Some Properties and Limitations of Electronically Steerable Phased Array Antennas*

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This paper is a treatment on linear and planar phased arrays of current sources, whose amplitudes are uniform and scan-invariant. By recognition that the radiation impedance of an array element is an analytic function of a complex scan variable, a powerful mathematical tool becomes available for the investigation of some important properties of the impedance as a function of scan. For example, it is proven that in a finite array the impedance seen by such a scan-invariant current source cannot be perfectly matched over a continuous scanning range using lossless, linear, passive and time-invariant elements. This result is extended to the infinite-array case by treating the latter as a periodic structure, and assuming that the Green's function of the unit cell is analytic with respect to the scan variable. The theory includes both linear and planar arrays. Among other results it is shown that the element impedance in an infinite array must be of a specific mathematical form. It is hoped that by recognizing the limitations imposed thereby, useful guidelines will be established for achieving optimal match of an array into space.

I. INTRODUCTION

The class of antennas widely known as phased arrays includes essentially two types of radiators: stationary and steerable ones. The first operates at fixed amplitude and fixed relative phase between the array elements. Consequently, the antenna characteristics, such as radiation pattern, input impedance, and mutual coupling between elements, remain unchanged during the entire operational lifetime of the antenna. The steerable antenna is characterized by time varying ex-

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citation. The relative phase between adjacent elements is varied either mechanically or electronically to bring about a variation in the orientation of the beam. In most instances scanned arrays are large in size and may contain several thousand elements. Their illumination has a linear phase taper. As a result the antenna characteristics become scan dependent. The relationship between scan angle and various parameters of interest such as gain, element impedance, and mutual coupling between elements have been the subject of intense investigation in recent years.[1, 2] One particular direction has been towards improvement of the impedance match over wide scanning ranges.[3] At present the merit of a matching technique can be determined only relatively to other techniques. To the best of the authors' knowledge an absolute mathematical criterion, based on physical realizability requirements, has not been formulated. Some investigators[4, 5] claim that a perfect match of an infinite array for all scan angles (at which the active impedance is not infinite, zero or purely reactive) can be achieved by an infinite set of interconnecting network elements. However, the proof is based on the assumption that the scan-dependent equivalent load impedance at the array-space interface remains unchanged after the sources have been interconnected by coupling elements. Although this assumption has been successfully applied[3, 5] to improve the matching capability of an infinite array, it is incorrect to use it in a perfect matching scheme.

In this paper a new mathematical approach to phased array analysis is presented. The model for the analysis is a phased array of ideal current sources (electric or magnetic) of scan-invariant uniform amplitude. This model is further discussed in Section II. The analysis itself is based on the general laws of antenna theory and on those properties which are common to all phased arrays represented by the model.

The first part of the theory is devoted to finite arrays and is treated in Section III. The starting point of the theory is a theorem which establishes that the radiation impedance of an element in a finite array is an analytic function of the scan angle. Further, it is shown that an element in a linear or planar phased array cannot be perfectly matched over a continuous scanning range by using lossless, linear, passive and time-invariant elements. Then it is demonstrated that the directions in space of the beams' maxima are eigenvalues of a Laplacian differential operator with periodic boundary conditions which are related to the phase taper of the array, and several useful properties of those eigenvalues are derived.

The second part of the theory appears in Section IV and is devoted to infinite arrays, which play an important role in the analysis of large phased arrays. The investigation is based on a transformation between the scan angle and a complex variable $s = \alpha + j\beta$, which can be interpreted on $0 < \alpha \leqq 1$, $\beta = 0$ as the trigonometric sine function of the angle between the plane of the array and the direction in which a chosen grating lobe propagates. It is subsequently shown that the element impedance, as a function of s, is restricted to a specific mathematical form. Recognition of the limitations imposed thereby may provide new insight into the behavior of such arrays.

II. PRELIMINARY REMARKS

The model chosen for the following treatment is a linear or planar phased array excited by a set of ideal current generators of uniform amplitude and linear phase taper. The description *ideal* implies that the sources have no internal impedance and are invariant under any loading. This means that except for the relative phasing between contiguous generators the currents are scan independent. Frequently in antenna analysis induced currents are replaced by equivalent sources by application of the equivalence principle.[6] Such currents are not part of the sources. The induced currents are accounted for automatically by fulfillment of the requirement that the tangential component of the electric field has to vanish on all conductors. In general, the source-current amplitude in each element of the array may be a function of scan. However, this dependence is generally unknown and is often neglected in theoretical work. The types of excitations commonly used are the "free excitation" and "forced excitation".* The first assumes a generator with a scan-invariant internal impedance which is capable of delivering scan-invariant incident power. In the latter a constant terminal voltage or current is maintained. As pointed out by Oliner and Malech free excitation is easier to realize in high-frequency technology than forced excitation. The latter, however, is more tractable here. The results of this study remain valid for scan-dependent excitation as well, provided the current density of the source is a smoothly varying function of scan angle and can be analytically continued into a complex scan-angle plane.

Under the assumption that the array is excited by a uniform amplitude and a linear phase taper, the current density excitation function

* A. A. Oliner and R. G. Malech, Ref., 1, pp. 209–211.

of an M-element linear array (Fig. 1) is given by

$$\mathbf{J}(x, y, z, \psi) = \begin{cases} \mathbf{J}_0(x - ma, y, z)e^{im\psi}, & ma \leqq x \leqq (m+1)a, \\ & m = 0, 1, \cdots, M-1, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and that of an $M \times N$ element planar array of rectangular symmetry (Fig. 2) is given by

$$\mathbf{J}(x, y, z, \psi_x, \psi_y) = \begin{cases} \mathbf{J}_0(x - ma, y - nb, z)e^{i(m\psi_x + n\psi_y)} \\ \qquad ma \leqq x \leqq (m+1)a, \\ \qquad nb \leqq y \leqq (n+1)b, \\ \qquad m = 0, 1, 2, \cdots, M-1, \\ \qquad n = 0, 1, 2, \cdots, N-1, \\ 0 \qquad \text{otherwise.} \end{cases} \tag{2}$$

The above currents can be either electric or magnetic the latter being regarded as equivalent to ideal electric voltage sources.

Note that the spherical coordinate systems in Fig. 1 and 2 differ from those commonly used in phased array analysis. The poles are located at endfire instead of broadside and the ranges of colatitude and azimuth are such that the upper hemisphere is spanned by $0 \leqq \theta \leqq \pi$, $0 \leqq \varphi < \pi$. This convention is chosen for reasons of mathematical convenience. The results derived in Section III are valid for linear as well as planar arrays. The inclusion of both cases in a single treatment is facilitated by a generalized notation for the current density excitation function. The steering phases $m\psi$ and $m\psi_x + n\psi_y$ are replaced by an equivalent "steering coefficient" $\sigma_{mn}(\varphi_{pq})$ in the plane of scan oriented at azimuth angle φ_{pq}. The steering coefficient

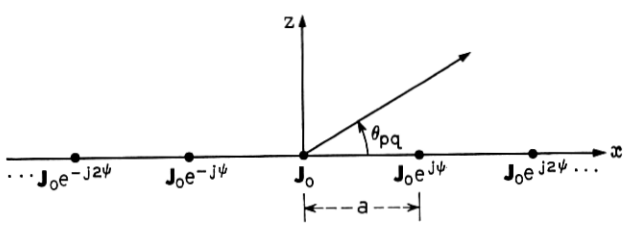

Fig. 1 — Linear phased array.

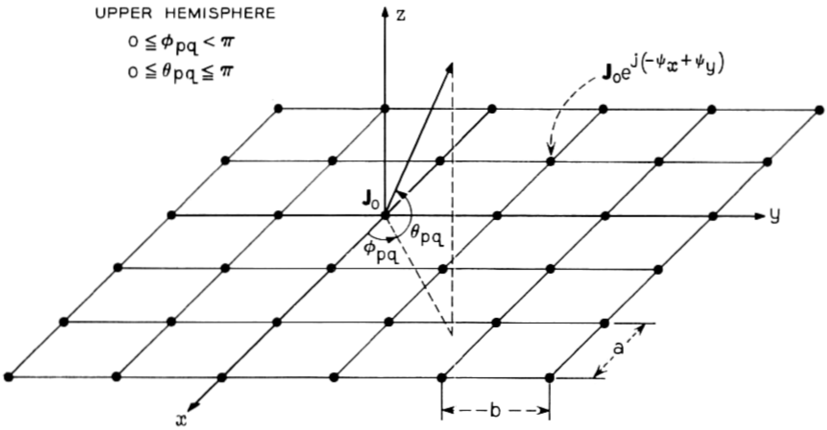

Fig. 2 — Planar phased array.

is derived by its relationship to the direction of a beam's maximum, which is determined for linear arrays by the equation

$$\psi + 2p\pi = ka \cos \theta_{p0} \qquad p = 0, \pm 1, \pm 2, \cdots \pm \infty \tag{3}$$

and for planar arrays by

$$\psi_x + 2p\pi = ka \cos \theta_{pq} \cos \varphi_{pq} \qquad p = 0, \pm 1, \pm 2, \cdots \pm \infty \tag{4a}$$

$$\psi_y + 2q\pi = kb \cos \theta_{pq} \sin \varphi_{pq} \qquad q = 0, \pm 1, \pm 2, \cdots \pm \infty, \tag{4b}$$

where k is the wave number in the medium, and θ_{pq} is as shown in Fig. 1 and 2. The steering coefficient is then defined by

$$\sigma_{mn}(\varphi_{pq}) = k(ma \cos \varphi_{pq} + nb \sin \varphi_{pq})$$
$$p, q = 0, \pm 1, \pm 2, \cdots \pm \infty. \tag{5}$$

Equations (1) and (2) can now be written as

$$\mathbf{J}(x, y, z, \theta_{pq}) = \begin{cases} \mathbf{J}_0(x - ma, y - nb, z) \exp (j\sigma_{mn} \cos \theta_{pq}), \\ \qquad ma \leqq x \leqq (m + 1)a, \\ \qquad nb \leqq y \leqq (n + 1)b, \\ \qquad m = 0, 1, \cdots, M - 1, \\ \qquad n = 0, 1, \cdots, N - 1, \\ 0 \qquad \text{otherwise} \end{cases} \tag{6}$$

at $\varphi_{pq} =$ const.

Under the above generalization the excitation function for the linear array becomes a special case, $q = 0$, $N = 1$, $\varphi_{pq} = 0$, and the period in the y-direction extends from $-\infty$ to $+\infty$; or alternatively $p = 0$, $M = 1$, $\varphi_{pq} = \pi/2$ and the period in the x-direction extending from $-\infty$ to $+\infty$. Since the phase constant $\exp\{j\sigma_{mn}(\varphi_{pq}) \cos \theta_{pq}\}$ is independent of (p, q), any θ_{pq} may be chosen as the independent variable of scan. The subscript pq will henceforth be omitted whenever the mathematical expressions are independent of (p, q).

The time dependence $e^{j\omega t}$ is assumed throughout the analysis. In a steerable array the phase taper is time dependent. However, it is understood that the rate of change of the phase taper is very small in comparison to the angular frequency, i.e., $d\psi/dt \ll \omega$, since only under that condition do the classical concepts of directivity and radiation impedance remain meaningful. If $\psi(t)$ is a step function it is assumed that the time interval is long enough to allow all transients to reach negligible values before a new step is initiated.

The formal solution of the array problem is obtained from Maxwell's Equations via a vector potential $\mathbf{A}(x, y, z, \theta)$ which is a solution of the inhomogeneous reduced wave equation

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}(x, y, z, \theta), \tag{7}$$

where μ is the permeability of the medium. The magnetic field is given by

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}, \tag{8a}$$

and the electric field (under Lorentz gauge) by

$$\mathbf{E} = -j\omega\left(\mathbf{A} + \frac{1}{k^2} \nabla\nabla \cdot \mathbf{A}\right). \tag{8b}$$

The solution to (7) over infinite space V can be written in closed form in terms of a dyadic Green's function[7]

$$\mathbf{A}(x, y, z, \theta) = \mu \int_V \bar{\bar{\mathcal{G}}}(x, y, z \mid \xi, \eta, \zeta) \cdot \mathbf{J}(\xi, \eta, \zeta, \theta)\, d\xi\, d\eta\, d\zeta, \tag{9}$$

where $\bar{\bar{\mathcal{G}}}(x, y, z \mid \xi, \eta, \zeta)$ is a solution of

$$\frac{\partial^2\bar{\bar{\mathcal{G}}}}{\partial x^2} + \frac{\partial^2\bar{\bar{\mathcal{G}}}}{\partial y^2} + \frac{\partial^2\bar{\bar{\mathcal{G}}}}{\partial z^2} + k^2\bar{\bar{\mathcal{G}}} = -\bar{\bar{I}}\,\delta(x - \xi)\,\delta(y - \eta)\,\delta(z - \zeta), \tag{10}$$

$\bar{\bar{I}}$ being the unit dyadic $\mathbf{a}_x\mathbf{a}_x + \mathbf{a}_y\mathbf{a}_y + \mathbf{a}_z\mathbf{a}_z$. The boundary conditions which $\bar{\bar{\mathcal{G}}}$ has to satisfy are derivable via the Vector Green's Theorem*

* P. M. Morse and H. Feshbach, Ref. 7, p. 1767.

by imposition of the requirement that the tangential component of the electric field has to vanish on all conductors. This guarantees that all induced currents are accurately determined.

It can be shown[8] that the average complex power delivered by the mnth element in the array is

$$P_{mn} = -\frac{1}{2} \int_{V_{mn}} \mathbf{E} \cdot \mathbf{J}_{mn}^* \, dv, \tag{11}$$

where

$$\mathbf{J}_{mn}(x, y, z, \theta) = \mathbf{J}(x, y, z, \theta),$$

$$ma \leqq x \leqq (m + 1)a, \qquad nb \leqq y \leqq (n + 1)b \tag{12}$$

the asterisk (*) denotes complex conjugate, and V_{mn} is a simply connected volume occupied by $\mathbf{J}_{mn}$. If S_{mn} is a surface obtained by taking a cross section through V_{mn}, the total current, I_{mn}, flowing through the cross section S_{mn} is

$$I_{mn} = \iint_{S_{mn}} \mathbf{J} \cdot \mathbf{ds}. \tag{13}$$

The element radiation impedance, Z_{mn}, is defined in terms of the complex power by

$$P_{mn} = \tfrac{1}{2} \mid I_{mn} \mid^2 Z_{mn} . \tag{14}$$

By (10) and (13) via (8b) and (9), the element radiation impedance can be defined directly in terms of the array geometry and the excitation:

$$Z_{mn}(\theta) = \frac{1}{\mid I_{mn} \mid^2} \int_{V_{mn}} \int_{V} \mathbf{J}_{mn}^*(x, y, z, \theta) \cdot \bar{\bar{G}}(x, y, z \mid \xi, \eta, \zeta) \cdot \mathbf{J}(\xi, \eta, \zeta, \theta) \, d\tau \, dv, \tag{15a}$$

where $d\tau = d\xi d\eta d\zeta$, $dv = dxdydz$, and

$$\bar{\bar{G}}(x, y, z \mid \xi, \eta, \zeta) = j\omega\mu\left(\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla\right) \cdot \bar{\bar{\mathcal{G}}}(x, y, z \mid \xi, \eta, \zeta). \tag{15b}$$

Operator ∇ operates on (x, y, z). The quantity $|I_{mn}|^2$ is introduced for the purpose of normalization, and may depend on the choice of the cross section S_{mn}.

The definition of the impedance includes both linear and planar array elements. It is consistent with the commonly known definition of impedance[9] if the latter is viewed as a relation between the average complex power delivered by the generator and the rms current flowing into the load. The definition given by (15) is necessary in view of the fact that in a system excited by distributed currents, a terminal voltage in the time domain is not always uniquely defined. In a system excited by magnetic currents, (15) defines the element admittance if the permeability μ is replaced by the permittivity ϵ and the electric currents by their magnetic counterparts.

In the following theoretical discussion, it is assumed that the phased arrays are excited by a uniform amplitude and a linear phase taper.

III. FINITE ARRAYS

Theorem 1: The element radiation impedance in a finite, steerable, linear or planar phased array of scan-invariant current sources, radiating into a linear, lossless, passive and time-invariant system, is an entire function[10] *of the scan angle θ in any given plane of scan, with an essential singularity* at $\theta \to \infty$.*

Proof: By (15a)

$$| I_{rs} |^2 Z_{rs} = \int_{V_{rs}} \int_V \mathbf{J}_{rs}^*(x, y, z, \theta) \cdot \bar{\bar{G}}(x, y, z \mid \xi, \eta, \zeta) \cdot \mathbf{J}(\xi, \eta, \zeta, \theta) \, d\tau \, dv. \tag{16}$$

On expanding (16) in a double sum of integrals over all cells $\{(m, n)\}$, $m = 0, 1, \cdots, M-1$; $n = 0, 1, \cdots, N-1$, and using the relationships of (6) followed by a change of variable in each term of the sum, one obtains

$$Z_{rs}(\theta) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \tilde{z}_{mnrs} \exp [j(\sigma_{mn} - \sigma_{rs}) \cos \theta], \tag{17a}$$

where

$$\tilde{z}_{mnrs} = \frac{1}{| I_{rs} |^2} \int_{V_{oo}} \int_{V_{oo}} \mathbf{J}_0^*(x, y, z)$$
$$\cdot \bar{\bar{G}}(x + ra, y + sb, z \mid \xi + ma, \eta + nb, \zeta) \cdot \mathbf{J}_0(\xi, \eta, \zeta) \, d\tau \, dv. \tag{17b}$$

In any given plane of scan φ is constant, so that

$$\sigma_{mn} - \sigma_{rs} = k[(m - r)a \cos \varphi + (n - s)b \sin \varphi] = \sigma_{m-r, n-s} \tag{18}$$

is independent of θ. Both $\cos \theta$ and the exponential function are entire functions.† Consequently, the exponential function appearing in (17a)

* R. V. Churchill, Ref. 10, Sec. 68, p. 157; Sec. 112, p. 270.
† R. V. Churchill, Ref. 10, Sec. 21, p. 47; Sec. 23, p. 50.

is an entire function of an entire function, which is likewise entire[11] (entire functions are also called integral functions). $Z_{rs}(\theta)$ is a finite sum of entire functions and is also entire.

The nature of the essential singularity at $\theta \to \infty$ is obtained by first expanding *cos θ* in the complex θ-plane

$$\cos(\theta_r + j\theta_i) = \cos\theta_r \cosh\theta_i - j\sin\theta_r \sinh\theta_i \,. \tag{19}$$

Then, if $|\theta_i| \to \infty$ in such a way that $(\sigma_{mn} - \sigma_{rs})\theta_i > 0$, the *mn*th term behaves as $\exp\{|\sigma_{mn} - \sigma_{rs}| \sin\theta_r \exp[|\theta_i|]\}$ Q.E.D. Note that even when $\mathbf{J}_0(x, y, z, \theta)$ is scan dependent, $Z_{rs}(\theta)$ is analytic provided $\mathbf{J}_0(x, y, z, \theta)$ is analytic. However, other isolated singularities may exist.

Corollary 1a: Re $\{Z_{rs}\}$ *and* $\frac{\partial}{\partial\theta} Re\{Z_{rs}\}$ *are entire functions of* θ *each with an essential singularity at* $\theta \to \infty$. Proof appears in Appendix A.

Theorem 2: The power radiated by an element in a finite, steerable, linear or planar phased array of scan-invariant current sources, radiating into a lossless, linear, passive and time-invariant system cannot be kept constant over a continuous scanning range with lossless, linear, passive and time-invariant network elements and scatterers only.

Proof: Let $\bar{\bar{G}}(x, y, z \mid \xi, \eta, \zeta)$ be the dyadic Green's function of the entire system including all equalizing elements. The radiation impedance of the *mn*th element of the array is given by (15a) for a lossless, linear, passive, time-invariant system. If the array is radiating constant power over a continuous scanning range, the real part of the radiation impedance, $R_{rs}(\theta) = \text{Re}\{Z_{rs}\}$, must remain constant in that range and

$$\frac{\partial}{\partial\theta}[R_{rs}(\theta)] = 0, \qquad \theta_1 \leqq \theta_r \leqq \theta_2 \,, \qquad \theta_i = 0 \tag{20}$$

where $\theta = \theta_r + j\theta_i$. By Corollary 1a, $\frac{\partial}{\partial\theta}[R_{rs}(\theta)]$ is analytic in the closed θ-plane and has an essential singularity at $\theta \to \infty$. However, if the derivative vanishes along the line $\theta_1 \leqq \theta_r \leqq \theta_2$ it must vanish everywhere in the θ-plane*. Hence, it cannot have an essential singularity at infinity. The contradiction implies that $R_{rs}(\theta)$ cannot be constant over a continuous scanning range. Q.E.D.

Equations (3) and (4) specify the directions of the beams' maxima, however, not all of them correspond to real directions in space. Whereas φ_{pq} is real for all (p, q), θ_{pq} can be either real or imaginary, as may be

* P. M. Morse and H. Feshbach, Ref. 7, Vol. I, p. 390.

seen from the solution of (4):

$$\varphi_{pq} = \tan^{-1} \frac{(\psi_y + 2q\pi)a}{(\psi_x + 2p\pi)b}, \qquad 0 \leqq \varphi_{pq} < \pi \quad (21a)$$

$$\theta_{pq} = \cos^{-1} \frac{\psi_x + 2p\pi}{ka \cos \varphi_{pq}} = \cos^{-1} \frac{\psi_y + 2q\pi}{kb \sin \varphi_{pq}}, \; 0 \leqq \theta_{pq} \leqq \pi. \quad (21b)$$

If θ_{pq} is real it is said that the beam is in real space. By way of mathematical generalization it is said that all those beams having an imaginary θ_{pq} are in "imaginary space". If $\theta_{pq} = 0$, or $\theta_{pq} = \pi$, it is said that the beam is in a grazing position between real and imaginary space. It can easily be verified from (21) that for a given phasing (ψ_x, ψ_y) every pair (p, q) corresponds to a unique direction $(\varphi_{pq}, \theta_{pq})$ in the complex domain $0 \leqq \varphi < \pi$, $0 \leqq \text{Re}\{\theta\} \leqq \pi$. These directions are the characteristic directions of the system. They are directly related, through (4), to the eigenvalues of

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \Gamma^2 F(x, y) = 0 \quad (22)$$

with the following periodic boundary conditions

$$F(x, y) = F(x + a, y) \exp(-j\psi_x), \quad (23a)$$

$$\frac{\partial F}{\partial x}(x, y) = \frac{\partial F}{\partial x}(x + a, y) \exp(-j\psi_x), \quad (23b)$$

$$F(x, y) = F(x, y + b) \exp(-j\psi_y), \quad (23c)$$

$$\frac{\partial F}{\partial y}(x, y) = \frac{\partial F}{\partial y}(x, y + b) \exp(-j\psi_y). \quad (23d)$$

The eigenfunctions, which form a complete orthogonal set in the interval $0 \leqq x \leqq a$, $0 \leqq y \leqq b$ are

$$F_{pq}(x, y) = \exp\left[j(\psi_x + 2p\pi)\frac{x}{a}\right] \exp\left[j(\psi_y + 2q\pi)\frac{y}{b}\right],$$
$$p, q = 0, \pm 1, \pm 2, \cdots \pm \infty. \quad (24)$$

By (4) they can also be written as

$$F_{pq}(x, y) = \exp\{jk \cos \theta_{pq}(x \cos \varphi_{pq} + y \sin \varphi_{pq})\}. \quad (25)$$

The eigenvalues $\{\Gamma_{pq}\}$ are

$$\Gamma_{pq} = k \cos \theta_{pq} \qquad p, q = 0, \pm 1, \pm 2, \cdots \pm \infty. \quad (26)$$

The results thus derived lead to several interesting conclusions which are summarized in the following lemmas.

Lemma 1: Every steerable linear or planar phased array with a linear phase taper has only a finite number of beams in real space. Proof appears in Appendix *B*.

For every pair of phasing (ψ_x, ψ_y) there exists an infinite set of characteristic directions $\{\theta_{pq}, \varphi_{pq}\}$. As the array is scanned by varying the values of (ψ_x, ψ_y) in the intervals $-\pi \leqq \psi_x \leqq \pi$, $-\pi \leqq \psi_y \leqq \pi$ some characteristic directions will go through a grazing position going from imaginary to real space or vice versa. We shall call such characteristic directions "transitive characteristic directions".* Since the condition for a grazing position is $|\cos \theta_{pq}| = 1$, it follows from Lemma 1 that the number of transitive characteristic directions is finite.

Lemma 2: The radiation impedance of an element in a linear or planar phased array can be expanded by an infinite series over all characteristic directions of the system. Proof appears in Appendix C.

IV. INFINITE ARRAYS

In analyzing large arrays it has been found useful to approximate the behavior of the center elements by the behavior of identical elements in an infinite array of the same geometry.[12] This approximation is motivated by the fact that the performance of the center elements is strongly affected through mutual coupling by contiguous elements, but very weakly by elements far away.[13]

The formulation of the infinite array problem may be obtained from the results derived for finite-size arrays by letting the number of elements M and N approach infinity. The infinite array problem can also be treated as a periodic structure by application of Floquet's theorem. In the following, the latter approach is adopted, but first it is demonstrated that both methods are consistent.

The electric field of an infinite array as given by (8b) must satisfy the same periodicity conditions as the source function, i.e.,

$$\mathbf{E}(x + ma, y + nb, z) = \mathbf{E}(x, y, z) \exp [j\sigma_{mn}(\varphi) \cos \theta]. \qquad (27)$$

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* Note the distinction made between "grazing position" and "transitive characteristic direction". A beam associated with a transitive characteristic direction may attain a grazing position for a particular phasing, but may also point in other directions.

On the other hand, the electric field

$$\mathbf{E}(x, y, z, \theta) = -\int_V \bar{\bar{G}}(x, y, z \mid \xi, \eta, \zeta) \cdot \mathbf{J}(\xi, \eta, \zeta, \theta) \, d\tau \tag{28}$$

can be expanded in an infinite sum of integrals using the relationships of (6):

$$\mathbf{E}(x, y, z, \theta) = -\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp(j\sigma_{mn} \cos \theta) \cdot \int_{V_{00}} \bar{\bar{G}}(x, y, z \mid \xi + ma, \eta + nb, \zeta) \cdot \mathbf{J}_0(\xi, \eta, \zeta) \, dv, \tag{29}$$

where V_{00} is the volume occupied by $\mathbf{J}_0$. Define a new Green's function

$$\bar{\bar{G}}_0(x, y, z \mid \xi, \eta, \zeta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \exp(j\sigma_{mn} \cos \theta) \bar{\bar{G}}(x, y, z \mid \xi + ma, \eta + nb, \zeta) \tag{30}$$

and notice that

$$\bar{\bar{G}}_0(x, y, z \mid \xi + Ma, \eta + Nb, \zeta) = \exp(-j\sigma_{MN} \cos \theta) \bar{\bar{G}}_0(x, y, z \mid \xi, \eta, \zeta) \tag{31}$$

since by (5)

$$\sigma_{m+M,n+N} = \sigma_{mn} + \sigma_{MN} \, . \tag{32}$$

From (27) and (31) it follows that $\bar{\bar{G}}_0(x, y, z \mid \xi, \eta, \zeta)$ can be expanded by the eigenfunctions (25) as

$$\bar{\bar{G}}_0 = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \bar{\bar{g}}_{pq}(z \mid \zeta) F_{pq}(x, y) F_{pq}^*(\xi, \eta), \tag{33a}$$

where

$$\bar{\bar{g}}_{pq} = \frac{1}{ab} \int_0^a \int_0^b \bar{\bar{G}}_0 F_{pq}(\xi, \eta) F_{pq}^*(x, y) \, dx \, dy \, d\xi \, d\eta. \tag{33b}$$

Substituting (30) via (33a) into (15a) for the center element, $m = n = 0$, one obtains

$$Z_{00} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} z_{pq} \, , \tag{34a}$$

where

$$z_{pq} = \frac{1}{|I_{00}|^2} \int_{V_{00}} \int_{V_{00}} \mathbf{J}_0^*(x, y, z) \cdot \bar{\bar{g}}_{pq}(z \mid \zeta) F_{pq}^*(\xi, \eta) F_{pq}(x, y) \cdot \mathbf{J}_0(\xi, \eta, \zeta) \, d\tau \, dv. \tag{34b}$$

Equation (34) is an alternate representation to (86) for the radiation impedance of the infinite array element and it demonstrates that Lemma 2 is valid for infinite arrays as well.

By substituting the new representation for $\bar{\bar{G}}_0$, (30), (33), into (29) and noting that the electric field satisfies the homogeneous reduced wave equation in the source-free region, one obtains for the unbounded space

$$\mathbf{E}(x, y, z) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \boldsymbol{\varepsilon}_{pq} F_{pq}(x, y) \exp(-\gamma_{pq} \mid z \mid) \qquad \mid z \mid > d_{\max}, \tag{35a}$$

where

$$\gamma_{pq} = \sqrt{\Gamma_{pq}^2 - k^2} = jk \sin \theta_{pq} \tag{35b}$$

$$\boldsymbol{\varepsilon}_{pq} e^{-\gamma_{pq}|z|} = -\int_{V_{00}} \bar{\bar{g}}_{pq}(z \mid \zeta) \cdot \mathbf{J}_0(\xi, \eta, \zeta) F_{pq}^*(\xi, \eta) \, d\tau, \tag{35c}$$

and $d_{\max}$ is the projection on the z-axis of the largest distance between two points on the surface enclosing V_{00}. It can be seen that the electric field in the source-free region, above the central area of a large array may be approximated by a finite number of homogeneous plane waves propagating in the real characteristic directions, and an infinite number of nonhomogeneous plane waves, exhibiting exponential decay in the direction perpendicular to the plane of the array. The latter are interpreted as waves propagating in the imaginary characteristic directions.

In an infinite array all elements are embedded in an identical environment, and therefore the power radiated by each element is the same. There is no net power flow into a unit cell through the "side walls". Consequently, the quantity $\mathrm{Re}\{|I_{00}|^2 z_{pq}\}$ of (34b) is equal to the power propagated by the plane wave (p, q) within a unit cell in the direction perpendicular to the plane of the array. By Lemma 1 there is only a finite number of plane waves with transitive characteristic directions (see footnote p. 1571). Let them be distinguished from all other plane waves by assignment of the subscript $(p, q) = (\tau, \nu)$.

$$\mathbf{E}_{\tau\nu} = \boldsymbol{\varepsilon}_{\tau\nu} F_{\tau\nu}(x, y) \exp(-jk \mid z \mid \sin \theta_{\tau\nu}), \qquad \mid z \mid > d_{\max} \tag{36}$$

$$\mathbf{H}_{\tau\nu} = \boldsymbol{\mathcal{H}}_{\tau\nu} F_{\tau\nu}(x, y) \exp(-jk \mid z \mid \sin \theta_{\tau\nu}), \qquad \mid z \mid > d_{\max}, \tag{37}$$

where $F_{\tau\nu}(x, y)$ is given by (26), and $\boldsymbol{\varepsilon}_{\tau\nu}$ by (35c). If

$$\boldsymbol{\mathfrak{D}}_{\tau\nu} = jk[\cos \theta_{\tau\nu} \cos \varphi_{\tau\nu} \mathbf{a}_x + \cos \theta_{\tau\nu} \sin \varphi_{\tau\nu} \mathbf{a}_y - \sin \theta_{\tau\nu} \mathbf{a}_z] \tag{38}$$

then

$$\mathcal{H}_{\tau\nu} = \frac{j}{\omega\mu} \mathfrak{D}_{\tau\nu} \times \mathcal{E}_{\tau\nu} . \tag{39}$$

The power radiated by a (τ, ν) plane wave per unit cell into the upper hemisphere is

$$P_{\tau\nu} = \frac{1}{2} \text{Re} \int_0^a \int_0^b (\mathbf{E}_{\tau\nu} \times \mathbf{H}_{\tau\nu}^*) \cdot \mathbf{a}_z \, dx \, dy. \tag{40}$$

Substitution of (36) through (39) into (40) gives

$$P_{\tau\nu} = \frac{ab}{\eta_0} \sin \theta_{\tau\nu}^* \left[| \mathcal{E}_{\tau\nu} \cdot \mathbf{a}_x |^2 + | \mathcal{E}_{\tau\nu} \cdot \mathbf{a}_y |^2 + \frac{\sin \theta_{\tau\nu}}{\sin \theta_{\tau\nu}^*} | \mathcal{E}_{\tau\nu} \cdot \mathbf{a}_z |^2 \right] \tag{41}$$

where $\eta_0 = (\mu/\epsilon)^{\frac{1}{2}}$. From (41) a radiation resistance per wave is defined as

$$R_{\tau\nu} = \frac{P_{\tau\nu}}{| I_{00} |^2}. \tag{42}$$

Since the entire system is passive and lossless, then by conservation of energy, the power $P_{\tau\nu}$ must originate from the element itself. Hence,

$$R_{\tau\nu} = \text{Re} \{z_{\tau\nu}\}, \qquad \theta_{\tau\nu} \quad \text{real}, \tag{43}$$

where $z_{\tau\nu}$ is given by (34b).

From (41) it follows that when a wave (τ, ν) is in real space $R_{\tau\nu}$ is real, and when it is in imaginary space $R_{\tau\nu}$ is imaginary (in which case $\text{Re}\{z_{\tau\nu}\} = 0$). Hence, of all the elements comprising the source's load, $R_{\tau\nu}$ appears either resistive or reactive, depending upon the scan angle. Such properties of a load, which are unknown in lumped network theory, are a consequence of the losslessness postulate. When propagation is possible power is carried away from the source. When propagation is inhibited there is no net loss of power and the load must be reactive. By Lemma 1 only $\text{Re}\{z_{\tau\nu}\}$ has those properties. All other z_{pq}, $(p, q) \neq (\tau, \nu)$ and $\text{Im}\{z_{\tau\nu}\}$ always retain their dissipative or reactive characteristics. Further, there is only a finite number of terms having $\text{Re}\{z_{pq}\} > 0$. In practical phased arrays the spacing between the elements and the scanning range are such that only one such term exists at a time.

The following two definitions summarize the properties described above:

Definition 1: An O-type network function is a scan-dependent immittance (impedance or admittance) which is seen by the source as resistive when the beam is in real space and as reactive when the beam is in imaginary space, and it behaves like an open circuit for impedance and like a short circuit for admittance in the grazing position.

Definition 2: An E-type network function is a scan-dependent immittance (impedance or admittance) which remains either resistive or reactive when the beam passes through the grazing position.

The motivation behind the nomenclature introduced by the two definitions will become clear later, in Theorems 4 and 5. The O-type and E-type immittances are of distinct mathematical form. To arrive at it consider first the following transformation:*

$$s = \sin\theta_{mn} \tag{44}$$

$$\chi = \cos\varphi_{mn}\,, \tag{45}$$

where (m, n) is one particular transitive characteristic direction out of all (τ, ν). Given s and χ all other characteristic directions are uniquely determined. By (4)

$$\psi_x = ka\chi\sqrt{1 - s^2} - 2m\pi \tag{46}$$

$$\psi_y = kb\sqrt{1-\chi^2}\sqrt{1 - s^2} - 2n\pi, \tag{47}$$

where $(1 - \chi^2)^{\frac{1}{2}} \geqq 0$ for all possible χ and $(1 - s^2)^{\frac{1}{2}} > 0$ if $0 \leqq \theta_{mn} < \pi/2$, and $(1 - s^2)^{\frac{1}{2}} < 0$ if $\pi/2 < \theta_{mn} \leqq \pi$. Then by substitution of (47) into (22) all other characteristic directions are found:

$$\tan\varphi_{pq} = \frac{kab(1-\chi^2)^{\frac{1}{2}}(1 - s^2)^{\frac{1}{2}} + 2(q - n)\pi a}{kab\chi(1 - s^2)^{\frac{1}{2}} + 2(p - m)\pi b} \tag{48a}$$

$$\cos\theta_{pq} = f_{pq}(s), \tag{48b}$$

where

$$f_{pq}(s) = \frac{ka\chi(1 - s^2)^{\frac{1}{2}} + 2(p - m)\pi}{ka\cos\varphi_{pq}}. \tag{48c}$$

This suggests that when characteristic direction (m, n) is scanned in a plane $\chi =$ const, each of the components z_{pq} of the total input impedance as given by (34) can be expressed as a function of the same variable s. The conformal mapping between the θ_{mn}-plane and the s-plane is shown in Fig. 3. In view of the branch cut $-1 \leqq \alpha \leqq 1$ it will be understood that $s = \alpha$ denotes $s = \alpha - j0$ if $0 \leqq \theta_{mn} \leqq \pi/2$ and $s = \alpha + j0$ if $\pi/2 \leqq \theta_{mn} \leqq \pi$. Let $s = s_{\tau\nu}$ be the value at which characteristic direc-

* Recall that θ_{mn} and φ_{mn} are not in the conventional spherical coordinate system (see p. 1564).

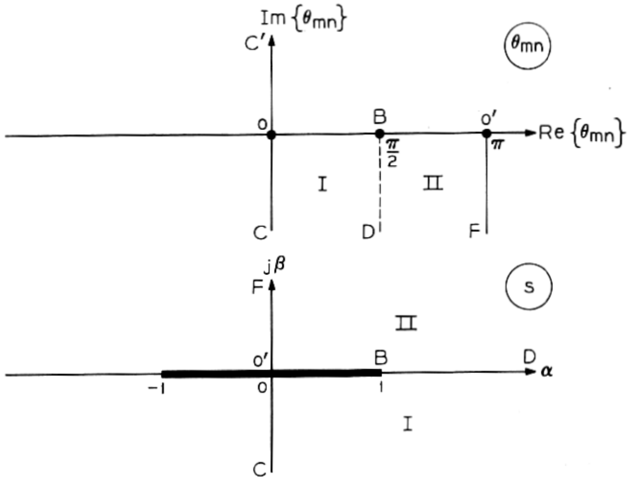

Fig. 3 — Conformal mapping $s = \sin \theta_{mn}$.

tion (τ, ν) is in grazing position. At this value

$$f_{\tau\nu}^2(s_{\tau\nu}) = 1. \tag{49}$$

Of all values $\{s_{\tau\nu}\}$ there is at least one which satisfies (49) for $s_{\tau\nu} = 0$. From (48) it is obvious that $f_{mn}^2(0) = 1$, and there may be other transitive characteristic directions $(\tau, \nu) \neq (m, n)$ which attain their grazing positions at $s_{\tau\nu} = 0$.

Theorem 3: In an obstacle-free space, the impedance function $z_{pq}(s)$, associated with the characteristic direction (p, q), is an analytic function of the complex variable $s = \alpha + j\beta$, with branch points at $s = s_{\tau\nu}$ and an essential singularity at $|s| \to \infty$. If $(p, q) = (m, n)$ then $z_{mn}(s)$ may have a simple pole at $s = s_{mn} = 0$.

Proof: The general definition of z_{pq} is given by (34b) in which the θ_{pq} dependence is contained in the Green's function component $\bar{g}_{pq}(z \mid \zeta)F_{pq}^*(\xi, \eta)F_{pq}(x, y)$. The Green's function is derived from (10) via (15b). Green's function $\bar{\mathcal{G}}(x, y, z \mid \xi, \eta, \zeta)$ satisfies the same periodic boundary conditions as $\bar{G}_0(x, y, z \mid \xi, \eta, \zeta)$ and can be expanded in a series similar to (33a):

$$\bar{\mathcal{G}} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \bar{C}_{pq}(z \mid \zeta)F_{pq}(x, y)F_{pq}^*(\xi, \eta). \tag{50}$$

By substitution of (50) into (10) and use of the orthogonality property of $F_{pq}(x, y)$ one obtains a differential equation for $\bar{\bar{C}}_{pq}(z \mid \zeta)$

$$\frac{d^2\bar{\bar{C}}_{pq}}{dz^2} - \gamma_{pq}^2 \bar{\bar{C}}_{pq}(z \mid \zeta) = -\bar{\bar{I}} \frac{\delta(z - \zeta)}{ab}$$

$$\gamma_{pq} = jk \sin \theta_{pq} \tag{51}$$

with the additional requirement that as $\mid z \mid \to \infty$, $\bar{\bar{C}}_{pq}$ behaves as an outgoing or evanescent wave. The solution of (51) for free space is

$$\bar{\bar{C}}_{pq}(z \mid \zeta) = \bar{\bar{I}} \frac{1}{2jabk \sin \theta_{pq}} \exp \{-jk \mid z - \zeta \mid \sin \theta_{pq}\} \tag{52}$$

$\bar{\bar{g}}_{pq}(z \mid \zeta)$ is obtained from $\bar{\bar{C}}_{pq}(z \mid \zeta)$ through an operator $\bar{\bar{\mathfrak{R}}}_{pq}$:

$$\bar{\bar{g}}_{pq}(z \mid \zeta) = j\omega\mu \bar{\bar{\mathfrak{R}}}_{pq} \cdot \bar{\bar{C}}_{pq}(z \mid \zeta), \tag{53a}$$

where

$$\bar{\bar{\mathfrak{R}}}_{pq} = \bar{\bar{I}} + \frac{1}{k^2} \mathfrak{D}_{pq}\mathfrak{D}_{pq} \tag{53b}$$

$\mathfrak{D}_{pq}$ being given by (38). Substitution of (52) into (53a) followed by substitution into (34b) gives

$$z_{pq} = \frac{\omega\mu}{2abk \mid I_{00} \mid^2 \sin \theta_{pq}} \int_{V_{00}} \int_{V_{00}} \mathbf{J}_0^*(x, y, z) \cdot \bar{\bar{\mathfrak{R}}}_{pq} \cdot \mathbf{J}_0(\xi, \eta, \zeta) \exp \{jk \cos \theta_{pq}[(x - \xi) \cos \varphi_{pq} + (y - \eta) \sin \varphi_{pq}] - jk \mid z - \zeta \mid \sin \theta_{pq}\} \, d\tau \, dv. \tag{54}$$

The integrand is an entire function of θ_{pq} with an essential singularity at $\mid \text{Im} \{\theta_{pq}\} \mid \to \infty$. Hence,* if $\mathbf{J}_0(x, y, z)$ is piecewise continuous, the integral is also an entire function with the same essential singularity. By (48)

$$\sin \theta_{pq} = [1 - f_{pq}^2(s)]^{\frac{1}{2}}. \tag{55}$$

By Lemma 1,

$$f_{pq}^2(s) \neq 1 \quad \text{if} \quad (p, q) \neq (\tau, \nu). \tag{56}$$

From Fig. 3 it is readily seen that $\mid s \mid < \infty$ when $\mid \theta_{pq} \mid < \infty$ which implies, via (48), (55) that $\mid \cos \theta_{pq} \mid < \infty$ and $\mid \sin \theta_{pq} \mid < \infty$ as long as $\mid s \mid < \infty$. Thus, the singularities introduced by the transformation (44), (45) are the branch points at $s = s_{\tau\nu}$. Also if $(p, q) = (m, n)$†

* E. J. Copson, Ref. 11, Sec. 5.5, pp. 107–109.
† Recall that (m, n) is the characteristic direction which defines the transformation from (ψ_x, ψ_y) into (s, χ), (44)–(47).

z_{pq} may have a simple pole at $s = 0$. (Note, for example, that for horizontal polarization, $\mathbf{J}_0 = \mathbf{a}_x J_0$, there is a simple zero in the plane of scan corresponding to $\varphi_{mn} = 0$, at $s = 0$.) Q.E.D.

The above proof can be applied separately to the real and imaginary parts of the right-hand side of (54). If $z_{pq} = R_{pq}(s) + jX_{pq}(s)$, then $R_{pq}(s)$ and $X_{pq}(s)$ are analytic functions of s, real on the real axis of s, with an essential singularity at $| s | \to \infty$, branch points at $s = s_{\tau\nu}$, and possibly simple poles at $s = 0$.

In systems other than obstacle-free space, the normalized complex power $z_{pq}(s)$ has different forms. Except for isolated values of s, the radiated power and the stored energy per unit cell are bounded and continuous functions of s over those portions of the real and imaginary axes of the s-plane which have physical meaning. Hence, it is reasonable to postulate that an analytic continuation of z_{pq} as a function of scan can be made into a region of the complex s-plane which includes portions of both the real and imaginary axes. It may be of interest to note that the impedance function $z_{pq}(s)$ derived by L. Stark[14] for the planar dipole array over a ground plane is analytic. The regularity of $z_{pq}(s)$ depends directly on the regularity of $\bar{g}_{pq}(z \mid \zeta; s)$. The singularities of z_{pq} in the s-plane are determined by the boundary conditions which $\bar{g}_{pq}(z \mid \zeta; s)$ satisfies.

Theorem 4: An E-type immittance function $V(s)$ is an even function of s.

Proof: Let the complex variable s be defined with respect to the transitive characteristic direction (m, n). Once (m, n) is chosen, the proper branch of $(1 - s^2)^{\frac{1}{2}}$ in (48) is uniquely determined. Let (k, l) denote all other transitive characteristic directions which reach their transitive position simultaneously with (m, n). Formally, this implies

$$f_{\tau\nu}^2(0) = 1 \qquad (\tau, \nu) = (m, n), (k, l). \tag{57}$$

As a consequence of Definition 2 and Lemma 1, $V(s)$ is recognizable as

$$V(s) = \begin{cases} R_{pq}(s) & (p, q) \neq (m, n), (k, l), \\ X_{pq}(s) & \text{all} \quad (p, q), \end{cases} \tag{58}$$

where $R_{pq}(s) + jX_{pq}(s) = z_{pq}(s)$, z_{pq} given by (54). Thus, (58) establishes the connection between the defined E-type function and physical quantities corresponding to $R_{pq}(s)$ and $X_{pq}(s)$. Consider Definition 2 which states

$$V(s) - V^*(s) = 0 \qquad s = \alpha \qquad 0 < \alpha < 1, \tag{59a}$$

$$V(s) - V^*(s) = 0 \qquad s = j\beta. \tag{59b}$$

Since $V(s)$ is analytic and also real on the real axis of s, (59) may be rewritten as*

$$V(s) - V(s^*) = 0 \qquad s = \alpha \qquad 0 < \alpha < 1, \tag{60a}$$

$$V(s) - V(s^*) = 0 \qquad s = j\beta. \tag{60b}$$

On the real axis

$$V(\alpha) - V(\alpha) = 0. \tag{61a}$$

On the imaginary axis

$$V(j\beta) - V(-j\beta) = 0. \tag{61b}$$

By analytic continuation† of (61b) from the imaginary axis to a point s in the complex plane one obtains

$$V(s) - V(-s) = 0. \tag{62}$$

Hence, $V(s)$ is an even function of s. Q.E.D.

Theorem 5: An O-type immittance $W(s)$ is an odd function of s. The proof is similar to that of Theorem 4 and it appears in Appendix D.

It has been shown in Theorem 2 that a finite phased array cannot be perfectly matched over a continuous scanning range. The proof is limited to finite arrays and cannot be directly extended to infinite arrays since the representation of the element impedance by (17a) does not guarantee convergence in the complex θ-plane if the limits of the summations are extended to infinity. In treating the infinite array, the element impedance is derived by symmetry considerations from which it is concluded that the net complex power radiated from each element is conserved entirely within the unit cell of that element. It has been shown that the two definitions are consistent. Although the problem of whether an infinite array can be perfectly matched is of academic interest only, it is worthwhile noting that as for finite arrays, the answer in this case is *also* negative. To show this the reader may recall that the impedance has been defined as normalized power and postulated to be an analytic function of the scan variable $s = \alpha + j\beta$. The normalization constant is $|I_{00}|^2$ given by (13). If the complex power as a function of scan is represented by

$$\hat{P}(s) = | I_{00} |^2 [R(s) + jX(s)], \tag{63}$$

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* P. M. Morse and H. Feshbach, Ref. 7, Vol. I, p. 393.
† Morse and Feshbach, *Op. Cit.*, p. 389.

then by Lemma 1, the term $R(s)$ is a finite sum of analytic functions of the complex variable s. Consequently, $R(s)$ is an analytic function of s. In general, it may be represented as

$$R(s) = E(s) + \mathcal{O}(s), \tag{64}$$

where $E(s)$ is an even function of s and $\mathcal{O}(s)$ is an odd function of s. Under conditions of perfect match over a continuous range, constant power, P_r, is radiated over that range. Since $R(s)$ is analytic it implies $R(s) = P_r \mid I_{00} \mid^{-2}$ everywhere in the s-plane. Since a constant is even, $\mathcal{O}(s) = 0$. Further, $E(s)$ must have a branch cut on the real axis of the s-plane in the interval $[-1,1]$. But the branch cut does not exist if $E(s) = P_r \mid I_{00} \mid^{-2}$. The contradiction implies that $\hat{P}(s)$ in (63) cannot equal a constant over a continuous range of s.

Theorem 6: The resistance and reactance functions of an element, or their derivatives, in an infinite linear or planar phased array of current sources are discontinuous when a grating lobe is in a grazing position.

Proof: In an infinite array the grating lobes are plane waves propagating in the characteristic directions. By Theorems 4 and 5 the element impedance $Z(s)$ in an obstacle-free space can be written as

$$Z(s) = P(s) + \frac{Q(s)}{s}. \tag{65}$$

For real values of s, $P(s)$ is an even complex function of s bounded at $s = 0$, and $Q(s)$ is an even real function of s nonzero at $s = 0$. On the real axis of s

$$Z(\alpha) = P(\alpha) + \frac{Q(\alpha)}{\alpha}. \tag{66a}$$

On the imaginary axis of s

$$Z(j\beta) = P(j\beta) - j\,\frac{Q(j\beta)}{\beta}. \tag{66b}$$

A grating lobe is in its transitive position at $s = 0$. The pole discontinuities are established by showing that

$$\operatorname{Re}\,\{\lim_{\alpha\to 0} Z(\alpha) - \lim_{\beta\to 0} Z(j\beta)\} = \lim_{\alpha\to 0} \frac{Q(\alpha)}{\alpha} = \infty \tag{67a}$$

$$\operatorname{Im}\,\{\lim_{\alpha\to 0} Z(\alpha) - \lim_{\beta\to 0} Z(j\beta)\} = \lim_{\beta\to 0} \frac{Q(j\beta)}{\beta} = \infty\,. \tag{67b}$$

The pole discontinuity has to be interpreted as an invalid mathematical solution at the transitive position. It is a result of the idealization introduced by the concept of an "infinite array." If $R_{mn}(s)$ has a simple zero at $s = 0$, as is the case when a horizontally polarized array is placed above a ground plane, then the active impedance in the neighborhood of $s = 0$ can be written as

$$Z(s) = R(s) + jX(s), \tag{68a}$$

where $R(s)$ and $X(s)$ are real functions of s (real for s real).

$$R(s) = \sum_{i=0}^{\infty} a_i s^i \tag{68b}$$

$$X(s) = \sum_{i=0}^{\infty} b_{2i} s^{2i}. \tag{68c}$$

When the beam whose transitive characteristic direction is in real space, $s = \alpha$

$$R_\alpha \triangleq R(\alpha) = \sum_{i=0}^{\infty} a_i \alpha^i \tag{69a}$$

and when it is in imaginary space, $s = j\beta$

$$R_\beta \triangleq \text{Re}\,\{R(j\beta)\} = \sum_{i=0}^{\infty} (-1)^i a_{2i} \beta^{2i}. \tag{69b}$$

The discontinuity in the derivative of the resistance is

$$\lim_{\alpha \to 0} \frac{dR_\alpha}{d\alpha} - \lim_{\beta \to 0} \frac{dR_\beta}{d\beta} = a_1. \tag{70}$$

Similarly, the reactance

$$X_\alpha \triangleq X(\alpha) = \sum_{i=0}^{\infty} b_{2i} \alpha^{2i} \tag{71a}$$

$$X_\beta \triangleq \text{Im}\,\{Z(j\beta)\} = \sum_{i=0}^{\infty} (-1)^i [b_{2i}\beta^{2i} + a_{2i+1}\beta^{2i+1}] \tag{71b}$$

$$\lim_{\alpha \to 0} \frac{dX_\alpha}{d\alpha} - \lim_{\beta \to 0} \frac{dX_\beta}{d\beta} = -a_1 . \tag{72}$$

The proof can be generalized for any order algebraic singularity or zero at $s = 0$. For example, if there is a zero of multiplicity N the discontinuity will be in the Nth derivatives of the resistance and reactance. A noninteger order zero yields a discontinuity after a sufficient number of differentiations. Q.E.D.

V. SUMMARY AND CONCLUSIONS

A new mathematical approach to phased arrays has been adopted to investigate and discover various properties of the radiation impedance of an array element as a function of scan angle. The underlying idea of the method is the treatment of the impedance as an analytic function of a complex scan variable, which enables one to prove that an array element subject to the model chosen cannot be perfectly matched over a continuous scanning range by using lossless, linear, passive and time-invariant elements.

The first half of the theory is devoted to finite arrays. It is shown that the directions (in space) of the beams' maxima are eigenvalues of a Laplacian differential operator with periodic boundary conditions, which are related to the phase taper of the array. It is proven that there exists only a finite number of real eigenvalues. The known concept of imaginary space is then adopted to accommodate the imaginary eigenvalues. Furthermore, it is demonstrated that all beams except a finite number are completely confined either to real space or to imaginary space, and that only a finite number of beams may attain a grazing position. The unique properties of the latter beams have been found to play an important role in the investigation of infinite arrays, to which the second half of the theory is devoted.

The interest in infinite arrays, apart from its academic aspect, stems from the good approximation it provides for the behavior of the center portion of a large finite array. It has been found that the infinite array element impedance as a function of scan is restricted to a specific mathematical form. It is the authors' hope that recognition of the limitations imposed by that form may provide useful guidelines in achieving optimal match of an array to space.

VI. ACKNOWLEDGMENT

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APPENDIX A

Proof of Corollary 1a

Corollary 1a: $Re\{Z_{rs}\}$ *and* $\dfrac{\partial}{\partial\theta} Re\{Z_{rs}\}$ *are entire functions of* θ *each with an essential singularity at* $\theta \rightarrow \infty$.

Proof: Denoting

$$\tilde{z}_{mnrs} = \rho_{mnrs} + j\chi_{mnrs} \tag{73}$$

one obtains from (17a)

$$\begin{aligned}\operatorname{Re}\{Z_{rs}(\theta)\} &\triangleq R_{rs}(\theta) \\ &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \{\rho_{mnrs} \cos [B_{mnrs}(\theta)] - \chi_{mnrs} \sin [B_{mnrs}(\theta)]\},\end{aligned} \tag{74}$$

where

$$B_{mnrs}(\theta) = (\sigma_{mn} - \sigma_{rs}) \cos \theta \tag{75}$$

and

$$\begin{aligned}\frac{\partial}{\partial \theta} R_{rs}(\theta) = -\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} B'_{mnrs}(\theta) \{\rho_{mnrs} \sin [B_{mnrs}(\theta)] \\ + \chi_{mnrs} \cos [B_{mnrs}(\theta)]\}.\end{aligned} \tag{76}$$

Since $\cos \theta$ is an entire function of θ, $\cos[B_{mnrs}(\theta)]$ and $\sin[B_{mnrs}(\theta)]$ are entire functions of an entire function, and are therefore entire. The existence of the essential singularity can be demonstrated in a similar fashion to that in Theorem 1. Q.E.D.

APPENDIX B

Proof of Lemma 1

Lemma 1: Every steerable linear or planar phased array with a linear phase taper has only a finite number of beams in real space.

Proof: A beam (p, q) is in real space if $|\cos \theta_{pq}| \leqq 1$. Dividing (4a) by ka and (4b) by kb, squaring and adding, one obtains

$$\left(\frac{\psi_x + 2p\pi}{ka}\right)^2 + \left(\frac{\psi_y + 2q\pi}{kb}\right)^2 \leqq 1 \tag{77}$$

or

$$\left(\frac{\psi_x}{2\pi} + p\right)^2 \left(\frac{\lambda}{a}\right)^2 + \left(\frac{\psi_y}{2\pi} + q\right)^2 \left(\frac{\lambda}{b}\right)^2 \leqq 1. \tag{78}$$

Necessary conditions for the above inequality to be satisfied are

$$\left|\frac{\psi_x}{2\pi} + p\right| \frac{\lambda}{a} \leqq 1 \tag{79}$$

$$\left|\frac{\psi_y}{2\pi} + q\right| \frac{\lambda}{b} \leqq 1. \tag{80}$$

Since

$$-\frac{1}{2} \leqq \frac{\psi_x}{2\pi} \leqq \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} \leqq \frac{\psi_y}{2\pi} \leqq \frac{1}{2},$$

$$|p| \leqq \frac{a}{\lambda} + \frac{1}{2} \tag{81}$$

$$|q| \leqq \frac{b}{\lambda} + \frac{1}{2}. \tag{82}$$

Hence, both p and q are bounded. Q.E.D.

APPENDIX C

Proof of Lemma 2

Lemma 2: The radiation impedance of an element in a linear or planar phased array can be expanded by an infinite series over all characteristic directions of the system.

Proof: The current density excitation function of a finite-size array given by (1), (2) satisfies the periodic boundary conditions (23) in the finite domain occupied by the array. Let this domain be denoted by D. The current density can, therefore, be uniquely expanded in D in terms of the eigenfunctions (25):

$$\mathbf{J}(x, y, z) = U(x, y, D) \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathbf{j}_{pq}(z) F_{pq}(x, y), \tag{83}$$

where

$$\mathbf{j}_{pq}(z) = \frac{1}{ab} \int_0^a \int_0^b \mathbf{J}_0(x, y, z) F_{pq}^*(x, y)\, dx\, dy \tag{84}$$

and $U(x, y, D)$ is a two-dimensional unit step function

$$U(x, y, D) = \begin{cases} 1 & (x, y) \text{ in } D, \\ 0 & \text{otherwise.} \end{cases} \tag{85}$$

Substitution of (31a) into (15a) yields

$$Z_{mn} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \hat{z}_{mnpq}, \tag{86a}$$

where

$$\hat{z}_{mnpq} = \frac{1}{|I_{mn}|^2} \int_{V_{mn}} \int_V \mathbf{J}_{mn}^*(x, y, z) \cdot \bar{\bar{G}}(x, y, z \mid \xi, \eta, \zeta) \cdot \mathbf{j}_{pq}(\zeta) F_{pq}(\xi, \eta) U(\xi, \eta, D)\, d\tau\, dv. \tag{86b}$$

Q.E.D.

APPENDIX D

Proof of Theorem 5

Theorem 5: An O-type immittance $W(s)$ is an odd function of s.

Proof: Let the complex variable s be defined with respect to the transitive characteristic direction (m, n). Let (k, l) be all other transitive characteristic directions which reach their transitive position simultaneously with (m, n). Then as a consequence of Definition 1 and Lemma 1

$$W(s) = R_{pq}(s) \qquad (p, q) = (m, n), (k, l), \tag{87}$$

where $R_{pq}(s) = \text{Re}\{z_{pq}\}$, z_{pq} given by (54). Thus, (87) establishes the connection between the defined O-type function and a physical quantity corresponding to $R_{pq}(s)$. From Definition 1

$$W(s) - W^*(s) = 0 \qquad s = \alpha, \qquad 0 < \alpha < 1 \tag{88}$$

$$W(s) + W^*(s) = 0 \qquad s = j\beta. \tag{89}$$

Since $W(s)$ is real on the real axis of s, (88), (89) may be rewritten as

$$W(s) - W(s^*) = 0 \qquad s = \alpha \tag{90}$$

$$W(s) + W(s^*) = 0 \qquad s = j\beta. \tag{91}$$

On the real axis

$$W(\alpha) - W(\alpha) = 0. \tag{92}$$

On the imaginary axis

$$W(j\beta) + W(-j\beta) = 0. \tag{93}$$

By analytic continuation of (93) from the imaginary axis to a point s in the complex plane one obtains

$$W(s) + W(-s) = 0. \tag{94}$$

Hence, $W(s)$ is an odd function of s. Q.E.D.

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