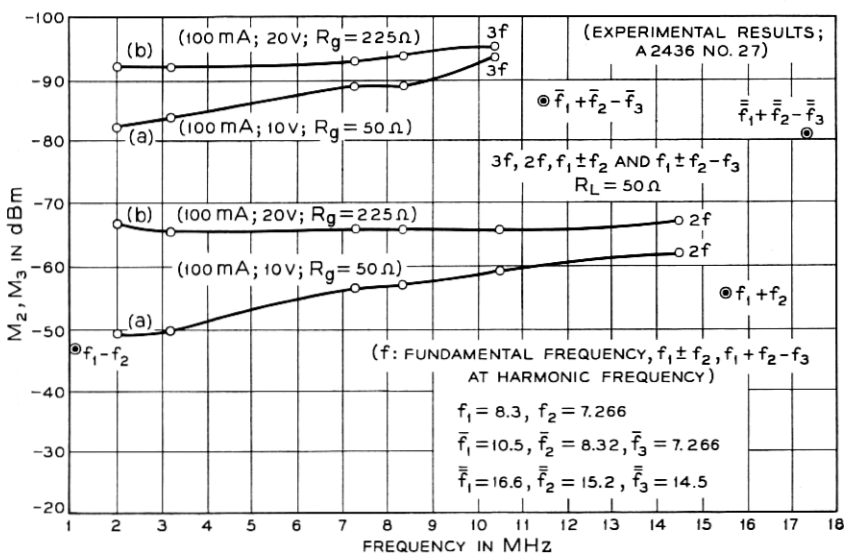


Fig. 8—Variation of  $M_2$ ,  $M_3$  with frequency.

$M_2$  due to  $a + b$  product is better than  $M_2$  of  $a - b$  product by 10 dB with the two tones at 8.32 and 7.266 MHz. These measurements were made with the transistor biased at 100 mA, 10V and with  $R_L = 50\Omega$  and  $R_s = 50\Omega$ . In curve (a) of Fig. 8, the fundamental tone was increased from 2 MHz to 10.5 MHz and signals at  $2f$  and  $3f$  were measured. It is seen that both  $M_2$  and  $M_3$  improved with increase in frequency. A theoretical explanation on the basis of dominant terms (in this range of parameter values) in (28) and (29) is given below. In (28) as well as in (29) the terms in brackets are multiplied by a frequency-dependent term

$$\frac{(r_b + R_s)(K_1 + sC_2) + 1}{(r_b + R_s)(K_1(1 - \alpha_1) + sC_2) + 1}.$$

In this range of frequency ( $s$  = harmonic frequency), if  $K_1(1 - \alpha_1) \leq |sC_2|$  and if  $|(r_b + R_s)sC_2| > 1$  but  $K_1 > |sC_2|$ , then the above term reduces to  $K_1/sC_2$  which decreases with increase in frequency. However, the avalanche terms ( $\hat{M}_2$ ,  $\hat{M}_3$ , etc.) involve the terms  $sC_2$  [in (28) and (29)] and  $S_1C_2$  in the numerator. Thus, if the avalanche terms are dominant, as at higher voltages, there should be no net contribution due to avalanche terms alone. The exponential terms  $[K_2/(K_1)^2$  and  $K_3/(K_1)^3]$  are multiplied by the factor

$$\frac{(r_b + R_s) \cdot sC_2 + 1}{(r_b + R_s) \cdot (K_1(1 - \alpha_1) + sC_2) + 1}.$$

This term is independent of frequency if  $(sC_2(r_b + R_s) + 1) > 1$ . Thus the above discussion shows that distortion will improve with increase in frequency at lower voltages and if  $|sC_2(r_b + R_s) + 1| < 1$ . To verify this statement, the voltage was increased to 20 volts and the input resistance changed to  $225\Omega$ . The plots of  $M_2$  and  $M_3$  with frequency, as measured, are given in curves labeled (b) in Fig. 8. It is seen that  $M_2$  and  $M_3$  do not improve with increase in frequency. The small improvement can be attributed to the  $h_{FE}$  terms.

In general, increase in frequency increases distortion; this is especially true for the common base configuration. But as shown above, for certain ranges of frequency and certain values of source impedance, distortion can improve with frequency.

### 5.3.2 Effect of Load Resistance, $R_L$

The load resistance is an external parameter which the circuit designer can vary; hence, it is useful to know its effect on distortion. The second and the third harmonic terms are multiplied by  $1/\sqrt{R_L}$  and  $1/R_L$

terms, respectively; it shows that the distortion can be reduced by increasing  $R_L$ . However, the avalanche terms  $\hat{M}_1$ ,  $\hat{M}_2$ , and  $\hat{M}_3$  are multiplied by the  $R_L$  term, so that increasing  $R_L$  will increase the contribution from the avalanche terms. Thus, an increase in  $R_L$  may increase distortion or reduce it due to cancellation. (The contribution from the collector capacitance terms also increases with increase in load ( $R_L$ ).) Because of the above interaction, for a given set of parameters and input frequencies and the harmonic frequency of interest there exists an optimum load  $R_L$ ; this, of course, can be determined using the computer program. In Fig. 9, the measured values of  $M_2$  and  $M_3$  at different values of  $R_L$  are plotted; in both cases increasing  $R_L$  reduces distortion until the optimum value is reached and then distortion increases with increase in  $R_L$ .

### 5.3.3 Effect of Source Impedance, $Z_o(s)$

Source impedance is another important external parameter. The source impedance affects the exponential nonlinearities  $K_2/K_1^2$  in (28) and  $K_3/K_1^3$  in (29) by the factor

$$\frac{(r_b + Z_o(s))sC_2 + 1}{(r_b + Z_o(s)) \cdot [sC_2 + K_1(1 - \alpha_1)] + 1}.$$

At low frequencies, this nonlinearity is reduced by the factor  $1/[(1 - \alpha)(R_o + r_b)K_1 + 1]$ . Thus, an increase in  $R_o$  will reduce distortion from this source. However, the contribution from other nonlinearities are increased by

$$\frac{1 + K_1(R_o + r_b)}{1 + K_1(R_o + r_b)(1 - \alpha_1)}.$$

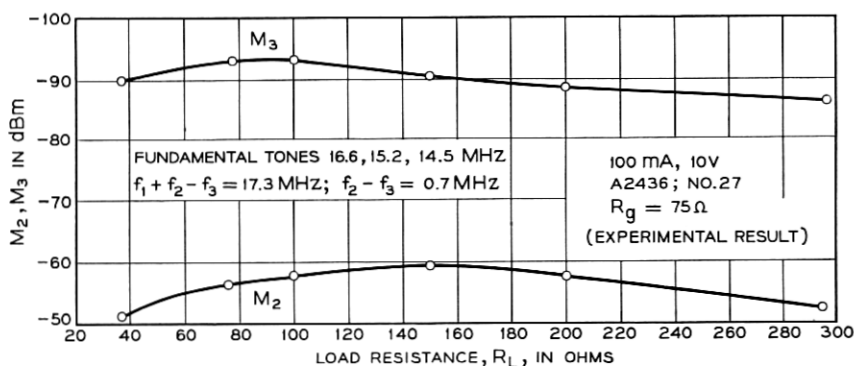


Fig. 9—Variation of  $M_2$ ,  $M_3$  with load resistance.

It is seen from this expression that if  $K_1(R_o + r_b)(1 - \alpha_1)$  is greater than 1, the other nonlinearities are not affected by the increase in  $R_o$ . Thus, if the exponential term is dominant, increasing  $R_o$  reduces distortion at low frequencies. At higher harmonic frequencies if  $|\underline{g}C_2(Z_o(\underline{g}) + r_b)| > 1$ , the distortion terms are independent of the source impedance since the  $[r_b + Z_o(\underline{g})]$  term in the numerator and denominator cancel. This is well illustrated in the measured results of Fig. 10. The second harmonic frequency being 0.7 MHz,  $|\underline{g}C_2(R_o + r_b)|$  is not much greater than one up to  $R_o = 100\Omega$ ; hence, the second harmonic distortion improves with increase in source resistance up to  $140\Omega$ . Further increase in  $R_o$  does not cause much change in distortion. The third harmonic frequency is 17.3 MHz; hence, a change in  $R_o$  does not affect  $M_3$  appreciably. ( $|\underline{g}C_2 \cdot (R_o + r_b)| > 1$ )

#### 5.3.4 Effect of Bias Current

Increase in bias current usually reduces distortion due to the following reasons. The increase in emitter bias current reduces the exponential terms

$$\frac{K_2}{(K_1)^2} \left( \alpha \frac{1}{I_E} \right) \quad \text{and} \quad \frac{K_3}{(K_1)^3} \left[ \alpha \frac{1}{(I_E)^2} \right].$$

Fig. 11 shows the effect of bias current on  $h_{FE}$  terms;  $\alpha_2$  decreases with increase in  $I_C$  by

$$\frac{1}{I_C} \log \left( \frac{I_C e}{I_{C \max}} \right);$$

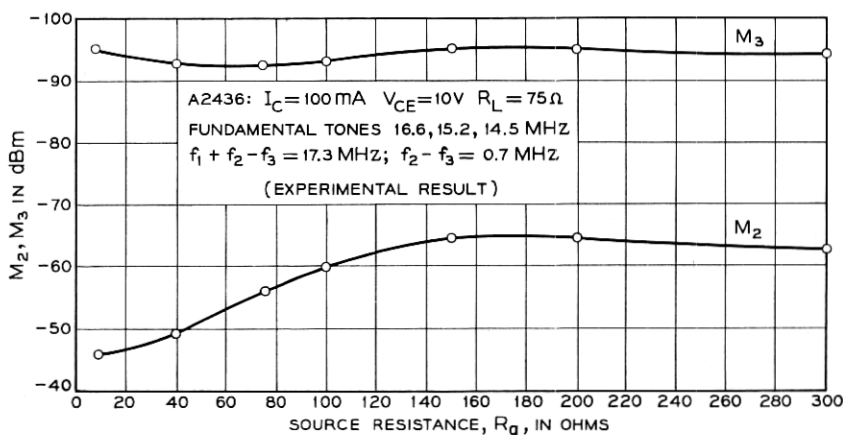


Fig. 10—Variation of  $M_2$ ,  $M_3$  with source resistance.

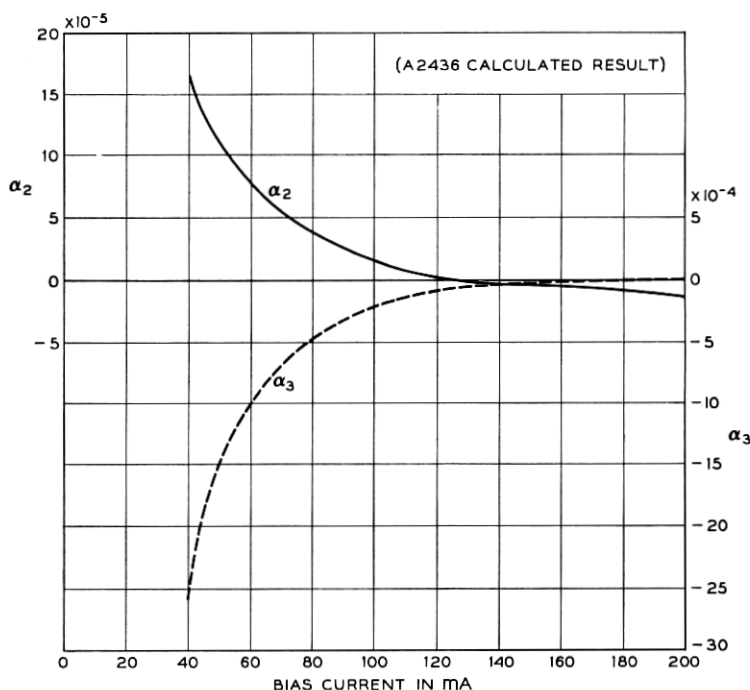
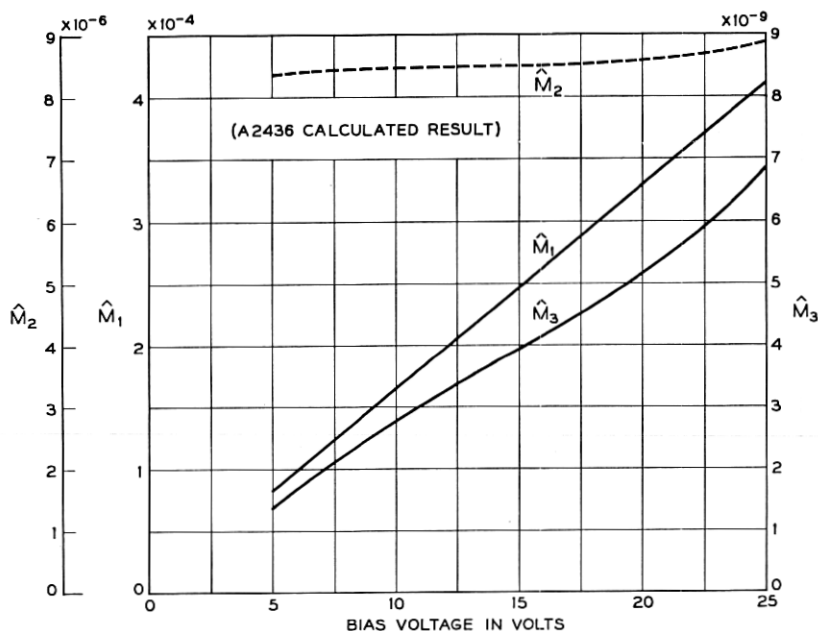
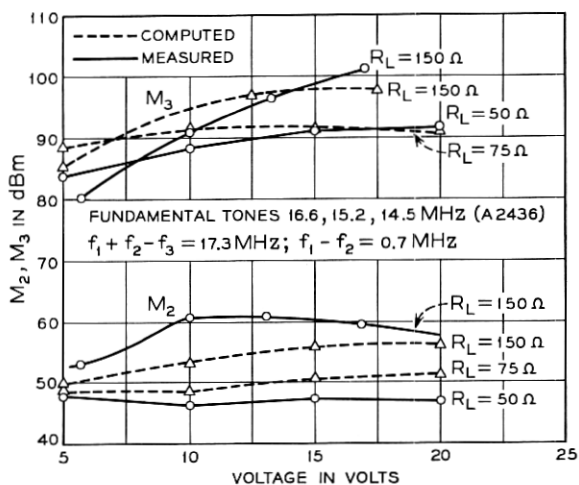


Fig. 11 — Variation of  $\alpha_2$ ,  $\alpha_3$  with bias current.

it becomes zero at  $I_C = I_{C \max}/e$ , and then becomes negative, and increases with further increase in  $I_C$ . The coefficient  $\alpha_3$  decreases with bias current  $I_E$ . Thus, in general, an increase in bias current has the effect of reducing both the second and third harmonic distortion (Fig. 7) (at least until  $\alpha_2 = 0$ ).

### 5.3.5 Effect of Bias Voltage

Whereas exponential and  $h_{FE}$  terms are functions of bias current, the avalanche and collector capacitance nonlinearities are affected by the bias voltage. The coefficient  $\tilde{M}_2$  increases with the voltage; but  $\tilde{M}_1$  and  $\tilde{M}_3$  increase much more rapidly (Fig. 12). (Both the collector capacitance nonlinear coefficients  $\gamma_2$ ,  $\gamma_3$  decrease with the increase in bias voltage.) The effects of change in bias voltage are especially pronounced at higher load resistance since avalanche (and collector capacitance) terms become dominant. The third harmonic distortion decreases more with the increase in voltage (Fig. 13) than the second harmonic distortion does.

Fig. 12—Variation of  $\hat{M}_1$ ,  $\hat{M}_2$ ,  $\hat{M}_3$  with bias voltage.Fig. 13—Variation of  $M_2$ ,  $M_3$  with voltage at 100 mA.



The physical significance of the closed form expressions has been qualitatively discussed. Precise quantitative estimates can and have been obtained using the computer program. For example, the effect of varying linear parameters by fifty percent of their original values was studied. The results show that the distortion does not critically depend on the linear parameters (Figs. 14 to 17). The other transistor parameters such as  $I_{C \max}$ ,  $V_{CE0}$ ,  $n$ , etc., can also be varied.

## VI. ANALYSIS OF CASCADED TRANSISTORS

It is often stated that in a multi-stage amplifier, the output stage alone determines the over-all distortion. Even though this statement is true to some extent, it is frequently found in practice that the effects

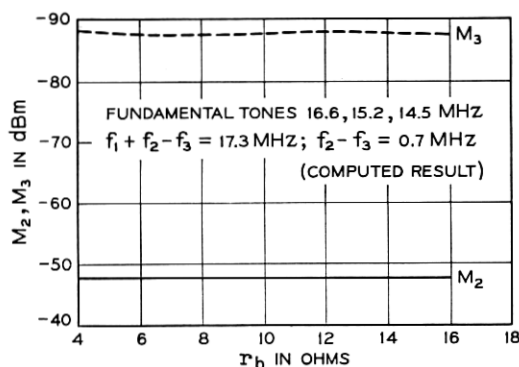


Fig. 14—Variation of  $M_2$ ,  $M_3$  with  $r_b$ .

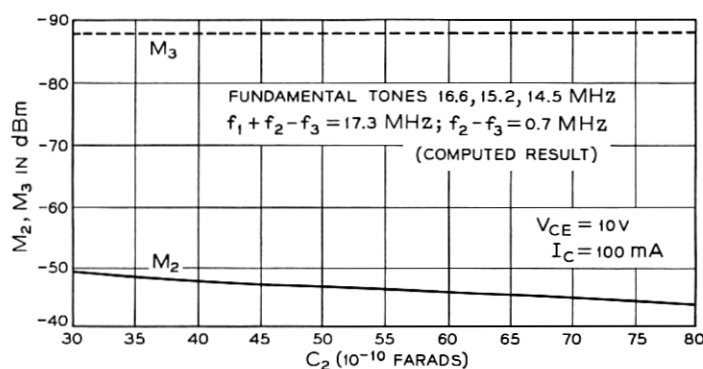
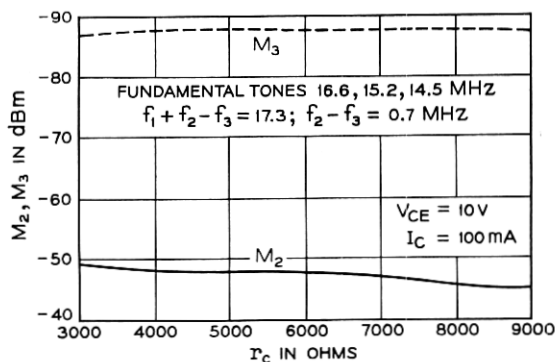
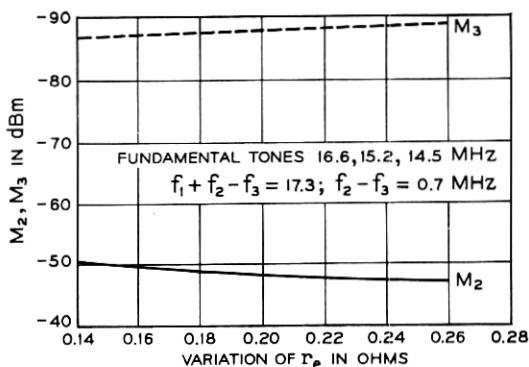


Fig. 15—Variation of  $M_2$ ,  $M_3$  with  $C_2$ .

Fig. 16 — Variation of  $M_2$ ,  $M_3$  with  $r_e$ .Fig. 17 — Variation of  $M_2$ ,  $M_3$  with  $r_e$ .

of the previous stages cannot be ignored and sometimes the previous stage is dominant. This is especially true if both minimum noise figure (which requires lower bias current) and modulation requirements are to be met by a two stage amplifier. Two analysis tools based on Volterra series are presented here which enable the study of such cascaded stages.

The first approach makes use of the cascaded formulae mentioned earlier; this method illustrates the cascade phenomenon and permits derivation of simple cascade rules.

Consider two cascaded transistors (Fig. 1); let the output voltage ( $v_2$ ) of the first transistor be denoted by  $D(v_g)$ ; the output voltage ( $v_3$ ) of the second stage by  $E(v_2)$  and  $F(v_g)$ . The aim is to compute the kernels  $F_1(s_1)$ ,  $F_2(s_1, s_2)$ ,  $F_3(s_1, s_2, s_3)$  knowing  $D$  and  $E$ . To calculate  $D(s_1)$ , etc., it is necessary to know the load impedance of the first stage which

is the input impedance of the second transistor. This can be computed; thus, for a given generator impedance and bias conditions,  $D(s_1)$ ,  $D_2(s_1, s_2)$ ,  $D_3(s_1, s_2, s_3)$  can be determined.  $E(s_1)$ ,  $E_2(s_1, s_2)$  and  $E_3(s_1, s_2, s_3)$  can be computed for a given load and bias conditions with  $R_o = 0$  (voltage  $v_2$  is directly impressed across the second). Now expression  $v_3$  in terms of  $v_o$  is given by

$$v_3 = F(v_o) = E(v_2) = E(D(v_o)) = (E \circ D)(v_o). \quad (30)$$

It is seen that  $F$  is related to  $E$  and  $D$  by the cascade formulae whose physical significance is discussed below.

### 6.1 Linear Term

The linear term is given by

$$F_1(s) = D_1(s)E_1(s) \quad (31)$$

which states that the overall gain in dB is the gain of the first stage in dB plus the gain of the last stage in dB.

### 6.2 Second Harmonic Term

The second-degree kernel is given by

$$F_2(s_1, s_2) = E_1(s_1 + s_2)D_2(s_1, s_2) + E_2(s_1, s_2) \prod_{i=1}^2 D_1(s_i). \quad (32)$$

The first term of the formula states that a given harmonic product from the first transistor  $D_2(j\omega_a \pm j\omega_b)$  is amplified by the second transistor at the harmonic frequency  $E_1(j\omega_a \pm j\omega_b)$ . The second term shows that the two fundamental tones are amplified by the first transistor  $[D_1(j\omega_a)D_1(\pm j\omega_b)]$  and then the second transistor acts on these tones to produce distortion  $E_2(j\omega_a, \pm j\omega_b)$ .

Equation (32) is related to the second harmonic distortion ( $M_2$ ) as follows:

$$M_2 = 20 \log \frac{\sqrt{10^{-3}R_L}}{2} \left| \frac{F_2(s_1, s_2)}{F_1(s_1)F_1(s_2)} \right| \quad (33)$$

$$= 20 \log \frac{\sqrt{10^{-3}R_L}}{2} \left| \frac{\left( \frac{D_2(s_1, s_2)}{\prod_{i=1}^2 D_1(s_i)} \right) \frac{E_1(s_1 + s_2)}{E_1(s_1)E_1(s_2)} + \frac{E_2(s_1, s_2)}{\prod_{i=1}^2 E_1(s_i)}}{1} \right|. \quad (34)$$

The second term is the second harmonic distortion of the last stage. The first term expresses the contribution from the first stage; it approxi-

mately equals

$$\left[ \begin{array}{c} \text{First stage second harmonic} \\ \text{distortion in dBm} \end{array} \right] - \left[ \begin{array}{c} \text{gain of the last stage} \\ \text{in dB} \end{array} \right]. \quad (35)$$

This shows that if the gain of the last stage is high, the contribution from the first stage is small. Equation (35) is approximate in two respects; it neglects the frequency effects and the phase addition of the contributions from the first and the second stage. In (35), the second stage gain in question is actually the ratio

$$\frac{E_1(s_1 + s_2)}{E_1(s_1)E_1(s_2)}$$

which involves the two fundamental and the harmonic frequencies. As an example, a shaping network which was introduced increased the gain (18 dB) at the harmonic frequency (0.7 MHz) and decreased the gain at fundamental tones 15.2 MHz (8 dB) and 14.5 MHz (8 dB) with the result the overall distortion was poorer by 34 dB.

### 6.3 Third Harmonic Distortion Term

The third harmonic kernel  $F_3(s_1, s_2, s_3)$  is given by

$$F_3(s_1, s_2, s_3) = E_1(s_1 + s_2 + s_3)D_3(s_1, s_2, s_3) + 2E_2(s_1, s_2 + s_3) \cdot D_1(s_1)D_2(s_2, s_3) + E_3(s_1, s_2, s_3) \prod_{i=1}^3 D_1(s_i). \quad (36)$$

The first term shows that the third harmonic product of the first stage  $[D_3(s_1, s_2, s_3)]$  is amplified by the last stage at the harmonic frequency  $[E_1(s_1 + s_2 + s_3)]$ . The second term is the interaction term; it arises when the second-degree kernel of the last stage  $[E_2(s_1, s_2 + s_3)]$  acts on the sum of the fundamental  $[D_1(s_1)]$  and the second harmonic output of the first stage  $[D_2(s_2, s_3)]$ . The last term shows that the second stage third-degree kernel  $[E_3(s_1, s_2, s_3)]$  acts on the fundamental tones amplified at the respective frequencies by the first stage  $[D_1(s_1)D_1(s_2)D_1(s_3)]$ .

From (36), the overall third harmonic distortion is related to that of the individual transistors by

$$M_3 = 20 \log \frac{1}{4} 10^{-3} R_L \left| \left[ \frac{E_1(s_1 + s_2 + s_3)}{\prod_{i=1}^3 E_1(s_i)} \right] \left[ \frac{D_3(s_1, s_2, s_3)}{\prod_{i=1}^3 D_1(s_i)} \right] + 2 \left[ \frac{E_2(s_1, s_2 + s_3)}{\prod_{i=1}^2 E_1(s_i)} \right] \frac{1}{E_1(s_3)} \left[ \frac{D_2(s_2, s_3)}{\prod_{i=1}^2 D_1(s_i)} \right] + \left[ \frac{E_3(s_1, s_2, s_3)}{\prod_{i=1}^3 E_1(s_i)} \right] \right|. \quad (37)$$

The first term is the contribution from the third harmonic term of the first stage; it is given approximately by

$$\left[ \begin{array}{l} \text{Third harmonic distortion} \\ \text{of the first stage in dBm} \end{array} \right] - 2 \left[ \begin{array}{l} \text{Gain of the last} \\ \text{stage in dB} \end{array} \right]. \quad (38)$$

The interaction term approximately equals

$$\begin{aligned} & \left[ \begin{array}{l} \text{Second harmonic distortion} \\ \text{of the first stage in dBm} \end{array} \right] + \left[ \begin{array}{l} \text{Second harmonic distortion} \\ \text{of the second stage in dBm} \end{array} \right] \\ & + 6 \text{ dB} - \left[ \begin{array}{l} \text{Gain of the last} \\ \text{stage in dB} \end{array} \right]. \end{aligned} \quad (39)$$

The third term in (37) is the third harmonic distortion of the last stage in dBm.

It is seen that the effect of the first stage and the interaction term can be reduced by increasing the gain of the last stage. Equation (39) illustrates that the second harmonic distortion of each stage should be good. This may become a limitation if the first stage is biased at lower currents.

In the above simplified expressions [(38) and (39)] phase addition and frequency effects have not been considered. In (38), 2 (gain in dB) actually represents

$$20 \log \left| \frac{E_1(j\omega_a \pm j\omega_b \pm j\omega_c)}{E_1(j\omega_a)E_1(\pm j\omega_b)E_1(\pm j\omega_c)} \right|.$$

In (39) the second harmonic distortion is to be measured with two tones, one at the fundamental and the other at the harmonic frequency

$$\frac{E_2(s_1, s_2 + s_3)}{E_1(s_1)E_1(s_2 + s_3)}$$

and then multiplied by the ratio of the gain

$$\frac{E_1(s_2 + s_3)}{E_1(s_1)}.$$

Moreover, the kernel must be made symmetrical by taking the average of three possible combinations.

Thus, the simplified expressions (35), (38), and (39) are exact if the transistor performance is not frequency dependent; in general, they

can be used to get a qualitative picture. Equations (34) and (37) are indeed exact and take into account the frequency dependence. The computer program is being extended to calculate (34) and (37).

An alternate approach to calculate the distortion of cascaded stages is to analyze the nonlinear equivalent circuit of cascaded transistors using the nodal technique illustrated in Section IV. The nodal equations are derived first; next each nodal voltage is expressed in terms of the Volterra series of the input voltage; the resulting vector matrix equations are successively solved. Since the procedure is similar, the details are omitted.

Two common-collector stages were cascaded using this approach. (Fig. 18). The measured values at 120 mA, 10 V with 75 ohm source and load impedances were  $-87$  dBm and  $-112$  dBm for the second and the third harmonic distortion, respectively. The computed distortion values are  $-88.7$  dBm for second and  $-116.6$  dBm for third harmonic distortion. Thus, good agreement with experimental result is obtained.

The cascade formulas are simple, physically meaningful and yield rules of thumb to judge the effect of the first stages. The nodal approach is more complicated. However, the advantage of the nodal approach is that it is general and can be used for an amplifier. For example, a cascade of common-emitter and common-collector stages involves five nodes; if shunt feedback is used at the input and at the output, the same program can be used to analyze this amplifier. (Cascade formulas do not take feedback into account.) In general, the nodal approach can be extended to study frequency-dependent nonlinear network with  $n$  nodes, if the nonlinearities are small.

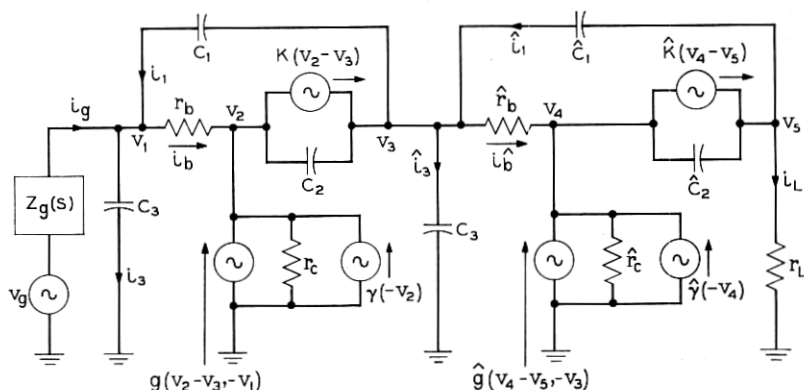


Fig. 18—Common-collector—common-collector nonlinear equivalent circuit.

## VII. ENGINEERING APPLICATIONS

A few pertinent practical applications of the work are described below. These results were either first predicted by the model and then verified in the laboratory or first experimentally observed and then confirmed by analysis.

In the initial design of L4 repeater a common-emitter—common-emitter—common-collector configuration<sup>28</sup> was used in the power amplifier. The third harmonic modulation performance was not as good as desired. This led first to the study of the output common-collector stage. As shown in Fig. 19, the increase in source impedance increases the distortion of the common-collector stage. Since the preceding common-emitter stage output impedance is high, the common-collector performance was not optimum. Secondly, the preceding common-emitter stage was studied because the gain of the common-collector stage is low. (see Section VI) As shown in Fig. 20, increase in load impedance beyond optimum  $R_L$  degrades its performance radically. Since the common-collector input impedance is high, the common-emitter stage performance was not optimum either. Thus, in the redesign work by Ken Tantarelli, the common-collector output stage is not being used.

Another interesting application feature was the improvement in modulation performance of the common-emitter stage with increase in voltage. As shown in Fig. 8, it is a function of load impedance, and at about  $150\Omega$ , maximum improvement was obtained.

New coaxial systems are currently being studied to operate at higher frequencies. Different configurations have been examined for the output stage. The model showed that common-collector and common-base performance is poorer with an increase in frequency and thus the use of these stages as output stages was questioned (unless transistors with

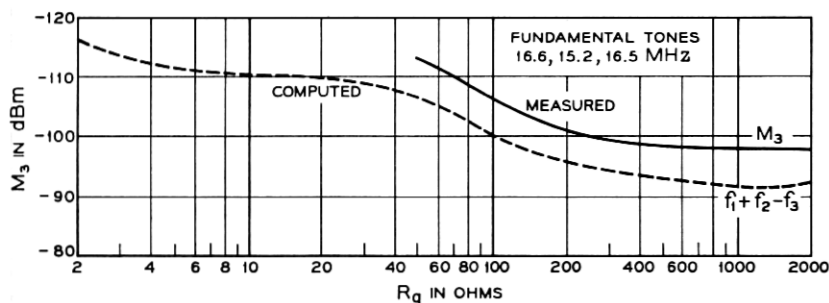


Fig. 19—Common-collector;  $M_3$  variation with source impedance;  $I_c = 120$  mA;  $V_{ce} = 10V$ ;  $R_L = 75 \Omega$ .

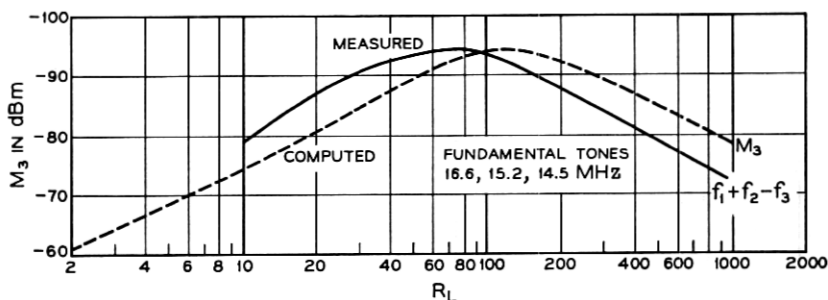


Fig. 20—Common-emitter;  $M_3$  variation with load impedance;  $I_c = 120$  mA;  $V_{ce} = 10$  V;  $R_g = 75 \Omega$ .

higher  $f_i$ 's are available). Recently when a new, high-frequency modulation test set was built, experiments confirmed the prediction. The third harmonic coefficient ( $M_3$ ) for  $a + b - c$  product was 8 dB poorer at 36.5 MHz (due to signals at 36.5 MHz, 40.1 MHz, and 43.1 MHz) compared to the value at 17.3 MHz (due to signals at 14.5, 15.2, and 16.6 MHz). The common-emitter configuration modulation performance suffered only about one dB degradation.

#### VIII. CONCLUSION AND ACKNOWLEDGMENT

This paper has presented a useful analysis tool for investigating the frequency-dependent nonlinear behavior of transistors. A digital program for all the three configurations has been prepared. The results obtained compare favorably with experimental results. The closed-form expressions yield a qualitative picture of distortion. The Volterra series proves useful in examining cascaded transistors; a few rules of thumb are derived and a general nodal analysis which can be extended to cascaded stages with feedback is developed. The practical applications cited show that the technique can be useful in the computer-aided optimal design of linear transistor feedback amplifiers.

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## APPENDIX A

## A.1 Higher-Dimensional Transforms

The second-degree case is illustrated as an example. From (4),

$$y_2(t) = \int_0^t \int_0^t c_2(t - \tau_1, t - \tau_2) \prod_{i=1}^2 x(\tau_i) d\tau_i. \quad (40)$$

If the system is physically realizable,  $c_2(t - \tau_1, t - \tau_2) = 0$ , for  $\tau_i > t$ . Hence, the limits can be extended to  $\infty$ .

$$y_2(t) = \int_0^\infty \int_0^\infty c_2(t - \tau_1, t - \tau_2) \prod_{i=1}^2 x(\tau_i) d\tau_i. \quad (41)$$

Introducing dummy variables  $t_1$  and  $t_2$ , the two-dimensional transform is taken

$$\begin{aligned} Y_2(s_1, s_2) &= \int_0^\infty \int_0^\infty y_2(t_1, t_2) \exp(-s_1 t_1) \exp(-s_2 t_2) dt_1 dt_2 \\ &= \int_0^\infty dt_1 \int_0^\infty dt_2 \left[ \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 c_2(t_1 - \tau_1, t_2 - \tau_2) \right. \\ &\quad \left. \cdot \prod_{i=1}^2 x(\tau_i) d\tau_i \right] \exp(-s_1 t_1) \exp(-s_2 t_2). \end{aligned} \quad (42)$$

Substituting  $t_1 - \tau_1 = m_1$ ,  $t_2 - \tau_2 = m_2$ , and using the fact that  $c_2(m_1, m_2) = 0$  for  $m_i < 0$  yields

$$\begin{aligned} Y_2(s_1, s_2) &= \int_0^\infty dm_1 \int_0^\infty dm_2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 c_2(m_1, m_2) x(\tau_1) x(\tau_2) \\ &\quad \cdot \exp(-s_1 m_1) \exp(-s_1 \tau_1) \exp(-s_2 m_2) \exp(-s_2 \tau_2) \end{aligned} \quad (43)$$

$$= C_2(s_1, s_2) X(s_1) X(s_2). \quad (44)$$

## A.2 The Output of the Kernels to Sinusoidal Inputs

For the second-degree case, consider two sinusoidal signals at frequencies  $f_a$  and  $f_b$ . The input  $x(\tau)$  equals,

$$x(\tau) = \left[ \frac{\exp(j\omega_a \tau) + \exp(-j\omega_a \tau)}{2} \right] + \left[ \frac{\exp(j\omega_b \tau) + \exp(-j\omega_b \tau)}{2} \right]. \quad (45)$$

From (41)

$$y(t) = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 c_2(t - \tau_1, t - \tau_2)$$

$$\cdot \left[ \frac{\exp(j\omega_a \tau_1) + \exp(-j\omega_a \tau_1)}{2} + \frac{\exp(j\omega_b \tau_1) + \exp(-j\omega_b \tau_1)}{2} \right] \\ \cdot \left[ \frac{\exp(j\omega_a \tau_2) + \exp(-j\omega_a \tau_2)}{2} + \frac{\exp(j\omega_b \tau_2) + \exp(-j\omega_b \tau_2)}{2} \right]. \quad (46)$$

Considering one cross term only,

$$\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 c_2(t - \tau_1, t - \tau_2) \\ \cdot \frac{1}{4} \exp(j\omega_a \tau_1) \exp(j\omega_b \tau_2). \quad (47)$$

Substituting  $m_1 = t - \tau_1$ ,  $m_2 = t - \tau_2$  and carrying out the integration yields

$$\frac{1}{4} C_2(j\omega_a; j\omega_b) \exp[j(\omega_a + \omega_b)t]. \quad (48)$$

This term occurs twice as does its complex conjugate.

Hence, the output due to the  $a + b$  term alone is

$$y_{a+b}(t) = |C_2(j\omega_a, j\omega_b)| \cos[(\omega_a + \omega_b)t + \varphi_{a+b}]. \quad (49)$$

The  $2\omega_a$  term and its conjugate occur only once in (46); hence, it is 6 db better. The response of the third harmonic kernel to three sinusoidal inputs is similarly treated.

### A.3 Cascade Relations

For the system shown in Fig. 1, the cascade formula are given below. The cascade relations can be symbolically written as

$$Z = F(x) = E(y) = E(D[x]) = (E \circ D)(x). \quad (50)$$

The formula are

$$F_1(s_1) = E_1(s_1)D_1(s_1) \quad (51)$$

$$F_2(s_1, s_2) = E_1(s_1 + s_2)D_2(s_1, s_2) + E_2(s_1, s_2) \prod_{i=1}^2 D_1(s_i) \quad (52)$$

$$F_3(s_1, s_2, s_3) = E_1(s_1 + s_2 + s_3)D_3(s_1, s_2, s_3) \\ + 2E_2(s_1, s_2 + s_3)D_1(s_1)D_2(s_2, s_3) + E_3(s_1, s_2, s_3) \prod_{i=1}^3 D_1(s_i). \quad (53)$$

A physical interpretation of the formula for cascaded transistors is given in Section VI. The procedure for deriving the cascade relation is as follows: the output  $Z(t)$  of the last stage is expressed in terms of the Volterra series of its input. (Only two terms are considered)

$$Z(t) = \int_0^\infty e_1(t - \tau_1) y(\tau_1) d\tau_1 \\ + \int_0^\infty \int_0^\infty e_2(t - \tau_1, t - \tau_2) \prod_{i=1}^2 y(\tau_i) d\tau_i. \quad (54)$$

The output of the first stage  $y(t)$  is related to its input by

$$y(\tau) = \int_0^\infty d_1(\tau - \sigma_1) x(\sigma_1) d\sigma \\ + \int_0^\infty \int_0^\infty d_2(\tau - \sigma_1, \tau - \sigma_2) \prod_{i=1}^2 x(\sigma_i) d\sigma_i \quad (55)$$

Substituting (55) in (54), terms of the same degree are collected; as an example, the first second-degree term equals

$$\int d\tau e_1(t - \tau) \iint d_2(\tau - \sigma_1, t - \sigma_2) \prod_{i=1}^2 x(\sigma_i) d\sigma_i. \quad (56)$$

Taking the two-dimensional transforms yields

$$E_1(s_1 + s_2) D_2(s_1, s_2) \prod_{i=1}^2 X(s_i). \quad (57)$$

## APPENDIX B

### The Nonlinear Parameters

From (15),

$$I_c = f(I_E) h(V_{CB}). \quad (58)$$

A two-dimensional Taylor's series expansion of (58) is taken;  $i_c$  is expressed by  $K(v_2)$  and  $v_{CB} = v_3 - v_1$ . Hence,

$$i_c = g(v_2, v_3 - v_1) = g_1(v_2, v_3 - v_1) \\ + g_2(v_2, v_3 - v_1) + g_3(v_2, v_3 - v_1), \quad (59)$$

where

$$g_1(v_2, v_3 - v_1) = \alpha_1 \hat{M}_0 K_1 v_2 + \hat{M}_1 (v_3 - v_1), \quad (60)$$

$$g_2(v_2, v_3 - v_1) = \alpha_2 \hat{M}_0 K_1^2 (v_2)^2 + m_2 (v_3 - v_1)^2 \\ + \alpha_1 \hat{M}_0 K_2 (v_2)^2 + \alpha_1 \hat{M}_1 K_1 (v_2) (v_3 - v_1), \quad (61)$$

and

$$g_3(v_2, v_3 - v_1) = \alpha_3 \hat{M}_0 (K_1)^3 (v_2)^3 + m_3 (v_3 - v_1)^3 \\ + \alpha_1 \hat{M}_2 K_1 v_2 (v_3 - v_1)^2 + \alpha_2 \hat{M}_1 K_1^2 v_2^2 (v_3 - v_1) \\ + \alpha_1 \hat{M}_0 K_3 (v_2)^3 + 2\alpha_2 \hat{M}_0 K_1 K_2 (v_2)^3 + \alpha_1 \hat{M}_1 K_2 (v_2)^2 (v_3 - v_1). \quad (62)$$

The avalanche coefficients are given below.

$$\hat{M}_0 = \frac{1}{1 - \left(\frac{V_{CB}}{V_{CBO}}\right)^n} \quad (63)$$

$$\hat{M}_1 = (\hat{M}_0)' = \frac{n(V_{CB})^{n-1}}{(V_{CBO})^n} (\hat{M}_0)^2 \quad (64)$$

$$\hat{M}_2 = \frac{1}{2}(\hat{M}_1)' = \frac{1}{2}(n-1) \frac{\hat{M}_1}{V_{CB}} + \frac{(\hat{M}_1)^2}{\hat{M}_0} \quad (65)$$

$$\hat{M}_3 = \frac{1}{3}(\hat{M}_2)' = \frac{1}{3}\hat{M}_2 \left( \frac{(n-1)}{2V_{CB}} + \frac{2\hat{M}_1}{\hat{M}_0} \right) - \frac{\hat{M}_1}{3} \left[ \frac{(n-1)}{2(V_{CB})^2} + \left( \frac{\hat{M}_1}{\hat{M}_0} \right)^2 \right] \quad (66)$$

The coefficients  $m_1$ ,  $m_2$ , and  $m_3$  equal  $m_i = I_C(\hat{M}_i/\hat{M}_0)$ ;  $i = 1, 2, 3$ , where  $I_C$  is the collector dc bias current.

The  $h_{FE}$  coefficients are given below:

$$\alpha_1 = \frac{h_{FE \max}}{h_{FE \max} + 1 + a \log^2 \frac{I_C}{I_{C \max}} + 2a \log e \log \frac{I_C}{I_{C \max}}} \quad (67)$$

$$\alpha_2 = -\frac{1}{2I_C} \frac{(\alpha_1)^3}{h_{FE \max}} 2a \log e \left[ \log \frac{I_C}{I_{C \max}} + \log e \right] \quad (68)$$

$$\alpha_3 = \frac{\alpha_1}{6} \left[ \frac{-2\alpha_2}{I_C} + 12 \frac{(\alpha_2)^2}{(\alpha_1)^2} - \frac{1}{(I_C)^2} \frac{(\alpha_1)^3}{h_{FE \max}} 2a(\log e)^2 \right] \quad (69)$$

The collector capacitance coefficients are given by

$$\gamma_1 = k(V_{CB})^{-\frac{1}{3}} \quad (70)$$

$$\gamma_2 = \frac{-1}{6} k(V_{CB})^{-\frac{4}{3}} \quad (71)$$

$$\gamma_3 = \frac{k}{27} (V_{CB})^{-\frac{7}{3}} \quad (72)$$

From (62) for  $g_3(v_2; v_3 - v_1)$ ,  $g_3$  is obtained by replacing  $B_1(s_i)$  for  $v_2$  and  $C_1(s_i) - A_1(s_i)$  for  $(v_3 - v_1)$ ; moreover, the kernel must also be symmetrical. Since the procedure is the same as for  $g_2$  it is omitted. The interaction terms are given below:

$$\begin{aligned} g_{23} = & 2\alpha_2 \hat{M}_0 K_1^2 \overline{B_1(s_1)B_2(s_2, s_3)} \\ & + 2m_2 \overline{[C_1(s_1) - A_1(s_1)][C_2(s_2, s_3) - A_2(s_2, s_3)]} \\ & + \alpha_1 \hat{M}_1 K_1 \overline{B_2(s_1, s_2)[C_1(s_3) - A_1(s_3)]} \end{aligned}$$

$$+ \alpha_1 \hat{M}_1 K_1 \overline{B_1(s_1)[C_2(s_2, s_3) - A_2(s_2, s_3)]} \quad (73)$$

$$\hat{\gamma}_{23} = 2\gamma_2 \overline{[C_1(s_1) - B_1(s_1)][C_2(s_2, s_3) - B_2(s_2, s_3)]} \quad (74)$$

where—denotes symmetrical kernel.

#### APPENDIX C

A2436 is an n-p-n silicon transistor with overlay type of construction. Its  $f_T$  ranges from 800 to 1000 MHz. It is a power transistor with current capability of 1 amp and can handle 2.2 watts of power.

Typical parameter values for transistor type 2436 27 at 120 mA, 10V are given below:

$I_c$	= 0.12 amps.
$r_b$	= 13.6 ohms
$r_c$	= 5200 ohms
$C_1$	= $(6)10^{-12}$ farads
$C_2$	= $(3.97)10^{-9}$ farads
$C_3$	= $(9.2)10^{-12}$ farads
$Z_o$	= 50 ohms
$Z_L$	= 50 ohms
$V_{CB}$	= 10 volts
$V_{CBO}$	= 350 volts
$n$	= 2
$re_1$	= 0.2165 ohms
$a$	= 0.38
$h_{FE \max}$	= 122
$I_{C \max}$	= 0.633 amps.

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