# The Excitation of Planar Dielectric Waveguides at p-n Junctions, I

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*The fields excited within a planar dielectric waveguide by an externally incident electromagnetic field are studied in this paper. The dielectric waveguide fills the half space $z > 0$, while the half space $z < 0$ is air. The waveguide is formed by a nonuniform, anisotropic, nonabsorbing, dielectric medium. Different choices of the dielectric tensor for this medium yield different waveguides. Certain models which are particularly relevant to electro-optic diode waveguides and laser diode amplifiers are studied in some detail. An arbitary incident field will, in general, excite not only a finite number of propagating modes, but also a background of continuum modes. Integral representations of the total transmitted field within the waveguide as well as of the reflected field are obtained. The representation of the total transmitted field can be decomposed into a finite sum of discrete propagating modes, a continuum propagating field, and an evanescent field. Explicit evaluation of the fields depends on the solution of a pair of integral equations. In practice, the dielectric tensor of the waveguide differs but little from the dielectric constant of the surrounding material. An approximate solution is found for this case, and numerical results will appear in a following paper.*

## I. INTRODUCTION

Recently there has been great interest in the guiding of light by the p-n junction region in certain piezoelectric semiconductors, for it has been noted that the Pockels effect due to the electric field within the p-n junction can be used to modulate light which propagates parallel to the junction plane.[1–4] This effect was first observed, and has been most intensively studied, with visible light in GaP junctions,[1] but it has also been observed with infrared light in GaAs junctions.[1, 4]

All treatments of the effect so far have assumed that the p-n junction region, which has a higher dielectric constant than the surrounding, normal GaP, behaves like a dielectric waveguide.[1–5] A detailed analysis of this waveguide would require a knowledge of the optical properties in the neighborhood of the junction. However, since these properties change significantly in a fraction of a wavelength, it is extremely difficult to investigate them individually by experimental means. In order to get around this difficulty it has been necessary to adopt an indirect approach based on analyzing a number of different mathematical models and comparing their predictions with experiment.

As part of this program Nelson and McKenna[6] have investigated the possible discrete modes which can propagate in a number of different models and have studied in considerable detail the properties of the lowest-order mode of each polarization. Recent experimental work has made it increasingly clear, however, that a knowledge of the discrete modes alone is not enough to provide an understanding of these p-n junction dielectric waveguides. This is because a beam of light, when focused on the face of a junction waveguide, excites within the waveguide not only a finite number of discrete modes, but also a background of continuum modes. In many cases this background light is intense enough to mask important features of the discrete propagating modes. Thus, unless an understanding of this background light is available, the task of comparing the predictions of different mathematical models with experiment is almost impossible. An understanding of the electromagnetic boundary value problem involved also has great relevance to understanding what happens when light is introduced into a laser diode amplifier.

The purpose of this paper is to study in some detail a class of mathematical models of the excitation of dielectric waveguides. These models are simple enough so that the mathematical analysis can be performed and the background light can be investigated carefully. At the same time, it is felt that the models are realistic enough so that their predictions can be compared with experiment.

The models can be described as follows. The waveguide consists of the half space $z > 0$, as shown in Fig. 1, while the region $z < 0$ is air. The waveguide itself is assumed to be formed by a nonuniform, anisotropic, nonabsorbing dielectric. The components of the dielectric tensor are functions of the coordinate $x$ only, and for each value of $x$ the dielectric tensor is diagonal in the fixed coordinate system shown in Fig. 1. As an example, for the GaP electro-optic diode modulator studied in $NM$ this corresponds to the cases where the junction field is in the [111] or [100] directions. Each such model is determined by its

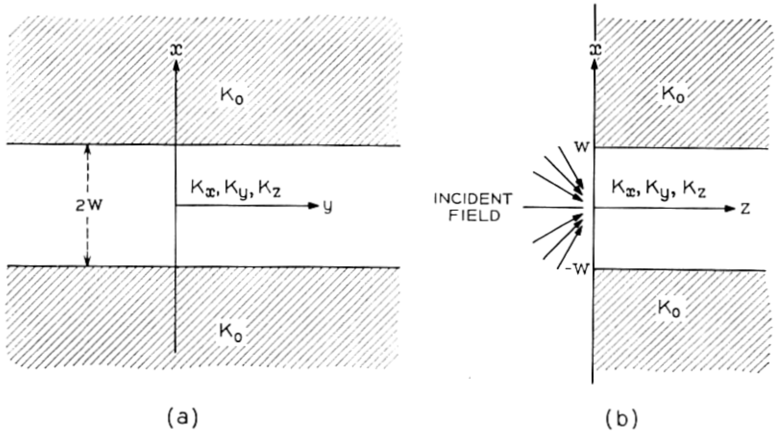


Fig. 1 — Symmetric step model illustrating the coordinate system used in all the models. The dielectric tensor is always diagonal in this fixed coordinate system.

dielectric tensor whose diagonal elements we will denote by $K_n(x)$ $(n = x, y, z)$.

It was shown in *NM* that the amount of absorption encountered in GaP electro-optic diode modulators was too small to affect significantly the shape of the modes. It is, therefore, felt that the study of absorptionless models here is well justified. It was also shown in *NM* that the detailed analytical form of the functions $K_n(x)$ is not important when only the lowest-order discrete mode of each polarization can propagate. The most important features of the discrete modes can be determined by studying models for which the functions $K_n(x)$ are step functions (piece-wise constant). Although it is possible to carry out a good deal of the analysis without specifying the functions $K_n(x)$, the final detailed results naturally depend on the choice of $K_n(x)$. We shall concentrate here on two models, the symmetric step model and the asymmetric step model. The symmetric step model is defined by the equations[6]

$$K_m(x) = K_m , \qquad |x| < w \tag{1}$$

$$= K_0 , \qquad |x| > w \tag{2}$$

and the asymmetric step model is defined by the equations[6]

$$K_m(x) = K_m , \qquad |x| < w \tag{3}$$

$$= K_1 , \qquad x < -w \tag{4}$$

$$= K_2 , \qquad x > w, \tag{5}$$

where $K_2 < K_1$, and $K_m > K_j \geqq 1$, $m = x, y, z$, $j = 0, 1, 2$ (see Fig. 2). In the case of the GaP electro-optic diode modulators there are relations of the form[6]

$$K_m = n^2(1 + \delta_m), \qquad (m = x, y, z) \tag{6}$$

$$K_0 = n^2(1 - \Delta), \qquad K_j = n^2(1 - \Delta_j), \qquad (j = 1, 2). \tag{7}$$

In (6) and (7) $n$ is the index of refraction of normal GaP, the quantities $\delta_m$ are linear in the junction field (the linear electro-optic effect), and $0 \leqq |\delta_m| < \Delta \ll 1$.

In Section II we will write down general integral representations for incident waves in the region $z < 0$, as well as integral representations for the resulting reflected and transmitted fields. These integral representations will involve a number of unknown functions. Some of these functions are determined directly from the structure of the waveguide and are independent of the incident field and the boundary condition at $z = 0$. The remaining unknown functions are determined by the incident field and the boundary conditions at $z = 0$. We show that these functions satisfy a set of linear integral equations. The results of Section II are independent of the specific form of $K_m(x)$ and the incident field. In Section III we explicitly calculate the unknown functions which depend only on the structure of the waveguide for the symmetric and asymmetric step models. In Section IV we obtain approximate solutions of the integral equations for a special class of

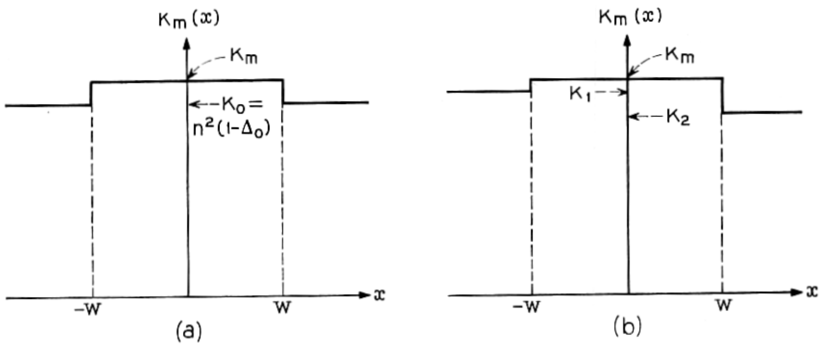


Fig. 2 — (a) The function $K_m(x)$ for the symmetric step model. (b) The function $K_m(x)$ for the asymmetric step model.

waveguide models. The remaining unknown functions are determined for these models in terms of the incident field. In a second paper on this subject we will give asymptotic expansions and numerical results for the fields within the waveguide for the symmetric and asymmetric models when they are excited by a Gaussian incident wave.

## II. A GENERAL DESCRIPTION OF THE FIELDS

In this section we study formal solutions of Maxwell's equations which describe an incident wave in the region $z < 0$ moving to the right and striking the waveguide from the left, a reflected wave in the region $z < 0$, and a transmitted wave in the region $z > 0$. The fields are assumed to be monochromatic and independent of the coordinate $y$. We write for the total electric and magnetic field vectors

$$\mathbf{E}(x, z, t) = \mathrm{Re}\,(\mathbf{e}(x, z)e^{i\omega t}), \qquad \mathbf{H}(x, z, t) = \mathrm{Re}\,(\mathbf{h}(x, z)e^{i\omega t}), \tag{8}$$

and for the total electric displacement and magnetic induction vectors

$$\mathbf{D}(x, z, t) = \mathrm{Re}\,(\mathbf{d}(x, z)e^{i\omega t}), \qquad \mathbf{B}(x, z, t) = \mathrm{Re}\,(\mathbf{b}(x, z)e^{i\omega t}), \tag{9}$$

where Re denotes the real part and $\omega = 2\pi f$ is the angular frequency of the radiation. Then Maxwell's equations are

$$\begin{aligned} \nabla \times \mathbf{e} &= -i\omega \mathbf{b}, & \nabla \cdot \mathbf{d} &= 0, \\ \nabla \times \mathbf{h} &= i\omega \mathbf{d}, & \nabla \cdot \mathbf{b} &= 0. \end{aligned} \tag{10}$$

From our assumptions about the model, the constitutive equations can be written as

$$\mathbf{b} = \mu_0 \mathbf{h}, \qquad \mathbf{d} = \epsilon_0 \mathbf{K} \cdot \mathbf{e}, \tag{11}$$

where $\epsilon_0$ and $\mu_0$ are, respectively, the permittivity and permeability of free space. The dielectric matrix $\mathsf{K} = \mathsf{K}(x, z)$ is the unit matrix for $z < 0$, and for $z > 0$ it is a diagonal matrix whose diagonal elements, $K_n(x)$ $(n = x, y, z)$, are functions of $x$ only. It is a straightforward matter to show that any solution of Maxwell's equations satisfying the above assumptions can be written as the linear combination of a *TE* solution and a *TM* solution. We consider these solutions separately.

### 2.1 *TE Fields*

We first look for *TE* solutions having the form

$$\mathbf{e}(x, z) = [0, e_y(x, z), 0], \qquad \mathbf{h}(x, z) = [h_x(x, z), 0, h_z(x, z)]. \tag{12}$$

In the region $z < 0$, $e_y$ must satisfy the Helmholtz equation

$$\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_y}{\partial z^2} + k^2 e_y = 0, \tag{13}$$

where the free-space wavenumber $k$ is defined by

$$k = \omega(\epsilon_0 \mu_0)^{\frac{1}{2}} = 2\pi/\lambda$$

and $\lambda$ is the free-space wavelength. The total field in $z < 0$ is the sum of the incident field $e_y^{(i)}$ and the reflected field $e_y^{(r)}$ and both $e_y^{(i)}$ and $e_y^{(r)}$ are solutions of (13). In the region $z > 0$ there is only the transmitted field which satisfies the equation

$$\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_y}{\partial z^2} + k^2 K_y(x) e_y = 0. \tag{14}$$

A solution of (13), which can be found by separation of variables, and which describes a general incident field due to sources in $z < 0$ at a finite distance from the plane $z = 0$, is

$$e_y^{(i)}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{E}_y^{(i)}(l) \exp\{-i\Omega(l)z - ilx\}\, dl, \tag{15}$$

where

$$\begin{aligned} \Omega(l) &= +\sqrt{k^2 - l^2}, \qquad |\, l \,| \leqq k \\ &= -i\sqrt{l^2 - k^2}, \qquad |\, l \,| \geqq k. \end{aligned} \tag{16}$$

The components of the magnetic field vector can be obtained with the aid of Maxwell's equations by differentiating (15). Let $\Sigma(z_0)$ denote the strip $-\infty < x < \infty$, $0 \leqq y \leqq 1$, lying in the plane $z = z_0$. Then the time averaged power incident on $\Sigma(z)$, $z \leqq 0$, is independent of $z$ and is

$$\begin{aligned} P_i &= -\tfrac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} e_y^{(i)}(x, z) h_x^{(i)}(x, z)^* \, dx \\ &= (4\pi\omega\mu_0)^{-1} \int_{-k}^{k} \sqrt{k^2 - l^2}\, |\, \mathcal{E}_y^{(i)}(l) \,|^2 \, dl, \end{aligned} \tag{17}$$

where * denotes complex conjugation. We will assume that

$$\int_{-\infty}^{\infty} |\, \mathcal{E}_y^{(i)}(l) \,|^2 \, dl < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |\, \Omega(l) \,|\, |\, \mathcal{E}_y^{(i)}(l) \,|^2 \, dl < \infty.$$

(15) is to describe an incident field due to sources at $z = -\infty$, then it is easy to see that we must have $\mathcal{E}_y^{(i)}(l) = 0$, $|\, l \,| > k$. Since the incident field must be specified, it will always be assumed that $\mathcal{E}_y^{(i)}(l)$ is known.

A solution of (13) describing a general wave reflected from the waveguide surface $z = 0$ is

$$e_y^{(r)}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{E}_y^{(r)}(l) \exp \{i\Omega(l)z - ilx\} \, dl. \tag{18}$$

We will always assume that the source of the incident radiation is perfectly absorbing so that $e_y^{(i)}(x, z) + e_y^{(r)}(x, z)$ is the total field in the region between the source and the surface of the waveguide at $z = 0$. It will be seen that because of the boundary conditions at $z = 0$, $\mathcal{E}_y^{(i)}(l)$ generally does not vanish outside some finite $l$ interval. Because of the factor $\exp \{i\Omega(l)z\}$, that part of the integral in (18) between the limits $-k$ and $k$, $\int_{-k}^{k} \{ \ \} \, dl$, represents a traveling field, while the remainder of the integral represents an evanescent field which damps out very rapidly in the negative $z$ direction. The time averaged power reflected back through the strip $\Sigma(z)$, $z \leqq 0$, is

$$P_r = (4\pi\omega\mu_0)^{-1} \int_{-k}^{k} \sqrt{k^2 - l^2} \, | \mathcal{E}_y^{(r)}(l) |^2 \, dl. \tag{19}$$

We now turn to the transmitted field. We use the method of separation of variables, and we seek transmitted waves which are linear superpositions of solutions of (14) of the form

$$e_y^{(t)}(x, z) \approx e_\nu(x) \exp \{-i\sqrt{-\nu} z\}. \tag{20}$$

In (20) $\nu$ is a real separation parameter, and if $\nu > 0$, $\sqrt{-\nu} = -i\sqrt{\nu}$. If (20) is substituted into (14) we get the eigenvalue equation

$$\frac{d^2 e_\nu}{dx^2} + (k^2 K_y(x) + \nu) e_\nu = 0. \tag{21}$$

Equation (21) defines a singular, self-adjoint, second-order boundary value problem on the interval $-\infty < x < \infty$. The theory of this equation is well known, and we refer the reader to Coddington and Levinson[7] for a detailed treatment. We give a summary here of those properties of such equations which we will need.

For all the models under consideration, the functions $K_m(x)$ are positive, bounded functions, which are bounded away from zero, and which are differentiable except for at most a finite number of step discontinuities. Equation (21), therefore, defines a problem which is called limit-point type at both plus and minus infinity. This means that for arbitrary, complex $\nu$, (21) possesses exactly one solution (up to a constant factor) which is square integrable over $0 < x < \infty$, and exactly one solution which is square integrable over $-\infty < x < 0$.

For a given real number $\nu$, let $\varphi_1(x, \nu)$ and $\varphi_2(x, \nu)$ be the two solutions of (21) which satisfy the conditions that $\varphi_j(x, \nu)$ and $\varphi_j'(x, \nu)$ be continuous and which satisfy the initial conditions

$$\varphi_1(0, \nu) = 1, \qquad \varphi_1'(0, \nu) = 0, \tag{22}$$

$$\varphi_2(0, \nu) = 0, \qquad \varphi_2'(0, \nu) = 1, \tag{23}$$

where $' = d/dx$. Equation (21) also determines a $2 \times 2$ matrix-valued function $\rho(\nu)$, $-\infty < \nu < \infty$, having the following properties: *(i)* $\rho(\nu)$ is Hermitian ($\rho_{jk}(\nu) = \rho_{kj}^*(\nu)$). *(ii)* $\rho(\nu) - \rho(u)$ is positive semidefinite if $\nu > \mu$. *(iii)* $\rho_{jk}(\nu)$ is of bounded variation on every finite interval. The matrix $\rho(\nu)$ is called the spectral density matrix and its construction is outlined in Section III. Then if $f(x)$ is any square integrable function ($\int_{-\infty}^{\infty} | f(x) |^2 \, dx < \infty$), we can define two transforms of $f(x)$, $g_j(\nu)$ ($j=1, 2$), such that

$$\lim_{L\to\infty} \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \left\{ g_j(\nu) - \int_{-L}^{L} f(x)\varphi_j(x, \nu) \, dx \right\} \cdot \left\{ g_k(\nu) - \int_{-L}^{L} f(x)\varphi_k(x, \nu) \, dx \right\}^* d\rho_{jk}(\nu) = 0. \tag{24a}$$

This is referred to as convergence in the mean with respect to the measure $\rho(\nu)$, and in the manner of Fourier transforms of $\mathfrak{L}^2$ functions, we write

$$g_j(\nu) = \int_{-\infty}^{\infty} f(x)\varphi_j(x, \nu) \, dx \qquad (j = 1, 2). \tag{24b}$$

In terms of these transforms, the Parseval equality

$$\int_{-\infty}^{\infty} | f(x) |^2 \, dx = \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} g_j(\nu)^* g_k(\nu) \, d\rho_{jk}(\nu), \tag{25}$$

and the expansion

$$f(x) = \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \varphi_j(x, \nu) g_k(\nu) \, d\rho_{jk}(\nu) \tag{26}$$

are valid. Equation (26) is defined in terms of convergence in the mean. The set of real points $\nu$ at which the functions $\rho_{jk}(\nu)$ are nonconstant is the spectrum of (21). The set of points where any $\rho_{jk}(\nu)$ is discontinuous is the point spectrum and for each such value of $\nu$, (21) has exactly one square integrable solution. The continuous spectrum is the set of points of continuity of $\rho(\nu)$ which are in the spectrum. In Section III we will exhibit the spectral density matrices for two important models.

We can now write down a formal expression for the transmitted field:

$$e_y^{(t)}(x, z) = \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \exp\{-i\sqrt{-\nu}z\}\varphi_j(x, \nu)g_k(\nu)\, d\rho_{jk}(\nu). \tag{27}$$

The two initial value solutions $\varphi_j(x, \nu)$ $(j = 1, 2)$, as well as the functions $\rho_{jk}(\nu)$ $(j, k = 1, 2)$ are determined, independently of the boundary conditions at $z = 0$, by (21) and we can assume that they are known. The two unknown functions $g_j(\nu)$ $(j = 1, 2)$ in (27) are determined by the field at $z = 0$, since with the aid of (24) we can write

$$g_j(\nu) = \int_{-\infty}^{\infty} e_y^{(t)}(x, 0)\varphi_j(x, \nu)\, dx. \tag{28}$$

It is clear that because of the factor $\exp\{-i\sqrt{-\nu z}\}$, the parts of the integrals $\int_{-\infty}^{0}$ in (27) represent the propagating portion of the transmitted field, while the parts $\int_0^\infty$ reprent the evanescent portion of the transmitted field. With the aid of the Parseval relation, (25), we can write down an expression for the time averaged power transmitted across any $\Sigma(z)$, $z \geqq 0$,

$$P_t = (2\omega\mu_0)^{-1} \sum_{j,k=1}^{2} \int_{-\infty}^{0} \sqrt{-\nu}g_j\,(\nu)^* g_k(\nu)\, d\rho_{jk}(\nu). \tag{29}$$

We now make use of the conditions that $e_y(x, z)$ and $h_x(x, z)$ must be continuous at $z = 0$ in order to write down a set of integral equations which determines $\mathcal{E}_y^{(r)}(l)$, $g_1(\nu)$, and $g_2(\nu)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} [\mathcal{E}_y^{(i)}(l) + \mathcal{E}_y^{(r)}(l)]e^{-ilx}\, dl = \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \varphi_j(x, \nu)g_k(\nu)\, d\rho_{jk}(\nu), \tag{30}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega(l)[\mathcal{E}_y^{(i)}(l) - \mathcal{E}_y^{(r)}(l)]e^{-ilx}\, dl$$
$$= \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \sqrt{-\nu}\varphi_j\,(x, \nu)g_k(\nu)\, d\rho_{jk}(\nu). \tag{31}$$

Although there appear to be only two equations in three unknown functions, because of (24) and (26), (30) and (31) are sufficient to determine the unknown functions. We indicate formally why this is true, although it will be clear from the results of Section IV that this scheme must be modified in specific cases. We do not go into these modifications, because in Section IV we use a different scheme to get approximate solutions. With the aid of (24b), solve (30) and (31) for

$g_j(\nu)$, giving the four equations

$$g_j(\nu) = \int_{-\infty}^{\infty} \varphi_j(x, \nu)\, dx \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathcal{E}_y^{(i)}(l) + \mathcal{E}_y^{(r)}(l)] e^{-ilx}\, dl, \tag{32}$$

$$\sqrt{-\nu}\, g_j(\nu) = \int_{-\infty}^{\infty} \varphi_j(x, \nu)\, dx \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega(l)[\mathcal{E}_y^{(i)}(l) - \mathcal{E}_y^{(r)}(l)] e^{-ilx}\, dl, \tag{33}$$

for $j = 1, 2$. On eliminating $g_j(\nu)$ between these equations we get the two equations in the unknown $\mathcal{E}_y^{(r)}(l)$.

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi_j(x, \nu)\, dx \frac{1}{2\pi} \int_{-\infty}^{\infty} (\sqrt{-\nu} + \Omega(l))\mathcal{E}_y^{(r)}(l) e^{-ilx}\, dl, \\ &= \int_{-\infty}^{\infty} \varphi_j(x, \nu)\, dx \frac{1}{2\pi} \int_{-\infty}^{\infty} (-\sqrt{-\nu} + \Omega(l))\mathcal{E}_y^{(i)}(l) e^{-ilx}\, dl, \end{aligned} \tag{34}$$

for $j = 1, 2$. Now from (26) we can write $f(x) = f_1(x) + f_2(x)$ where

$$f_k(x) = \sum_{j=1}^{2} \int_{-\infty}^{\infty} \varphi_j(x, \nu) g_k(\nu)\, d\rho_{jk}(\nu) \qquad (k = 1, 2), \tag{35}$$

$$\int_{-\infty}^{\infty} f_k(x)\varphi_j(x, \nu)\, dx = \delta_{jk} g_j(\nu) \qquad (j, k = 1, 2), \tag{36}$$

and $\delta_{jk}$ is the Kronecker delta function. It is this decomposition of an arbitrary $f(x)$ into components lying in the two subspaces spanned by $\varphi_1(x, \nu)$ and $\varphi_2(x, \nu)$ which is reflected in the two integral equations (34). The solution of (34) with given $j$ yields the component of the reflected field lying in the subspace spanned by the corresponding $\varphi_j(x, \nu)$. Let $\mathcal{E}_{yj}^{(r)}(l)$, $j = 1, 2$, denote the two solutions. Then $\mathcal{E}_y^{(r)}(l) = \mathcal{E}_{y1}^{(r)}(l) + \mathcal{E}_{y2}^{(r)}(l)$ describes the total reflected field. With this result $g_j(\nu)$ $(j = 1, 2)$ can be obtained from either (32) or (33). We have been unable to obtain exact solutions for the integral equations (30)–(31) for any of the models considered here. However, in Section IV approximate solutions are obtained for certain situations of interest.

### 2.2 *TM Fields*

We next seek TM solutions of Maxwell's equations of the form

$$\mathbf{e}(x, z) = (e_x(x, z), 0, e_z(x, z)), \qquad \mathbf{h}(x, z) = (0, h_y(x, z), 0). \tag{37}$$

In the region $z < 0$, $h_y$ must satisfy (13). In the region $z > 0$, $h_y$ must satisfy the equation

$$\frac{\partial}{\partial x}\left\{(1/K_z(x)) \frac{\partial h_y}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{(1/K_x(x)) \frac{\partial h_y}{\partial z}\right\} + k^2 h_y = 0. \tag{38}$$

Just as for the TE fields, a general incident field due to sources in $z < 0$ at a finite distance from the plane $z = 0$ is

$$h_y^{(i)}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{H}_y^{(i)}(l) \exp\{-i\Omega(l)z - ilx\}\, dl. \tag{39}$$

The time averaged power due to this wave which is incident on $\Sigma(z)$, $z \leqq 0$, is

$$P_i = \tfrac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} e_x^{(i)}(x, z) h_y^{(i)}(x, z)^*\, dx = (4\pi\omega\epsilon_0)^{-1} \int_{-k}^{k} \Omega(l) \mid \mathcal{H}_y^{(i)}(l) \mid^2 dl. \tag{40}$$

We assume that $\int_{-\infty}^{\infty} \mid \mathcal{H}_y^{(i)}(l) \mid^2 dl < \infty$ and $\int_{-\infty}^{\infty} \mid \Omega(l) \mid \mid \mathcal{H}_y^{(i)}(l) \mid^2 dl < \infty$. As for the *TE* field if the sources of the *TM* field are at $z = -\infty$ then $\mathcal{H}_y^{(i)}(l) = 0$, $\mid l \mid > k$. Furthermore, it will always be assumed that $\mathcal{H}_y^{(i)}(l)$ is known.

A solution of (13) describing a general reflected wave is

$$h_y^{(r)}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{H}_y^{(r)}(l) \exp\{i\Omega(l)z - ilx\}\, dl. \tag{41}$$

Just as in the case of the *TE* field, $h_y^{(r)}(x, z)$ can be split into a propagating field and an evanescent field. The time averaged power reflected back through the strip $\Sigma(z)$, $z \leqq 0$, is

$$P_r = (4\pi\omega\epsilon_0)^{-1} \int_{-k}^{k} \Omega(l) \mid \mathcal{H}_y^{(r)}(l) \mid^2 dl. \tag{42}$$

The transmitted field is again treated by separation of variables, and we write

$$h_y^{(t)}(x, z) \approx h_y(x) \exp\{-i\sqrt{-\nu}\, z\}.$$

Then $h_y(x)$ satisfies the eigenvalue equation

$$K_x(x) \frac{d}{dx} \left\{ (1/K_z(x)) \frac{dh_y}{dx} \right\} + (k^2 K_x(x) + \nu) h_y = 0. \tag{43}$$

Equation (43) is not in the canonical form of a self-adjoint boundary value problem. However, if we make the change of variables

$$u = \int_0^x \{K_x(t)\}^{-1}\, dt, \tag{44}$$

(43) is transformed to the equation

$$\frac{d}{du} \left[ \{K_x(u) K_z(u)\}^{-1} \frac{dh_y}{du} \right] + (k^2 K_x(u) + \nu) h_y = 0. \tag{45}$$

This equation defines a self-adjoint boundary value problem,[8] and even though the function $\{K_x(u)K_z(u)\}^{-1}$ may have step discontinuities, the techniques of Ref. 7 can be shown to be still valid. Equation (45) is limit-point at $u = \pm\infty$, and so on transforming back to the variable $x$, the following statements can be made.

For a given real number $\nu$, let $\psi_1(x, \nu)$ and $\psi_2(x, \nu)$ be the two solutions of (43) which satisfy the requirements that

$$\psi_j(x, \nu) \quad \text{and} \quad \{K_z(x)\}^{-1}\psi_j'(x, \nu)$$

be continuous for all $x$, and which satisfy the initial conditions

$$\psi_1(0, \nu) = 1, \qquad (1/K_z(0))\psi_1'(0, \nu) = 0, \tag{46}$$

$$\psi_2(0, \nu) = 1, \qquad (1/K_z(0))\psi_2'(0, \nu) = 1. \tag{47}$$

Equation (43) determines a $2 \times 2$ spectral density matrix $\sigma(\nu)$ whose construction is given in Section III. If $f(x)$ is any square integrable function of $x$, we define two transforms of $f(x)$,

$$h_j(\nu) = \int_{-\infty}^{\infty} f(x)\psi_j(x, \nu)\{K_z(x)\}^{-1}\,dx \qquad (j = 1, 2), \tag{48}$$

where equality in (48) is defined in terms of convergence in the mean with respect to the measure $\sigma(\nu)$. In terms of these transforms, the Parseval equality

$$\int_{-\infty}^{\infty} |\, f(x)\,|^2\, \{K_z(x)\}^{-1}\,dx = \sum_{j,k=1}^{2}\int_{-\infty}^{\infty} h_j(\nu)h_k(\nu)^*\,d\sigma_{jk}(\nu), \tag{49}$$

and the expansion

$$f(x) = \sum_{j,k=1}^{2}\int_{-\infty}^{\infty} \psi_j(x, \nu)h_k(\nu)\,d\sigma_{jk}(\nu). \tag{50}$$

are valid. The last equality is again defined in the sense of convergence in the mean.

We can write down a formal expression for the transmitted field

$$h_y^{(t)}(x, z) = \sum_{j,k=1}^{2}\int_{-\infty}^{\infty} \exp\{-i\sqrt{-\nu}\,z\}\psi_j(x, \nu)h_k(\nu)\,d\sigma_{jk}(\nu). \tag{51}$$

The two initial value solutions $\psi_j(x, \nu)$ $(j = 1, 2)$, as well as the functions $\sigma_{jk}(\nu)$ $(j, k = 1, 2)$ are determined, independently of the boundary conditions at $z = 0$, by (43) and we can assume that they are known. The two unknown functions $h_j(\nu)$ $(j = 1, 2)$ in (51) are determined by the field at $z = 0$ since with the aid of (48) we can write

$$h_j(\nu) = \int_{-\infty}^{\infty} h_y^{(t)}(x, 0)\psi_j(x, \nu)\{K_z(x)\}^{-1}\, dx. \tag{52}$$

With the aid of the Parseval relation, (49), we can write down an expression for the time averaged power transmitted across any $\Sigma(z)$, $z \geqq 0$:

$$P_t = (2\omega\epsilon_0)^{-1} \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \sqrt{-\nu}\, h_j(\nu)^* h_k(\nu)\, d\sigma_{jk}(\nu). \tag{53}$$

We can now make use of the conditions that $e_x(x, z)$ and $h_y(x, z)$ must be continuous at $z = 0$ in order to write down a set of integral equations from which $\mathcal{H}_y^{(r)}(l)$, $h_1(\nu)$, and $h_2(\nu)$ can be determined.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} [\mathcal{H}_y^{(i)}(l) + \mathcal{H}_y^{(r)}(l)]e^{-ilx}\, dl = \sum_{j,k=1}^{2} \int_{-\infty}^{\infty} \psi_j(x, \nu)h_k(\nu)\, d\sigma_{jk}(\nu), \tag{54}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega(l)[\mathcal{H}_y^{(i)}(l) - \mathcal{H}_y^{(r)}(l)]e^{-ilx}\, dl$$
$$= \sum_{j,k=1}^{2} \{1/K_z(x)\} \int_{-\infty}^{\infty} \sqrt{-\nu}\, \psi_j(x, \nu)h_k(\nu)\, d\sigma_{jk}(\nu). \tag{55}$$

Just as in the case of the TE field, the solution of (54) and (55) reduces to the solution of the two integral equations

$$\int_{-\infty}^{\infty} \psi_j(x, \nu)\, dx\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \{\sqrt{-\nu}/K_z(x) + \Omega(l)\}\mathcal{H}_y^{(r)}(l)e^{-ilx}\, dl$$
$$= \int_{-\infty}^{\infty} \psi_j(x, \nu)\, dx\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \{-\sqrt{-\nu}/K_z(x) + \Omega(l)\}$$
$$\cdot \mathcal{H}_y^{(i)}(l)e^{-ilx}\, dl, \qquad (j = 1, 2). \tag{56}$$

## III. THE SPECTRAL DENSITY MATRIX FOR SEVERAL MODELS

### 3.1 *General Outline of the Construction*

In Section II it was shown that the determination of the transmitted field for a given model depended on a knowledge of the initial value solutions $\varphi_j(x, \nu)$ and $\psi_j(x, \nu)$ $(j = 1, 2)$ and the spectral density matrices $\rho(\nu)$ and $\sigma(\nu)$. In this section we study these functions in some detail for two simple but important models, the symmetric step model and the asymmetric step model. These calculations illustrate the technique for treating the whole class of piecewise constant models.

We first outline the general construction of the spectral density matrices.[7] The solutions of (21) have the property that the functions

$\varphi_j(x, \nu)$, $\varphi_j'(x, \nu)$ $(j = 1, 2)$ are entire functions of $\nu$ for each fixed $x$, when $\nu$ is a complex variable. The first step is to determine the two functions of $\nu$, $m_\infty(\nu)$ and $m_{-\infty}(\nu)$ such that when Im $\nu > 0$, $\varphi_1(x, \nu) + m_\infty(\nu)\varphi_2(x, \nu)$ is a square integrable function of $x$ over $[0, \infty]$ and $\varphi_1(x, \nu) + m_{-\infty}(\nu)\varphi_2(x, \nu)$ is square integrable over $[-\infty, 0]$. The elements of the spectral density matrix are then given by the formula

$$\rho_{jk}(\nu) - \rho_{jk}(\mu) = \lim_{\epsilon \to +0} \frac{1}{\pi} \int_\mu^\nu \operatorname{Im} M_{jk}(\eta + i\epsilon)\, d\eta \tag{57}$$

where $\mu$ and $\nu$ are real, Im denotes the imaginary part, and for arbitrary complex $\nu$

$$M_{11}(\nu) = (m_{-\infty}(\nu) - m_\infty(\nu))^{-1}, \tag{58}$$

$$M_{12}(\nu) = M_{21}(\nu) = \tfrac{1}{2}(m_{-\infty}(\nu) + m_\infty(\nu))(m_{-\infty}(\nu) - m_\infty(\nu))^{-1}, \tag{59}$$

$$M_{22}(\nu) = m_{-\infty}(\nu)m_\infty(\nu)(m_{-\infty}(\nu) - m_\infty(\nu))^{-1}. \tag{60}$$

Equation (57) defines $\rho_{jk}(\nu)$ uniquely at points of continuity up to an arbitrary, additive constant. The functions $M_{jk}(\nu)$ $(j, k = 1, 2)$ are meromorphic if Im $\nu \neq 0$ and all their real poles are simple. The point spectrum consists exactly of the points which are real poles of one of the $M_{jk}(\nu)$. There are at most a countable number of such points. Let $\nu_0$ be a real pole of $M_{jk}(\nu)$ and let $a_{jk}$ be the residue there,

$$M_{jk}(\nu) = \frac{a_{jk}}{\nu - \nu_0} + \cdots . \tag{61}$$

Then it follows from (57) and (61) that

$$\rho_{jk}(\nu_0 + 0) - \rho_{jk}(\nu_0 - 0) = -\operatorname{Re}(a_{jk}). \tag{62}$$

If $\nu_0$ is not a pole of any $M_{jk}(\nu)$, and Im $M_{jk}(\nu_0) \neq 0$ for some $(j, k)$, then $\nu_0$ is a point of the continuous spectrum and

$$d\rho_{jk}(\nu_0) = \frac{1}{\pi} \operatorname{Im} M_{jk}(\nu_0). \tag{63}$$

If $\nu_0$ is not a pole of any $M_{jk}(\nu)$ and Im $M_{jk}(\nu) = 0$ for all $(j, k)$ in some neighborhood of $\nu_0$, then $\nu_0$ is not in the spectrum and

$$d\rho_{jk}(\nu) = 0 \qquad (j, k = 1, 2) \tag{64}$$

in a neighborhood of $\nu_0$.

### 3.2 *TE Fields for Symmetric Step Model*

We now apply these formulas to the symmetric step model for the case of the *TE* field. The functions $K_n(x)$ $(n = x, y, z)$ are defined by

(1 and 2). Equation (21) has constant coefficients in the two regions $|x| < w$ and $|x| > w$. Since $e_y(x, z)$ and $h_z(x, z)$ must be continuous at $x = \pm w$, the desired solution of (21) must be continuous and have a continuous derivative. We have

$$\varphi_1(x, \nu) = \cos(\omega_\nu x), \qquad |x| \leqq w \tag{65}$$

$$= \cos(\omega_\nu w) \cos\{\omega_0(|x| - w)\}$$
$$- (\omega_\nu/\omega_0) \sin(\omega_\nu w) \sin\{\omega_0(|x| - w)\}, \qquad |x| \geqq w \tag{66}$$

$$\varphi_2(x, \nu) = (1/\omega_\nu) \sin(\omega_\nu x), \qquad |x| \leqq w \tag{67}$$

$$= (1/\omega_\nu) \sin(\omega_\nu w) \cos\{\omega_0(x - w)\}$$
$$+ (1/\omega_0) \cos(\omega_\nu w) \sin\{\omega_0(x - w)\}, \qquad x \geqq w \tag{68}$$

$$\varphi_2(x, \nu) = -\varphi_2(-x, \nu), \qquad x \leqq -w \tag{69}$$

where

$$\omega_n = \{\nu + k^2 K_n\}^{\frac{1}{2}} \qquad (n = 0, x, y). \tag{70}$$

In (70) $\omega_n$ is defined as a single-valued function of $\nu$ in the complex plane cut along the real axis from $-k^2K_n$ to $\infty$. That branch is chosen which is positive real on the upper side of the cut. Simple calculations now yield

$$m_\infty(\nu) = -m_{-\infty}(\nu)$$
$$= \{\omega_\nu \sin(\omega_\nu w) + i\omega_0 \cos(\omega_\nu w)\}\{\cos(\omega_\nu w) - i(\omega_0/\omega_\nu) \sin(\omega_\nu w)\}^{-1}. \tag{71}$$

Therefore,

$$M_{11}(\nu) = -1/\{4M_{22}(\nu)\} = 1/\{2m_{-\infty}(\nu)\}, \tag{72}$$

$$M_{12}(\nu) = M_{21}(\nu) = 0. \tag{73}$$

In order to determine the spectrum, we begin by decomposing the whole real axis into the union of three intervals

$$I_1 = [-\infty, -k^2K_\nu], \; I_2 = (-k^2K_\nu, -k^2K_0), \; I_3 = [-k^2K_0, \infty]. \tag{74}$$

From (57) and (73) it is clear that $\rho_{12}(\nu)$ and $\rho_{21}(\nu)$ are constant for all $\nu$, hence

$$d\rho_{12}(\nu) = d\rho_{21}(\nu) = 0, \qquad -\infty \leqq \nu \leqq \infty. \tag{75}$$

It is easily seen that $M_{11}(\nu)$ and $M_{22}(\nu)$ are real and have no poles or zeros in $I_1$. Therefore, $I_1$ contains no points of the spectrum, and

$$\rho_{jj}(\nu) = \rho_{jj}(-\infty), \qquad d\rho_{jj}(\nu) = 0 \qquad \nu \,\varepsilon\, I_1 \qquad (j = 1, 2). \tag{76}$$

In the interval $I_2$, $M_{11}(\nu)$ and $M_{22}(\nu)$ can each have a finite number of poles, and from (72) it follows that the poles $M_{11}(\nu)$ are the zeros of $M_{22}(\nu)$ and vice versa. The real poles of $M_{11}(\nu)$ are the real solutions of

$$\omega_y \sin(\omega_y w) + i\omega_0 \cos(\omega_y w) = 0 \tag{77}$$

and the real poles of $M_{22}$ are the real solutions of

$$\cos(\omega_y w) - i(\omega_0/\omega_y) \sin(\omega_y w) = 0. \tag{78}$$

For $\nu \in I_2$, $\omega_y$ is real while $\omega_0$ is purely imaginary. If we let

$$b(\nu) = \omega_y(\nu), \qquad p(\nu) = -i\omega_0(\nu) = (-\nu - k^2K_0)^{\frac{1}{2}}, \tag{79}$$

then (77) in the single unknown $\nu$ can be replaced by the set of three equations

$$-\nu = k^2K_0 + p^2, \qquad -\nu = k^2K_y - b^2, \qquad b \tan bw = p, \tag{80}$$

in the two positive real unknowns $b$ and $p$ and the original unknown $\nu$. Similarly, (78) can be replaced by the set of equations

$$-\nu = k^2K_0 + p^2, \qquad -\nu = k^2K_y - b^2, \qquad b \cot bw = -p. \tag{81}$$

These equations are well known and their solutions have been determined.[6, 9] The set of equations (80) has a finite number of real solutions and always has at least one solution for all positive values of the parameters, $w$, $k$, $K_y - K_0$. These are the even modes of $NM$. We denote corresponding values of $\nu$ by $\nu_{1j}$, $j = 1, 2, \cdots, R_1$. The set of equations (81) also has a most finite number of solutions, although if $(wk)^2 \times (K_y - K_0)$ is small enough it has no real solutions. These are the odd modes of $NM$. We denote the values of $\nu$ corresponding to these roots $\nu_{2j}$, $j = 1, 2, \cdots, R_2$. The points $\nu_{1j}$, $\nu_{2j}$, which are all in the interval $I_2$, comprise the point spectrum of (21). Let

$$\delta\rho(\nu) = \lim_{\epsilon \to +0} \{\rho(\nu + \epsilon) - \rho(\nu - \epsilon)\}. \tag{82}$$

Then with the aid of (62) it is easy to show that

$$\delta\rho_{11}(\nu_{1j}) = p(\nu_{1j})/\{1 + wp(\nu_{1j})\}, \ \delta\rho_{22}(\nu_{1j}) = 0, \ j = 1, 2, \cdots, R_1, \tag{83}$$

$$\delta\rho_{11}(\nu_{2j}) = 0, \qquad \delta\rho_{22}(\nu_{2j}) = b^2(\nu_{2j})p(\nu_{2j})/\{1 + wp(\nu_{2j})\}, \quad j = 1, 2, \cdots, R_2. \tag{84}$$

With the aid of (65) through (69) and (77) through (79) it is readily shown that

$$\varphi_1(x, \nu_{1j}) = \cos(b(\nu_{1j})x), \qquad |x| \leqq w \tag{85}$$

$$= \cos(b(\nu_{1j})w) \exp\{p(\nu_{1j})(w - |x|)\}, \qquad |x| \geqq w \tag{86}$$

$$\varphi_2(x, \nu_{2j}) = \{1/b(\nu_{2j})\} \sin(b(\nu_{2j})x), \qquad |x| \leqq w \tag{87}$$

$$= \{1/b(\nu_{2j})\} \sin(b(\nu_{2j})w) \exp\{p(\nu_{2j})(w - x)\}, \qquad x \geqq w \tag{88}$$

$$\varphi_2(x, \nu_{2j}) = -\varphi_2(-x, \nu_{2j}). \qquad x \leqq -w \tag{89}$$

It is also true that

$$\int_{-\infty}^{\infty} \varphi_j(x, \nu_{jk})^2 \, dx = 1/\delta\rho_{jj}(\nu_{jk}), \quad k = 1, 2, \cdots, R_j, \quad j = 1, 2. \tag{90}$$

The remaining points in $I_2$ are not in the spectrum.

Finally, in the interval $I_3$ it is readily shown that $M_{11}(\nu)$ and $M_{22}(\nu)$ have no poles. It is shown easily then that the whole interval $I_3$ is in the continuous spectrum, and in this interval

$$d\rho_{jj}(\nu) = \rho'_{jj}(\nu)\, d\nu \qquad (j = 1, 2), \tag{91}$$

where

$$\rho'_{11}(\nu) = \frac{1}{2\pi} [\omega_\nu^2 \sin^2(\omega_\nu w) + \omega_0^2 \cos^2(\omega_\nu w)]^{-1} \omega_0 , \tag{92}$$

$$\rho'_{22}(\nu) = \frac{1}{2\pi} [\omega_\nu^2 \cos^2(\omega_\nu w) + \omega_0^2 \sin^2(\omega_\nu w)]^{-1} \omega_\nu^2 \omega_0 . \tag{93}$$

In summary, the spectrum of (21) consists of the points $\nu_{jk}$, $k = 1, 2, \cdots, R_j$, $j = 1, 2$, and the interval $I_3$. Equation (27) for the transmitted field can be written as

$$\begin{aligned} e_y^{(t)}(x, z) = {} & \sum_{j=1}^{2} \sum_{k=1}^{R_j} \delta\rho_{jj}(\nu_{jk}) g_j(\nu_{jk}) \varphi_j(x, \nu_{jk}) \exp\{-i\sqrt{-\nu_{jk}}\, z\} \\ & + \sum_{j=1}^{2} \int_{-k^2 K_0}^{0} \exp\{-i\sqrt{-\nu}\, z\} \varphi_j(x, \nu) g_j(\nu) \rho'_{jj}(\nu)\, d\nu \\ & + \sum_{j=1}^{2} \int_{0}^{\infty} \exp\{-\sqrt{\nu}\, z\} \varphi_j(x, \nu) g_j(\nu) \rho'_{jj}(\nu)\, d\nu . \end{aligned} \tag{94}$$

The terms in the first, double summation in (94) are just the possible *TE* modes which can be excited in the waveguide. The terms in the second summation represent the propagating continuum field while the terms in the last summation represent the evanescent part of the transmitted field. A useful interpretation of the propagating continuum field can be obtained as follows. Consider within the waveguide in the region $x < -w$ an incident plane wave of the form

$$e_\nu^{(0)}(x, z, \nu) = \exp\{-i\sqrt{-\nu}\, z - i\omega_0(\nu)x\}, \tag{95}$$

so that if $\theta$ is the direction of propagation of this wave (measured clockwise from the positive $z$ axis), then

$$\cos\theta = \sqrt{-\nu}/k\sqrt{K_0}\,, \quad \sin\theta = \omega_0(\nu)/k\sqrt{K_0}\,. \tag{95}$$

On striking the region of higher dielectric constant, $|x| < w$, part of this wave will be reflected and part of it will be transmitted through the region $|x| < w$. Denote by $\chi_+(x, z, \nu)$ this total electromagnetic field set up by the incident wave, (95). Similarly, denote by $\chi_-(x, z, \nu)$ the total electromagnetic field set up by the incident wave in the region $x > w$

$$e_\nu^{(0)}(x, z, \nu) = \exp\{-i\sqrt{-\nu}\, z + i\omega_0(\nu)x\}. \tag{97}$$

In Fig. (3) we give a schematic description of $\chi_+$ and $\chi_-$. Then it can be shown that for $-k^2K_0 \leqq \nu \leqq 0$,

$$\exp\{-i\sqrt{-\nu}\, z\}\varphi_j(x, \nu) = a_j(\nu)\chi_+(x, z, \nu) + b_j(\nu)\chi_-(x, z, \nu) \quad (j = 1, 2). \tag{98}$$

For the above values of $\nu$ the directions of propagation of the incident waves for $\chi_+$ and $\chi_-$ fill the interval $-\pi/2 \leqq \theta \leqq \pi/2$. Thus, the propagating continuum field is just a wave packet of plane waves appropriate to the medium defined by the dielectric tensor $K_n(x)$.

Similarly, the evanescent part of the field can be interpreted as a superposition of waves bound to the surface $z = 0$ and propagating in

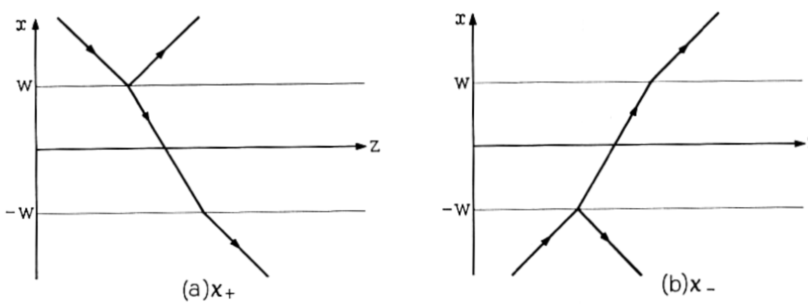


Fig. 3 — A schematic diagram of the plane waves appropriate to the dielectic medium in the symmetric step model. The wave $\chi_+$ is incident on the junction region from the positive $x$ direction, while $\chi_-$ is incident from the negative $x$ direction.

the positive and negative $x$ directions. The distinction between the propagating and evanescent parts of the transmitted field is further shown in the expression for the time averaged transmitted power, (29), which for the symmetric step model is

$$P_t = (2\omega\mu_0)^{-1} \sum_{j=1}^{2} \sum_{k=1}^{R_j} \sqrt{-\nu_{jk}} \mid g_j(\nu_{jk}) \mid^2 \delta\rho_{jj}(\nu_{jk})$$

$$+ (2\omega\mu_0)^{-1} \sum_{j=1}^{2} \int_{-k^2K_0}^{0} \sqrt{-\nu} \mid g_j(\nu) \mid^2 \rho'_{jj}(\nu)\, d\nu. \tag{99}$$

As this expression shows, the evanescent part of the field transmits no energy on the average.

### 3.3 *TM Fields For Symmetric Step Model*

The $TM$ fields of the symmetric step model can be treated similarly. Equation (43) has constant coefficients in the two regions $|x| < w$ and $|x| > w$. Since $e_z(x, z)$ and $h_y(x, z)$ must be continuous at $x = \pm w$, the solutions of (43) must be such that $\psi_j(x, \nu)$ and $\{1/K_z(x)\}\psi'_j(x, \nu)$ $(j = 1, 2)$ are continuous. We have

$$\psi_1(x, \nu) = \cos(K_r\omega_x x), \qquad |x| \leqq w \tag{100}$$

$$= \cos(K_r\omega_x w) \cos\{\omega_0(|x| - w)\}$$

$$- \{(\omega_x K_0)/(\omega_0 K_g)\} \sin(K_r\omega_x w) \sin\{\omega_0(|x| - w)\}, \quad |x| \geqq w \tag{101}$$

$$\psi_2(x, \nu) = \{K_g/\omega_x\} \sin(K_r\omega_x x), \qquad |x| \leqq w \tag{102}$$

$$= \{K_g/\omega_x\} \sin(K_r\omega_x w) \cos\{\omega_0(x - w)\}$$

$$+ \{K_0/\omega_0\} \cos(K_r\omega_x w) \sin\{\omega_0(x - w)\}, \qquad x \geqq w \tag{103}$$

$$\psi_2(x, \nu) = -\psi_2(-x, \nu), \qquad x \leqq -w \tag{104}$$

where

$$K_g = (K_x K_z)^{\frac{1}{2}}, \qquad K_r = (K_z/K_x)^{\frac{1}{2}}, \tag{105}$$

and $\omega_x$ and $\omega_0$ are defined in (70). Next,

$$m_\infty(\nu) = -m_{-\infty}(\nu) = \{(\omega_x/K_g) \sin(K_r\omega_x w) + i(\omega_0/K_0) \cos(K_r\omega_x w)\}$$

$$\cdot \{\cos(K_r\omega_x w) - i(K_g\omega_0/K_0\omega_x) \sin(K_r\omega_x w)\}^{-1}. \tag{106}$$

Therefore,

$$M_{11}(\nu) = -1/\{4M_{22}(\nu)\} = 1/\{2m_{-\infty}(\nu)\}, \tag{107}$$

$$M_{12}(\nu) = M_{21}(\nu) = 0, \tag{108}$$

and from (57) and (108) we have

$$d\sigma_{12}(\nu) = d\sigma_{21}(\nu) = 0, \qquad -\infty < \nu < \infty . \tag{109}$$

The spectrum in the case of TM fields is determined in the same way as in the case of TE fields, and we merely state the results. There are no points of the spectrum in the interval $I_1 = [-\infty, -k^2K_x]$,

$$\sigma_{jj}(\nu) = \sigma_{jj}(-\infty), \qquad d\sigma_{jj}(\nu) = 0, \qquad \nu \, \varepsilon \, I_1 \qquad (j = 1, 2). \tag{110}$$

The interval $I_2 = (-k^2K_x, -k^2K_0)$ contains a finite number of points in the point spectrum. The points of discontinuity of $\sigma_{11}(\nu)$ are the real solutions of

$$(\omega_x/K_g) \sin (K_r\omega_x w) + i(\omega_0/K_0) \cos (K_r\omega_x w) = 0, \tag{111}$$

while the points of discontinuity of $\sigma_{22}(\nu)$ are the real solutions of

$$\cos (K_r\omega_x w) - i(K_g\omega_0/K_0\omega_x) \sin (K_r\omega_x w) = 0. \tag{112}$$

If we let

$$b(\nu) = K_r\omega_x(\nu), \qquad p(\nu) = -i\omega_0(\nu) = (-\nu - k^2K_0)^{\frac{1}{2}}, \tag{113}$$

then (111) in the single unknown $\nu$ can be replaced by the set of equations

$$-\nu = k^2K_0 + p^2, \qquad -\nu = k^2K_x - K_xb^2/K_z , \qquad bK_0 \tan bw = pK_z , \tag{114}$$

in the two positive real unknowns $b$ and $p$ and the original unknown $\nu$. In the same way, (112) can be replaced by the set of equations

$$-\nu = k^2K_0 + p^2, \qquad -\nu = k^2K_x - K_xb^2/K_z , \qquad bK_0 \cot bw = -pK_z . \tag{115}$$

The set of (114) has a finite number of real solutions and for all positive values of the parameters $K_0/K_z$, $K_x/K_z$, $w$, $k^2(K_x - K_0)$ there is always at least one solution.[6, 9] These are the even modes of $NM$. The corresponding values of $\nu$ are denoted by $\nu_{1j}$, $j = 1, 2, \cdots, S_1$. The set of equations (115) also has at most a finite number of solutions, although if $(wk)^2 \, (K_x - K_0)$ is small enough it has no real solutions. These are the odd modes of $NM$. The corresponding values of $\nu$ are denoted by $\nu_{2j}$, $j = 1, 2, \cdots, S_2$. The points $\nu_{1j}$, $\nu_{2j}$ are the point spectrum of (43) and they all lie in the interval $I_2$. Furthermore,

$$\delta\sigma_{11}(\nu_{1j}) = \mathcal{S}(p(\nu_{1j})), \qquad \delta\sigma_{22}(\nu_{1j}) = 0, \qquad j = 1, 2, \cdots , S_1 , \tag{116}$$

$$\delta\sigma_{11}(\nu_{2j}) = 0, \; \delta\sigma_{22}(\nu_{2j}) = b(\nu_{2j})^2\mathcal{S}(p(\nu_{2j}))/K_z^2 , \; j = 1, 2, \cdots , S_2 , \tag{117}$$

where

$$S(p) = K_x p\left[wp + \frac{k^2 K_0 K_x (K_x - K_0)}{(K_x K_z - K_0^2)p^2 + k^2 K_0^2 (K_x - K_0)}\right]^{-1}. \tag{118}$$

From (100) through (104) and (111) through (113) it follows that

$$\psi_1(x, \nu_{1j}) = \cos(b(\nu_{1j})x), \qquad |x| \leqq w \tag{119}$$

$$= \cos(b(\nu_{1j})w) \exp\{p(\nu_{1j})(w - |x|)\}, \qquad |x| \geqq w \tag{120}$$

$$\psi_2(x, \nu_{2j}) = \{K_z/b(\nu_{2j})\} \sin(b(\nu_{2j})x), \qquad |x| \leqq w \tag{121}$$

$$= \{K_z/b(\nu_{2j})\} \sin(b(\nu_{2j})x) \exp\{p(\nu_{2j})(w - x)\}, \qquad x \geqq w, \tag{122}$$

$$\psi_2(x, \nu_{2j}) = -\psi_2(-x, \nu_{2j}). \qquad x \leqq -w \tag{123}$$

It is also true that

$$\int_{-\infty}^{\infty} \psi_j(x, \nu_{jk})^2 \{K_z(x)\}^{-1}\, dx = 1/\delta\sigma_{jj}(\nu_{jk}), \ k = 1, 2, \cdots, S_j, \quad j = 1, 2. \tag{124}$$

The remaining points in $I_2$ are not in the spectrum.

The continuous spectrum is the interval $I_3 = [-k^2K_0, \infty]$. For points of the continuous spectrum

$$d\sigma_{jj}(\nu) = \sigma'_{jj}(\nu)\, d\nu \qquad (j = 1, 2), \tag{125}$$

where

$$\sigma'_{11}(\nu) = \frac{1}{2\pi}[K_0^2\omega_x^2 \sin^2(K_r\omega_x w) + K_x K_z \omega_0^2 \cos^2(K_r\omega_x w)]^{-1} K_0 K_x K_z \omega_0, \tag{126}$$

$$\sigma'_{22}(\nu) = \frac{1}{2\pi}[K_0^2\omega_x^2 \cos^2(K_r\omega_x w) + K_x K_z \omega_0^2 \sin^2(K_r\omega_x w)]^{-1} K_0 \omega_x^2 \omega_0. \tag{127}$$

To summarize these results, the spectrum consists of the points $\nu_{jk}$, $k = 1, 2, \cdots, S_j$, $j = 1, 2$ and the interval $I_3$, and the transmitted field can be written in the form

$$\begin{aligned} h_y^{(t)}(x, z) = &\sum_{j=1}^{2}\sum_{k=1}^{S_j} \sigma_{jj}(\nu_{jk}) h_j(\nu_{jk}) \psi_j(x, \nu_{jk}) \exp\{-i\sqrt{-\nu_{jk}}\, z\} \\ &+ \sum_{j=1}^{2} \int_{-k^2K_0}^{0} \exp\{-i\sqrt{-\nu}\, z\} \psi_j(x, \nu) h_j(\nu) \sigma'_{jj}(\nu)\, d\nu \\ &+ \sum_{j=1}^{2} \int_{0}^{\infty} \exp\{-\sqrt{\nu}\, z\} \psi_j(x, \nu) h_j(\nu) \sigma'_{jj}(\nu)\, d\nu. \end{aligned} \tag{128}$$

Just as for the $TE$ fields, the terms in the first, double summation in (128) are the possible $TM$ modes which can be excited in the waveguide. The terms in the second summation represent the propagating continuum field while the terms in the last summation represent the evanescent part of the transmitted field. Just as for the $TE$ fields, the propagating part of the continuum field can be interpreted as a wave packet of reflected and refracted plane waves, and the evanescent part of the field can be interpreted in terms of surface waves at $z = 0$. Equation (53) for the transmitted energy is

$$P_t = (2\omega\epsilon_0)^{-1} \sum_{j=1}^{2} \sum_{k=1}^{S_j} \sqrt{-\nu_{jk}}\, |\, h_j(\nu_{jk})\, |^2\, \delta\sigma_{jj}(\nu_{jk}) + (2\omega\epsilon_0)^{-1} \sum_{j=1}^{2} \int_{-k^2K_0}^{0} \sqrt{-\nu}\, |\, h_j(\nu)\, |^2\, \sigma'_{jj}(\nu)\, d\nu. \tag{129}$$

### 3.4 *TE Fields For Asymmetric Step Model*

We now turn to the second of the two models which are studied in detail and examine the $TE$ fields for the asymmetric step model. The functions $K_n(x)$ $(n = x, y, z)$ are defined by (3) through (5). Equation (21) has constant coefficients in the regions $|x| < w$, $x > w$, $x < -w$, and we seek solutions which are continuous and have continuous first derivatives. Then

$$\varphi_1(x, \nu) = \cos(\omega_y x), \qquad |\, x\, | \leqq w \tag{130}$$

$$= \cos(\omega_y w) \cos\{\omega_2(x - w)\} - (\omega_y/\omega_2) \sin(\omega_y w) \sin\{\omega_2(x - w)\}, \qquad x \geqq w \tag{131}$$

$$= \cos(\omega_y w) \cos\{\omega_1(x + w)\} + (\omega_y/\omega_1) \sin(\omega_y w) \sin\{\omega_1(x + w)\}, \qquad x \leqq -w \tag{132}$$

$$\varphi_2(x, \nu) = (1/\omega_y) \sin(\omega_y w), \qquad |\, x\, | \leqq w \tag{133}$$

$$= (1/\omega_y) \sin(\omega_y w) \cos\{\omega_2(x - w)\} + (1/\omega_2) \cos(\omega_y w) \sin\{\omega_2(x - w)\}, \qquad x > w \tag{134}$$

$$= -(1/\omega_y) \sin(\omega_y w) \cos\{\omega_1(x + w)\} + (1/\omega_1) \cos(\omega_y w) \sin\{\omega_1(x + w)\}, \qquad x \leqq -w, \tag{135}$$

where

$$\omega_n(\nu) = (\nu + k^2K_n)^{\frac{1}{2}} \qquad (n = 1, 2, x, y). \tag{136}$$

As before $\omega_n$ is defined as a single-valued function of $\nu$ in the complex plane cut along the real axis from $-k^2K_n$ to $\infty$. Then

$$m_\infty(\nu) = \{\omega_\nu \sin(\omega_\nu w) + i\omega_2 \cos(\omega_\nu w)\} \cdot \{\cos(\omega_\nu w) - i(\omega_2/\omega_\nu)\sin(\omega_\nu w)\}^{-1}, \qquad (137)$$

$$m_{-\infty}(\nu) = -\{\omega_\nu \sin(\omega_\nu w) + i\omega_1 \cos(\omega_\nu w)\} \cdot \{\cos(\omega_\nu w) - i(\omega_1/\omega_\nu)\sin(\omega_\nu w)\}^{-1}. \qquad (138)$$

From (58) through (60) and (137) through (138) we obtain

$$M_{jk}(\nu) = N_{jk}(\nu)/D(\nu) \qquad (j, k = 1, 2), \qquad (139)$$

where

$$N_{11}(\nu) = -\tfrac{1}{2}[(1 - \omega_1\omega_2/\omega_\nu^2) + (1 + \omega_1\omega_2/\omega_\nu^2)\cos(2\omega_\nu w) - i\{(\omega_1 + \omega_2)/\omega_\nu\}\sin(2\omega_\nu w)], \qquad (140)$$

$$N_{12}(\nu) = N_{21}(\nu) = (i/2)(\omega_1 - \omega_2), \qquad (141)$$

$$N_{22}(\nu) = \tfrac{1}{2}\{(\omega_\nu^2 - \omega_1\omega_2) - (\omega_\nu^2 + \omega_1\omega_2)\cos(2\omega_\nu w) + i\omega_\nu(\omega_1 + \omega_2)\sin(2\omega_\nu w)\}, \qquad (142)$$

$$D(\nu) = (\omega_\nu + \omega_1\omega_2/\omega_\nu)\sin(2\omega_\nu w) + i(\omega_1 + \omega_2)\cos(2\omega_\nu w). \qquad (143)$$

To determine the spectrum we note first that in the interval $I_1 = [-\infty, -k^2K_\nu]$, the functions $M_{jk}(\nu)$ $(j, k = 1, 2)$ are analytic and real. This interval, therefore, contains no points of the spectrum and

$$d\rho_{jk}(\nu) = 0 \qquad (j, k = 1, 2), \qquad \nu \in I_1 . \qquad (144)$$

The only real poles of the functions $M_{jk}(\nu)$ are in the interval $I_2 = (-k^2K_\nu, -k^2K_1)$. These poles are the real solutions of $D(\nu) = 0$. In $I_2$, $\omega_\nu$ is real while $\omega_1$ and $\omega_2$ are purely imaginary. If we let

$$b(\nu) = \omega_\nu(\nu), \qquad p_n(\nu) = -i\omega_n(\nu) = (-\nu - k^2K_n)^{\frac{1}{2}} \qquad (n = 1, 2), \quad (145)$$

then the equation $D(\nu) = 0$ is equivalent to the set of four equations

$$-\nu = k^2K_1 + p_1^2, \qquad -\nu = k^2K_2 + p_2^2, \qquad -\nu = k^2K_\nu - b^2, \quad (146)$$

$$\tan 2bw = \{p_1/b + p_2/b\}/\{1 - (p_1/b)(p_2/b)\},$$

in the three positive real unknowns $b$, $p_1$, $p_2$ and the original unknown $\nu$. These equations and their solutions have also been studied in detail.[5, 6] In order that (146) have a solution, it is necessary and sufficient that

$$K_\nu > K_n \qquad (n = 1, 2), \tag{147}$$
$$2wk(K_\nu - K_1)^{\frac{1}{2}} > \tan^{-1} \{(K_1 - K_2)/(K_\nu - K_1)\}^{\frac{1}{2}}.$$

If conditions (147) are satisfied, $D(\nu) = 0$ has a finite number of real solutions, $\nu_j$, $j = 1, 2, \cdots, R$ which all lie in the interval $I_2$. This is the first significant difference between the symmetric and asymmetric step models. The symmetric step model always has at least one point in its point spectrum while the asymmetric step model may have no point spectrum.

We can write, assuming that (146) and (147) are satisfied.

$$\delta\rho_{jk}(\nu_l) = -N_{jk}(\nu_l)/D'(\nu_l), \qquad j, k = 1, 2, \qquad l = 1, 2, \cdots, R, \tag{148}$$

where $D'(\nu) = (d/d\nu)\, D(\nu)$. If we make use of (145), it is easy to show that

$$\{\delta\rho_{12}(\nu_l)\}^2 = \delta\rho_{11}(\nu_l)\,\delta\rho_{22}(\nu_l), \qquad l = 1, 2, \cdots, R. \tag{149}$$

Neither of the functions $\varphi_1(x, \nu_j)$ or $\varphi_2(x, \nu_j)$ is square integrable over $-\infty < x < \infty$ for $j = 1, 2, \cdots, R$. However, because of (149), they appear in (27) for $e_y^{(t)}(x, z)$ only in the combination

$$\Phi(x, \nu_j) = \sqrt{\delta\rho_{11}(\nu_j)}\, \varphi_1(x, \nu_j) + \{\delta\rho_{12}(\nu_j)/\sqrt{\delta\rho_{11}(\nu_j)}\}\varphi_2(x, \nu_j), \qquad j = 1, 2, \cdots, R. \tag{150}$$

If we define

$$\Phi_0(x, \nu_j) = \sqrt{\delta\rho_{11}(\nu_j)}\, \cos(b(\nu_j)x) + \{\delta\rho_{12}(\nu_j)/\sqrt{\delta\rho_{11}(\nu_j)}\, b(\nu_j)\} \sin(b(\nu_j)x), \tag{151}$$

then because of (146)

$$\Phi(x, \nu_j) = \Phi_0(x, \nu_j), \qquad |x| \leqq w \tag{152}$$
$$= \Phi_0(w, \nu_j) \exp\{p_2(\nu_j)(w - x)\}, \qquad x \geqq w \tag{153}$$
$$= \Phi_0(-w, \nu_j) \exp\{p_1(\nu_j)(w + x)\}. \qquad x \leqq -w \tag{154}$$

Thus, the functions $\Phi(x, \nu_j)$ are square integrable, and, as we shall see, are just the possible propagating modes in the wave guide. The remaining points in the interval $I_2$ are not in the spectrum.

The remainder of the real axis, the interval $-k^2K_1 \leqq \nu \leqq \infty$, forms the continuous spectrum. To show this, consider first the interval $I_3 = [-k^2K_1, -k^2K_2]$. In $I_3$, $\omega_\nu$ and $\omega_1$ are real, while $\omega_2$ is purely imaginary. The functions $M_{jk}(\nu)$ have no poles in $I_3$ and their imaginary parts are not zero. We introduce the notation

$$\omega_y(\nu) = b(\nu), \qquad \omega_1(\nu) = p_1(\nu), \qquad \omega_2(\nu) = ip_2(\nu), \qquad \nu \varepsilon I_3 . \tag{155}$$

Then we can write

$$d\rho_{jk}(\nu) = \frac{1}{\pi} \{p_1(\nu)/\Delta(\nu)\} r_j(\nu) r_k(\nu) \, d\nu \qquad (j, k = 1, 2), \tag{156}$$

where

$$r_1(\nu) = \cos bw + (p_2/b) \sin bw, \tag{157}$$

$$r_2(\nu) = b \sin bw - p_2 \cos bw, \tag{158}$$

$$\Delta(\nu) = \{b \sin 2bw - p_2 \cos 2bw\}^2 + \{(p_1 p_2/b) \sin 2bw + p_1 \cos 2bw\}^2. \tag{159}$$

For $\nu \varepsilon I_3$ it is clear from (131), (134), and (155) that $\varphi_1(x, \nu)$ and $\varphi_2(x, \nu)$ both grow exponentially as $x \to +\infty$. However, from (156) we see that in (27) for $e_\nu^{(t)}(x, z)$, the functions $\varphi_j(x, \nu)$ $(j = 1, 2)$ appear only in the combination

$$\Lambda(x, \nu) = r_1(\nu)\varphi_1(x, \nu) + r_2(\nu)\varphi_2(x, \nu) \tag{160}$$

when $\nu \varepsilon I_3$. However,

$$\Lambda(x, \nu) = \cos \{b(x - w)\} - (p_2/b) \sin \{b(x - w)\}, \qquad |x| \leqq w \tag{161}$$

$$= \exp \{p_2(w - x)\}, \qquad x \geqq w \tag{162}$$

$$= (\cos 2bw + (p_2/b) \sin 2bw) \cos \{p_1(x + w)\} + (1/p_1)(b \sin 2bw - p_2 \cos 2bw) \cdot \sin \{p_1(x + w)\}, \qquad x \leqq -w. \tag{163}$$

Equations (161) through (163) represent the second important difference between the symmetric and asymmetric step models. In the symmetric model all the components of the continuum field are oscillatory functions of $x$ on both sides of the waveguide while in the asymmetric model some of the components of the continuum field are exponentially damped on one side of the waveguide. The physical interpretation of $\Lambda(x, \nu)$ will be discussed later.

In the remaining interval, $I_4 = [-k^2K_2, \infty]$, the functions $\omega_n$ $(n = 1, 2, y)$ are all real and the functions $M_{jk}(\nu)$ $(j, k = 1, 2)$ have no poles. Therefore,

$$d\rho_{jk}(\nu) = \rho'_{jk}(\nu) \, d\nu, \tag{164}$$

where

$$\rho'_{11}(\nu) = \frac{1}{\pi}(\omega_1 + \omega_2)\{\omega_\nu^2 \cos^2(\omega_\nu w) + \omega_1\omega_2 \sin^2(\omega_\nu w)\}/\mathfrak{D}, \tag{165}$$

$$\rho'_{12}(\nu) = \rho'_{21}(\nu) = \frac{1}{\pi}\omega_\nu(\omega_1 - \omega_2)(\omega_\nu^2 + \omega_1\omega_2)\sin(\omega_{\nu w})\cos(\omega_\nu w)/\mathfrak{D}, \tag{165}$$

$$\rho'_{22}(\nu) = \frac{1}{\pi}\omega_\nu^2(\omega_1 + \omega_2)\{\omega_\nu^2 \sin^2(\omega_\nu w) + \omega_1\omega_2\cos^2(\omega_\nu w)\}/\mathfrak{D}, \tag{167}$$

$$\mathfrak{D}(\nu) = (\omega_\nu^2 + \omega_1\omega_2)^2 \sin^2(2\omega_\nu w) + \omega_\nu^2(\omega_1 + \omega_2)^2\cos^2(2\omega_\nu w). \tag{168}$$

The spectrum for the $TE$ fields of the asymmetric model consists of the (possibly empty) set of points $\nu_j$, $j = 1, 2, \cdots, R$ and the interval $-k^2K_1 \leqq \nu \leqq \infty$. The transmitted field can now be written in the following way.

$$e_y^{(t)}(x, z) = \sum_{j=1}^{R}\left\{\sum_{k=1}^{2}\{\delta\rho_{11}(\nu_j)\}^{-\frac{1}{2}}\delta\rho_{1k}(\nu_j)g_k(\nu_j)\right\}\exp\{-i\sqrt{-\nu_j}\,z\}\Phi(x, \nu_j)$$

$$+\frac{1}{\pi}\int_{-k^2K_1}^{-k^2K_2}\exp\{-i\sqrt{-\nu}\,z\}\Lambda(x, \nu)\left\{\sum_{j=1}^{2}r_j(\nu)g_j(\nu)\right\}\{p_1(\nu)/\Delta(\nu)\}\,d\nu$$

$$+\sum_{j,k=1}^{2}\int_{-k^2K_2}^{0}\exp\{-i\sqrt{-\nu}\,z\}\varphi_j(x, \nu)g_k(\nu)\rho'_{jk}(\nu)\,d\nu$$

$$+\sum_{j,k=1}^{2}\int_{0}^{\infty}\exp\{-\sqrt{\nu}\,z\}\varphi_j(x, \nu)g_k(\nu)\rho'_{jk}(\nu)\,d\nu. \tag{169}$$

The expression for $e_y^{(t)}(x, z)$ has been split up into a sum of parts in order to facilitate its physical interpretation. The first part represents the possible discrete, propagating modes which can be excited in the system. The form of these modes has been studied in detail elsewhere,[5,6] and as pointed out earlier, unless condition (147) is satisfied, no such modes can be excited. In order to interpret the second term, consider within the waveguide in the region $x < -w$ an incident plane wave of the form

$$e_y^{(0)}(x, z, \nu) = \exp\{-i\sqrt{-\nu}\,z - i\omega_1(\nu)x\}. \tag{170}$$

At the surface $x = -w$, part of this wave will be reflected and part will be transmitted. However, at the surface $x = w$, the wave will suffer total internal reflection. The total electromagnetic field set up by $e_y^{(0)}(x, z, \nu)$ is proportional to $\Lambda(x, \nu)\exp\{-i\sqrt{-\nu z}\}$. The second term is then just a superposition of plane waves which are totally reflected at $x = w$. In Fig. 4 we give a schematic description of these

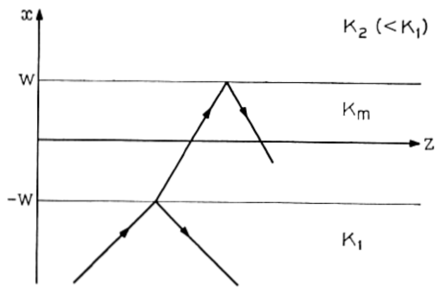


Fig. 4 — A schematic diagram of the totally reflected wave in the asymmetric step model. The wave is incident on the junction region at $x = -w$ where it is partly reflected and partly transmitted. The partly transmitted portion is then totally reflected at $x = w$.

waves. In microscopy the theory of the Becke line is based on just such a superposition of totally reflected plane waves.[10] The third term is a superposition of plane waves which are reflected and refracted at $x = \pm w$. The last term is a superposition of waves bound to the surface $z = 0$ and propagating in the positive and negative $x$ directions.

The time averaged, transmitted power is

$$P_t = (2\omega\mu_0)^{-1} \sum_{l=1}^{R} \sqrt{-\nu_l} \mid \sqrt{\delta\rho_{11}(\nu_l)}\, g_1(\nu_l)$$
$$+ \{\delta\rho_{12}(\nu_l)/\sqrt{\delta\rho_{11}(\nu_l)}\} g_2(\nu_l) \mid^2$$
$$+ (2\omega\mu_0\pi)^{-1} \int_{-k^2K_1}^{-k^2K_2} \sqrt{-\nu} \mid r_1(\nu)g_1(\nu) + r_2(\nu)g_2(\nu) \mid^2 \{p_1(\nu)/\Delta(\nu)\}\, d\nu$$
$$+ (2\omega\mu_0)^{-1} \int_{-k^2K_2}^{0} \sqrt{-\nu} \left\{ \sum_{j,k=1}^{2} g_j(\nu)^* g_k(\nu) \rho'_{jk}(\nu) \right\} d\nu. \tag{171}$$

### 3.5 *TM Fields For Asymmetric Step Model*

The *TM* fields for the asymmetric model present no new features, and we merely record the results. We have

$$\psi_1(x, \nu) = \cos(K_r\omega_x x), \qquad |x| \leqq w \tag{172}$$

$$= \cos(K_r\omega_x w)\cos\{\omega_2(x - w)\}$$
$$- (\omega_x K_2/\omega_2 K_g)\sin(K_r\omega_x w)\sin\{\omega_2(x - w)\}, \quad x \geqq w \tag{173}$$

$$= \cos(K_r\omega_x w)\cos\{\omega_1(x + w)\}$$
$$+ (\omega_x K_1/\omega_1 K_g)\sin(K_r\omega_x w)\sin\{\omega_1(x + w)\}, \quad x \leqq -w \tag{174}$$

$$\psi_2(x, \nu) = (K_g/\omega_x) \sin (K_r\omega_x x), \qquad |x| \leqq w \tag{175}$$

$$= (K_g/\omega_x) \sin (K_r\omega_x w) \cos \{\omega_2(x - w)\}$$
$$+ (K_2/\omega_2) \cos (K_r\omega_x w) \sin \{\omega_2(x - w)\}, \qquad x \geqq w \tag{176}$$

$$= -(K_g/\omega_x) \sin (K_r\omega_x w) \cos \{\omega_1(x + w)\}$$
$$+ (K_1/\omega_1) \cos (K_r\omega_x w) \sin \{\omega_1(x + w)\}, \qquad x \leqq -w \tag{177}$$

where $\omega_n(\nu)$ $(n = x, 1, 2)$ are defined in (136) and $K_g$ and $K_r$ are defined in (105). Next,

$$m_\infty(\nu) = \{(\omega_x/K_g) \sin (K_r\omega_x w) + i(\omega_2/K_2) \cos (K_r\omega_x w)\}$$
$$\cdot \{\cos (K_r\omega_x w) - i(\omega_2 K_g/\omega_x K_2) \sin (K_r\omega_x w)\}^{-1}, \tag{178}$$

$$m_{-\infty}(\nu) = -\{(\omega_x/K_g) \sin (K_r\omega_x w) + i(\omega_1/K_1) \cos (K_r\omega_x w)\}$$
$$\cdot \{\cos (K_r\omega_x w) - i(\omega_1 K_g/\omega_x K_1) \sin (K_r\omega_x w)\}^{-1}. \tag{179}$$

Then from (58) through (60), (178), and (179) we obtain

$$M_{jk}(\nu) = N_{jk}(\nu)/D(\nu) \qquad (j, k = 1, 2), \tag{180}$$

where

$$N_{11}(\nu) = -\tfrac{1}{2}[(1 - \omega_1\omega_2 K_g^2/\omega_x^2 K_1 K_2)$$
$$+ (1 + \omega_1\omega_2 K_g^2/\omega_x^2 K_1 K_2) \cos (2K_r\omega_x w)$$
$$- i(K_g/\omega_x)(\omega_1/K_1 + \omega_2/K_2) \sin (2K_r\omega_x w)], \tag{181}$$

$$N_{12}(\nu) = N_{21}(\nu) = (i/2)(\omega_1/K_1 - \omega_2/K_2), \tag{182}$$

$$N_{22}(\nu) = \tfrac{1}{2}[(\omega_x^2/K_g^2 - \omega_1\omega_2/K_1K_2)$$
$$- (\omega_x^2/K_g^2 + \omega_1\omega_2/K_1K_2) \cos (2K_r\omega_x w)$$
$$+ i(\omega_x/K_g)(\omega_1/K_1 + \omega_2/K_2) \sin (2K_r\omega_x w)], \tag{183}$$

$$D(\nu) = (\omega_x/K_g + \omega_1\omega_2 K_g/\omega_x K_1 K_2) \sin (2K_r\omega_x w)$$
$$+ i(\omega_1/K_1 + \omega_2/K_2) \cos (2K_r\omega_x w). \tag{184}$$

There are no points of the spectrum in $I_1 = [-\infty, -k^2K_x]$. The only real poles of the functions $M_{jk}(\nu)$ are in the interval $I_2 = (-k^2K_x, -k^2K_1)$. In $I_2$, $\omega_x$ is real while $\omega_1$ and $\omega_2$ are imaginary. If we let

$$b(\nu) = K_r\omega_x(\nu), \qquad p_n(\nu) = -i\omega_n(\nu), \qquad n = 1, 2, \tag{185}$$

then the equation determining the poles, $D(\nu) = 0$ is equivalent to the set of equations

$$-\nu = k^2K_1 + p_1^2\,, \quad -\nu = k^2K_2 + p_2^2\,, \quad -\nu = k^2K_x - \frac{K_x}{K_z}b^2,$$
$$\tan 2bw = (p_1K_z/bK_1 + p_2K_z/bK_2)/(1 - p_1p_2K_z^2/b^2K_1K_2). \tag{186}$$

In order that these equations have a solution, it is necessary and sufficient that[5, 6]

$$K_x > K_n \qquad (n = 1, 2),$$
$$2wk\{K_z(K_x - K_1)/K_x\}^{\frac{1}{2}} > \tan^{-1}\{K_xK_z(K_1 - K_2)/K_2^2(K_x - K_1)\}^{\frac{1}{2}}. \tag{187}$$

If conditions (187) are satisfied, $D(\nu) = 0$ has a finite number of real solutions in $I_2$, $\nu_j$, $j = 1, 2, \cdots, S$.

If (186) and (187) are satisfied, we can write

$$\delta\sigma_{jk}(\nu_l) = -N_{jk}(\nu_l)/D'(\nu_l), \qquad j, k = 1, 2, \qquad l = 1, 2, \cdots, S. \tag{188}$$

Just as for the $TE$ fields, it is true that

$$\{\delta\sigma_{12}(\nu_l)\}^2 = \delta\sigma_{11}(\nu_l)\,\delta\sigma_{22}(\nu_l), \qquad l = 1, 2, \cdots, S. \tag{189}$$

Because of (189) the functions $\psi_1(x, \nu_l)$ and $\psi_2(x, \nu_l)$ appear in (49) for $h_y^{(t)}(x, z)$ only in the combination

$$\Psi(x, \nu_j) = \sqrt{\delta\sigma_{11}(\nu_j)}\;\psi_1(x, \nu_j)$$
$$+ \{\delta\sigma_{12}(\nu_j)/\sqrt{\delta\sigma_{11}(\nu_j)}\}\,\psi_2(x, \nu_j), \qquad j = 1, 2, \cdots, S. \tag{190}$$

If we define

$$\Psi_0(x, \nu_j) = \sqrt{\delta\sigma_{11}(\nu_j)}\,\cos(b(\nu_j)x)$$
$$+ \{K_z\delta\sigma_{12}(\nu_j)/\sqrt{\sigma_{11}(\nu_j)}\;b(\nu_j)\}\sin(b(\nu_j)x), \tag{191}$$

then because of (186)

$$\Psi(x, \nu_j) = \Psi_0(x, \nu_j), \qquad |x| \leqq w \tag{192}$$
$$= \Psi_0(w, \nu_j)\exp\{p_2(\nu_j)(w - x)\}, \qquad x \geqq w \tag{193}$$
$$= \Psi_0(-w, \nu_j)\exp\{p_1(\nu_j)(w + x)\}. \qquad x \leqq -w \tag{194}$$

The remaining points in $I_2$ are not in the spectrum.

The remainder of the real axis, the interval $-k^2K_1 \leqq \nu \leqq \infty$ forms the continuous spectrum. In the subinterval $I_3 = [-k^2K_1\,, -k^2K_2)$, $\omega_x$ and $\omega_1$ are real while $\omega_2$ is imaginary. If we let

$$K_r\omega_x(\nu) = b(\nu), \qquad \omega_1(\nu) = p_1(\nu), \qquad \omega_2(\nu) = ip_2(\nu), \qquad \nu \in I_3\,, \tag{195}$$

then we can write

$$d\sigma_{jk}(\nu) = \frac{1}{\pi} \{p_1(\nu)/K_1\Delta(\nu)\}s_j(\nu)s_k(\nu)\, d\nu \qquad (j, k = 1, 2), \tag{196}$$

where

$$s_1(\nu) = \cos bw + (p_2K_z/bK_2) \sin bw, \tag{197}$$

$$s_2(\nu) = -(p_2/K_2) \cos bw + (b/K_z) \sin bw, \tag{198}$$

$$\Delta(\nu) = \{(b/K_z) \sin 2bw - (p_2/K_2) \cos 2bw\}^2 + \{(p_1p_2K_z/bK_1K_2) \sin 2bw + (p_1/K_1) \cos 2bw\}^2. \tag{199}$$

When $\nu \varepsilon I_3$, $\psi_1(x, \nu)$ and $\psi_2(x, \nu)$ appear in (51) for $h_\nu^{(t)}(x, z)$ only in the combination

$$\Xi(x, \nu) = s_1(\nu)\psi_1(x, \nu) + s_2(\nu)\psi_2(x, \nu). \tag{200}$$

We have

$$\Xi(x, \nu) = \cos \{b(x - w)\} - (p_2K_z/bK_2) \sin \{b(x - w)\}, \qquad | x | \leqq w \tag{201}$$

$$= \exp \{p_2(w - x)\}, \qquad x \geqq w \tag{202}$$

$$= \{\cos 2bw + (p_2K_z/bK_2) \sin 2bw\} \cos \{p_1(x + w)\} + (1/p_1)\{(bK_1/K_z) \sin 2bw - (p_2K_1/K_2) \cos 2bw\} \sin \{p_1(x + w)\}, \quad x \leqq -w. \tag{203}$$

In the remaining interval, $I_4 = [-k^2K_2, \infty]$, the functions $\omega_n$ ($n = 1, 2, x$) are all real and we can write

$$d\sigma_{jk}(\nu) = \sigma'_{jk}(\nu)\, d\nu, \tag{204}$$

where

$$\sigma'_{11}(\nu) = \frac{1}{\pi} (\omega_1/K_1 + \omega_2/K_2)\{(\omega_x^2/K_g^2) \cos^2 (K_r\omega_x w) + (\omega_1\omega_2/K_1K_2) \sin^2 (K_r\omega_x w)\}/\mathfrak{D}, \tag{205}$$

$$\sigma'_{12}(\nu) = \sigma'_{21}(\nu) = \frac{1}{\pi} (\omega_x/K_g)(\omega_1/K_1 - \omega_2/K_2) \cdot \{\omega_x^2/K_g^2 + \omega_1\omega_2/K_1K_2\} \sin (K_r\omega_x w) \cos (K_r\omega_x w)/\mathfrak{D}, \tag{206}$$

$$\sigma'_{22}(\nu) = \frac{1}{\pi} (\omega_x/K_g)^2(\omega_1/K_1 + \omega_2/K_2)\{(\omega_1\omega_2/K_1K_2) \cos^2 (K_r\omega_x w) + (\omega_x/K_g)^2 \sin^2 (K_r\omega_x w)\}/\mathfrak{D}, \tag{207}$$

$$\mathfrak{D} = \{(\omega_x/K_g)^2 + (\omega_1\omega_2/K_1K_2)\}^2 \sin^2 (2K_r\omega_x w) + (\omega_x/K_g)^2(\omega_1/K_1 + \omega_2/K_2)^2 \cos^2 (2K_r\omega_x w). \qquad (208)$$

To summarize, the spectrum for the $TM$ waves of the asymmetric model consists of the (possibly empty) set of points $\nu_l$, $l = 1, 2, \cdots, S$, and the interval $-k^2K_1 \leqq \nu \leqq \infty$. The transmitted field can be written as

$$h_y^{(t)}(x, z) = \sum_{j=1}^{S} \sum_{k=1}^{2} \{\delta\sigma_{11}(\nu_j)\}^{-\frac{1}{2}} \delta\sigma_{1k}(\nu_j)h_k(\nu_j) \exp\{-i\sqrt{-\nu_j}\, z\}\Psi(x, \nu_j)$$
$$+ \frac{1}{\pi}\int_{-k^2K_1}^{-k^2K_2} \exp\{-i\sqrt{-\nu}\, z\}\Xi(x, \nu) \sum_{j=1}^{2} s_j(\nu)h_j(\nu)\{p_1(\nu)/K_1\Delta(\nu)\}\, d\nu$$
$$+ \sum_{j,k=1}^{2} \int_{-k^2K_2}^{0} \exp\{-i\sqrt{-\nu}\, z\}\psi_j(x, \nu)h_k(\nu)\sigma'_{jk}(\nu)\, d\nu$$
$$+ \sum_{j,k=1}^{2} \int_{0}^{\infty} \exp\{-\sqrt{\nu}\, z\}\psi_j(x, \nu)h_k(\nu)\sigma'_{jk}(\nu)\, d\nu. \qquad (209)$$

The time averaged, transmitted power is

$$P_t = (2\omega\epsilon_0)^{-1} \sum_{l=1}^{S} \sqrt{-\nu_l} \left| \sqrt{\sigma_{11}(\nu_l)}\, h_1(\nu_l) + \{\sigma_{12}(\nu_l)/\sqrt{\sigma_{11}(\nu_l)}\}h_2(\nu_l) \right|^2$$
$$+ (2\omega\epsilon_0)^{-1} \int_{-k^2K_1}^{-k^2K_2} \sqrt{-\nu}\, | s_1(\nu)h_1(\nu) + s_2(\nu)h_2(\nu) |^2 \{p_1(\nu)/K_1\Delta(\nu)\}\, d\nu$$
$$+ (2\omega\epsilon_0)^{-1} \int_{-k^2K_2}^{0} \sqrt{-\nu} \sum_{j,k=1}^{2} h_j(\nu)^*h_k(\nu)\sigma'_{jk}(\nu)\, d\nu. \qquad (210)$$

## IV. APPROXIMATE SOLUTION OF THE INTEGRAL EQUATIONS

In Section II we obtained general expressions for the reflected and transmitted fields for the $TE$ fields in (18) and (27) and for the $TM$ fields in (41) and (51). In (27) and (51) there appear the functions $\varphi_j(x, \nu)$ and $\psi_j(x, \nu)$ and the spectral density matrices $\rho(\nu)$ and $\sigma(\nu)$. A technique for determining these quantities in certain cases was illustrated in Section III by explicitly calculating them for the symmetric and asymmetric step models. In order to complete the determination of the reflected and transmitted fields, the functions $\mathcal{E}_y^{(r)}(l)$, $\mathcal{H}_y^{(r)}(l)$, $g_k(\nu)$, and $h_k(\nu)$ must be calculated. In Section II we showed that these functions were determined by the integral equations (30)–(31) and (54)–(55).

We have been unable to solve these integral equations exactly for the general case. However, there are certain cases of great physical interest, such as the electro-optic diode modulator, where excellent approximate solutions can be obtained. Let

$$M_n = \max_x K_n(x), \qquad m_n = \min_x K_n(x), \qquad (n = x, y, z) \tag{211}$$

and assume that

$$(M_n - m_n)/m_n \ll 1 \qquad (n = x, y, z). \tag{212}$$

Then the incident field impinges on an essentially uniform, plane dielectric interface, and the reflected field can be calculated as if the region $z > 0$ were a uniform dielectric. Let $\bar{K}_n$ $(n = x, y, z)$ be suitably chosen, constant values for the dielectric tensor for $z > 0$. Then it is readily shown that for the $TE$ fields

$$\mathcal{E}_y^{(r)}(l) = R_e(l)\mathcal{E}_y^{(i)}(l), \tag{213}$$

and for the $TM$ fields

$$\mathcal{H}_y^{(r)}(l) = R_h(l)\mathcal{H}_y^{(i)}(l), \tag{214}$$

where the reflection coefficients are

$$R_e(l) = \{\Omega(l) - k_y\Omega(l/k_y)\}\{\Omega(l) + k_y\Omega(l/k_y)\}^{-1}, \tag{215}$$

$$R_h(l) = \{k_x\Omega(l) - \Omega(l/k_z)\}\{k_x\Omega(l) + \Omega(l/k_z)\}^{-1}, \tag{216}$$

$$k_n = (\bar{K}_n)^{\frac{1}{2}} \qquad (n = x, y, z), \tag{217}$$

and $\Omega(l)$ is defined in (16). In this approximation, the total fields at $z = 0$ for the $TE$ and $TM$ fields are, respectively,

$$e_y(x, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T_e(l)\mathcal{E}_y^{(i)}(l)e^{-ilx}\, dl, \tag{218}$$

$$h_y(x, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T_h(l)\mathcal{H}_y^{(i)}(l)e^{-ilx}\, dl, \tag{219}$$

where the transmission coefficients are

$$T_n(l) = 1 + R_n(l), \qquad n = e, h. \tag{220}$$

Now that $e_y(x, 0)$ and $h_y(x, 0)$ are known, $g_j(\nu)$ $(j = 1, 2)$ can be calculated from (28) and $h_j(\nu)$ $(j = 1, 2)$ can be calculated from (52), since $e_y(x, 0) = e_y^{(t)}(x, 0)$ and $h_y(x, 0) = h_y^{(t)}(x, 0)$.

We illustrate some features of the calculation of $g_k(\nu)$ and $h_k(\nu)$ with the symmetric and asymmetric step models. We first note that if these models are used to study an electro-optic diode modulator, typical values of the parameters defining the dielectric tensors in (1) through (7) are[6] $n = 3.31$, $\Delta \cong 10^{-3}$, $\delta_n \cong 2 \times 10^{-4}$ $(n = x, y, z)$, $\Delta_1 = 0.96\Delta$,

$\Delta_2 = 1.04\ \Delta$. Then $M_n - m_n \cong 1.4 \times 10^{-2}$, $m_n \approx 10.9$. Condition (212) is thus well satisfied.

For the symmetric step model we let $\bar{K}_n = K_0$ ($n = x, y, z$). If the functions $\mathcal{E}_y^{(i)}(l)$ and $\mathcal{H}_y^{(i)}(l)$ are sharply peaked about $l = 0$, then (218) and (219) can be further approximated by

$$e_y(x, 0) = T_e(0) \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{E}_y^{(i)}(l) e^{-ilx}\, dl = T_e(0) e_y^{(i)}(x, 0), \tag{221}$$

$$h_y(x, 0) = T_h(0) h_y^{(i)}(x, 0). \tag{222}$$

The calculation of $g_k(\nu)$ and $h_k(\nu)$ is now reduced to quadratures. If the incident field is not sharply peaked, we define

$$\Phi_j(l, \nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_j(x, \nu) e^{-ilx}\, dx, \tag{223}$$

$$\Psi_j(l, \nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_j(x, \nu) \{k_z(x)\}^{-1} e^{-ilx}\, dx, \tag{224}$$

so that

$$g_j(\nu) = \int_{-\infty}^{\infty} T_e(l) \mathcal{E}_y^{(i)}(l) \Phi_j(l, \nu)\, dl, \tag{225}$$

$$h_j(\nu) = \int_{-\infty}^{\infty} T_h(l) \mathcal{H}_y^{(i)}(l) \Psi_j(l, \nu)\, dl, \qquad j = 1, 2. \tag{226}$$

If $\nu$ is in the continuous spectrum, $\Phi_j(l, \nu)$ and $\Psi_j(l, \nu)$ are distributions which are easily determined with the aid of the relation[11]

$$\int_0^{\infty} e^{ix\sigma}\, dx = 1/(i\sigma) + \pi\delta(\sigma), \tag{227}$$

where $\delta(\sigma)$ is the delta function and when $1/\sigma$ appears under an integral sign, it is assumed that the Cauchy principal value is taken. If $\nu$ is in the point spectrum, $\Phi_j(l, \nu)$ and $\Psi_j(l, \nu)$ are ordinary functions.

For the asymmetric step model we let $\bar{K}_n = \frac{1}{2}(K_1 + K_2)$, ($n = x, y, z$). For this model, a straightforward application of (28) and (52) fails in general if $\nu$ is the point spectrum or if $\nu \in I_3$, because $\varphi_j(x, \nu)$ and $\psi_j(x, \nu)$ now grow exponentially as $x$ tends to either plus infinity or minus infinity. This apparent difficulty is merely a reflection of the manner of convergence of the integrals defining $g_k(\nu)$ and $h_k(\nu)$. For our purposes here, it is enough to note from (169) and (209) that when $\nu$ is in the point spectrum, the functions $g_k(\nu)$ and $h_k(\nu)$ do not appear independently, but only in the linear combinations

$$\sum_{k=1}^{2} \{\delta\rho_{11}(\nu_j)\}^{-\frac{1}{2}}\delta\rho_{1k}(\nu_j)g_k(\nu_j) = \int_{-\infty}^{\infty} e_\nu(x, 0)\Phi(x, \nu_j)\,dx, \quad j = 1, 2, \cdots, R, \tag{228}$$

$$\sum_{k=1}^{2} \{\delta\sigma_{11}(\nu_j)\}^{-\frac{1}{2}}\delta\sigma_{1k}(\nu_j)h_k(\nu_j) = \int_{-\infty}^{\infty} h_\nu(x, 0)\Psi(x, \nu_j)\,dx, \quad j = 1, 2, \cdots, S. \tag{229}$$

The integrals on the right of (218) and (219) are now well defined. Similarly, if $\nu \varepsilon I_3$, the relevant quantities to calculate are

$$\sum_{k=1}^{2} r_k(\nu)g_k(\nu) = \int_{-\infty}^{\infty} e_\nu(x, 0)\Lambda(x, \nu)\,dx, \tag{230}$$

$$\sum_{k=1}^{2} s_k(\nu)h_k(\nu) = \int_{-\infty}^{\infty} h_\nu(x, 0)\Xi(x, \nu)\,dx. \tag{231}$$

If $\nu \varepsilon I_4$, (28) and (52) can be applied directly. Now, all the techniques discussed in the case of the symmetric model can be applied here.

## V. SUMMARY

In Section I we have defined a class of dielectric waveguide models. The waveguide is formed by an anisotropic, nonuniform dielectric filling the half space $z > 0$. The dielectric tensor is diagonal in the fixed coordinate system of Fig. 1, and the diagonal matrix elements are functions of $x$ only, $K_n(x)$ $(n = x, y, z)$.

Integral representations for the incident, reflected, and transmitted fields were given in (15), (18), and (27), respectively, for the $TE$ fields, and in (39), (41) and (51), respectively, for the $TM$ fields. These representations are very general, holding for a large class of functions $K_n(x)$ and incident fields. These integral representations, however, contain the unknown functions $\varphi_j(x, \nu)$, $\psi_j(x, \nu)$, $\rho_{jk}(\nu)$ and $\sigma_{jk}(\nu)$ $(j, k = 1, 2)$, which are determined solely by the dielectric tensor, $K_n(x)$, and the unknown functions $g_k(\nu)$, $h_k(\nu)$, $(k = 1, 2)$, $\mathcal{E}_\nu^{(r)}(l)$, and $\mathcal{H}_\nu^{(r)}(l)$, which also depend on the incident field and the boundary conditions at $z = 0$. It was shown that this latter group of unknown functions are the solutions of two sets of integral equations, (30)–(31) for the $TE$ fields and (54)–(55) for the $TM$ fields. These equations are very complicated, and we have been unable to solve them exactly for any specific models of interest.

In Section III we gave a detailed calculation of the functions $\varphi_j(x, \nu)$, $\psi_j(x, \nu)$, $\rho_{jk}(\nu)$, and $\sigma_{jk}(\nu)$ $(j, k = 1, 2)$ for both the symmetric and asymmetric step models. These calculations are important in their own right, since the symmetric and asymmetric step models have been used extensively in the study of the electro-optic diode modulators.[1–6] However, these computations also illustrate the technique for treating the whole class of piecewise constant models. This is important, for it is not yet completely established which is the correct model to use in exploring the behavior of the electro-optic diode modulator, and it is felt that any actual physical situation can be well approximated by a piecewise constant model.

It should be noted that the success of the techniques used in this paper depends on being able to obtain exact analytic solutions of (21) and (43), or at least good analytic approximations to these solutions. There are a number of other models for which the exact solutions of (21) can be obtained, for example the continuous dielectric constant models described in Section III of *NM*. It is, however, much more difficult to find models, other than the piecewise constant models, for which (43) is solvable in terms of known functions. Nevertheless, the possibility remains of investigating the *TE* fields for a fairly wide varity of models.

The calculations of Section III provide a method of determining the discrete modes which is different from the methods used in earlier treatments.[5,6,9] These calculations showed also that the asymmetry of the background light is accentuated in the asymmetric step model by total internal reflection at the junction region boundary.

Finally, in Section IV it was shown that good approximations can be found for the functions $g_k(\nu)$, $h_k(\nu)$, $\mathcal{H}_\nu^{(r)}(l)$, and $\mathcal{E}_\nu^{(r)}(l)$ in certain cases of physical interest. In particular, these approximations are valid for the electro-optic diode modulator. These approximations do not depend on a particular choice of the incident field.

The final results of this paper then are integral representations for the fields for both the *TE* and *TM* fields. Of the various functions in the integrands, some have been determined exactly and good approximations have been found for the remainder for a number of important models and for arbitrary incident fields.

These integral representations are complicated in appearance, but when $z$ is large enough, asymptotic expansions of them can be found which lend themselves to numerical analysis. In a subsequent paper asymptotic expansions of the transmitted fields will be presented for the symmetric and asymmetric step models in the case that the incident field is Gaussian and numerical results for cases of experimental interest will be presented.

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