Error Probability for Binary Signaling Through a Multipath Channel

By R. T. AIKEN

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Error probability is considered for binary signaling through a multipath channel in which (i) the receiver observes a waveform comprising white Gaussian noise and the sum of (perhaps several) time-delayed, frequency-shifted, Rayleigh-faded versions of the transmitted waveform, (ii) the receiver decides with minimum error probability which of the two possible transmissions was sent. Results given herein for the exact minimum error probability necessarily depend upon a number of parameters and are cumbersome to use. By introducing bounds on the error probability, depending upon bounds on spectra of certain matrices, the number of parameters is reduced and the less cumbersome results become applicable to any one of a set of channels rather than to just one channel. The error-probability bounds are presented in terms of values of the distribution function, derived herein, of the difference of two chi-square random variables. The bounds are sharp when the spectra are narrow. For the case of widely orthogonal signals, any version of one possible transmission being orthogonal to any version of the other transmission, the bounds are given as a set of universal curves plotted versus signal-to-noise ratio for various values of the number of paths and of the spectral width of certain matrices. Spectral bounds can easily be computed when the versions for each transmission are nearly orthogonal. Returning to the general case, another bound is derived, by a technique due to Chernoff, which does not explicitly require spectral bounds which may neither be readily available nor be accurate approximations of eigenvalues. This bound is not as sharp as the previous bound for the case of small spectral width, but has promise for the large-width case.

I. INTRODUCTION

This paper considers error probability for the optimum reception of binary signals transmitted through a multipath channel having

P paths.* One of two possible signals is transmitted; the received waveform is the sum of P Rayleigh-faded, time-delayed, frequency-shifted versions of the transmitted signal, plus white Gaussian noise. That is to say, if the complex signal $\sqrt{2E_m}\, x_m(t)$ is transmitted, $m = 1, 2$, the contribution to the received waveform from the pth path is

$$y_m(t; p) = \sqrt{2E_m}\, a_p x_m(t - \tau_p) \exp [i(2\pi f_p t + \varphi_p)],$$

where a_p, φ_p, τ_p, and f_p are the Rayleigh-distributed amplitude, the uniformly-distributed phase, the fixed time delay, and the fixed frequency shift associated with the pth path. The received waveform is

$$z_m(t) = \sum_{p=1}^{P} y_m(t; p) + n(t),$$

where $n(t)$ is white Gaussian noise.

The above multipath situation is a special case of a more general communications situation in which a receiver observes a sample $z(t)$ of a zero-mean complex Gaussian process on the time interval [O, T], the covariance function $\langle z(s) z^*(t) \rangle_m$ having been selected from a set of two distinct functions by chance according to the prior probabilities $\{\alpha_m\}$, $m = 1, 2$, and the other second-moment function $\langle z(s) z(t) \rangle_m$ being zero. The receiver is to be designed so that its decision upon one of the two possible hypotheses is made with minimum average error probability P_e, where $P_e = \Sigma \alpha_m P_e(m)$ and $P_e(m)$ is the probability, when covariance indexed m is true, of deciding otherwise.

The receiver-design problem has been treated in Ref. 1, rigorously demonstrating that optimum processing involves quadratic filtering. However, the filter kernels, being the solutions of integral equations, are difficult to determine in general; moreover, the error probability is not evaluated. For the multipath channel, the first difficulty is overcome in Ref. 2 and the evaluation of binary error probability is considered in the present paper.

Section II presents the theory of a method that can be used to calculate error probability exactly. However, it is quickly appreciated that error probability depends in a cumbersome fashion upon a large number of parameters including the path strengths and the scalar products of the versions. To simplify this situation, this paper introduces bounds on the error probability which depend upon bounds on the spectra of certain matrices, the eigenvalues of which determine

* Each path could comprise a multitude of randomly phased subpaths having essentially the same delay and frequency-shift parameters.

error probability exactly. Thus, the bounds are applicable to any one of a set of channels rather than to just one channel.

Section III presents these error-probability bounds in terms of values of the distribution function of the difference of two chi-square random variables and then derives this distribution function. More specific results are obtained in Section IV for the case of widely orthogonal signals, any path's version of one of the two possible transmitted waveforms being orthogonal to any path's version of the other waveform. Here, easily computed spectral bounds can be given for the case in which the versions under each hypothesis are nearly orthogonal. Section V considers the case of well-resolved paths, making contact with diversity theory (Ref. 3, Chap. 7), and the case of on-off keying.

The error-probability bounds considered above require spectral bounds which may not always be easily computed and which may not be accurate approximations of the eigenvalues. A bound that circumvents these difficulties is obtained in Section VI with a technique due to Chernoff. Comparison of this bound with previous bounds is carried out analytically only for the case of well-resolved paths, but qualitative comparison is made for more general cases.

II. PROCEDURE TO OBTAIN ERROR PROBABILITY IN THE GENERAL CASE

2.1 *Notation*

The binary situation is a specialization of the case of M-ary signaling through the multipath channel in which the received process $z(t)$ can have one of M possible covariance functions, $\langle z(s)z^*(t)\rangle_m$, $m = 1, 2, \ldots M$, of the form

$$2E_m \sum_{p=1}^{P} \sigma_p b_p(s, m) b_p^*(t, m) + N_0 \delta(s - t),$$

a degenerate kernel plus a white-noise kernel (Ref. 2). Here $b_p(t, m) = \exp(i2\pi f_p t) x_m(t - \tau_p)$ is a time-doppler-shifted normalized version of the transmitted signal $\sqrt{2E_m}\, x_m(t)$; the path with index p has an average cross section of σ_p units, a delay of τ_p seconds, and a doppler-shift of f_p Hz. We put

$$\int dt \mid x_m(t) \mid^2 = \int dt \mid b_p(t, m) \mid^2 = 1,$$

so that the average energy received from the medium is

$$\frac{1}{2} \int dt\, 2E_m \sum \sigma_p \mid b_p(t, m) \mid^2 = E_m \sum \sigma_p = E_m ,$$

since we put $\sum \sigma_p = 1$.

The above covariance function can be written

$$2E_m \tilde{b}(s, m) \sigma b^*(t, m) + N_0 \delta(s - t),$$

where $b(t, m)$ is a vector with pth component $b_p(t, m)$ and σ is the diagonal matrix with pth entry σ_p, with $tr\ \sigma = 1$.

The optimum receiver decides according to the value of m that corresponds to the largest of M test statistics computed as follows. For each value of m, the receiver first generates the column vector $Z(m) = N_0^{-\frac{1}{2}} \int dt\, z(t) b^*(t, m)$ and then evaluates a test statistic comprising a Hermitian form in $Z(m)$ plus a bias constant. This test statistic is

$$[(N_0/2E_1)^{\frac{1}{2}} Z(m)]^{\dagger} (2E_m/N_0) H(m) [(N_0/2E_1)^{\frac{1}{2}} Z(m)] + (N_0/2E_1) \theta(m),$$

where the Hermitian combining matrix is

$$(2E_m/N_0) H(m) = (2E_m/N_0)[(2E_m/N_0) B(m) + \sigma^{-1}]^{-1},$$

the bias is given by,

$$\theta(m) = \log \frac{\alpha_m \det [(2E_1/N_0) B(1) + \sigma^{-1}]}{\alpha_1 \det [(2E_m/N_0) B(m) + \sigma^{-1}]},$$

$B(m)$ is the correlation-function matrix $\int dt\, b^*(t, m) \tilde{b}(t, m)$, and the hypotheses are ordered so that $E_1 = \max E_m$. The above test statistic is obtained from that given in Ref. 2 by subtracting $\log [\alpha_1 \det^{-1} \sigma \det^{-1} H^{-1}(1)]$ and multiplying all resulting terms by $N_0/2E_1$.

The above test statistic has a certain intuitive appeal. The components of the vector $Z(m)$ are the correlations of the received signal against the noise-free versions of the transmitted signal that would occur when message m is sent. That is to say, $Z(m)$ provides a measure of the projection of $z(t)$ on the P-dimensional subspace spanned by these versions. Moreover, the test statistic is a measure of the likelihood that this P-dimensional subspace is in fact the correct subspace. Then the optimum receiver strategy is decision according to the most likely of the M possible subspaces. Also, since P dimensions are involved, it might be anticipated that the results are related to the case of P-fold diversity, cf. Section 5.1.

Henceforth, only the binary case, $M = 2$, is considered. In this case, decision according to the larger of two test statistics is equivalent to decision according to the sign of their difference. The decision events can then be written in terms of one Hermitian form in a composite Gaussian vector

$$Z = (N_0/2E_1)^{\frac{1}{2}} \begin{pmatrix} Z(1) \\ Z(2) \end{pmatrix}$$

as follows. Let

$$Q = \begin{bmatrix} \frac{2E_1}{N_0} H(1) & O_{P \times P} \\ O_{P \times P} & -\frac{2E_2}{N_0} H(2) \end{bmatrix},$$

where $0_{P\times P}$ is the $p \times p$ zero matrix. Then the receiver decides upon $m = 2$ when $Z^\dagger QZ$ is less than $(N_0/2E_1)\theta(2)$, and decides upon $m = 1$ otherwise.

The conditional error probabilities are thus

$$P_e(1) = \Pr \{Z^\dagger QZ < (N_0/2E)\theta \mid 1\} = F_1\left[\left(\frac{N_0}{2E}\right)\theta\right],$$

$$P_e(2) = \Pr \{Z^\dagger QZ > (N_0/2E)\theta \mid 2\} = 1 - F_2\left[\left(\frac{N_0}{2E}\right)\theta\right],$$

where $E = E_1$, $\theta = \theta(2)$, and $F_m(x)$ is the distribution function of $Z^\dagger QZ$ conditioned upon the mth hypothesis.

2.2 *The Fundamental Matrices*

Since $Z^\dagger QZ$ is a function of a Gaussian vector, the distribution function $F_m(x)$ is determined by the conditional mean, $\langle Z\rangle_m$, which is the zero vector, and by the conditional covariance $L(m) = \langle ZZ^\dagger\rangle_m$, the other second-moment matrix $\langle Z\tilde{Z}\rangle_m$ being the $2P \times 2P$ zero matrix.

The conditional covariance matrix $L(m)$ is evaluated as follows. Let

$$L(m) = \begin{bmatrix} L^{11}(m) & L^{12}(m) \\ L^{21}(m) & L^{22}(m) \end{bmatrix},$$

where $L^{jk}(m) = (N_0/2E_1)\langle Z(j)Z^\dagger(k)\rangle_m$. Then, by the definition of $Z(j)$ and interchange of operations, we obtain

$$\langle Z(j)Z^\dagger(k)\rangle_m = \frac{1}{N_0} \iint ds\, dt\, b^*(s, j)\langle z(s)z^*(t)\rangle_m \tilde{b}(t, k)$$

$$= \frac{1}{N_0} \iint ds\, dt\, b^*(s, j) 2E_m \tilde{b}(s, m) \sigma b^*(t, m) \tilde{b}(t, k)$$

$$+ \iint ds\, dt\, b^*(s, j)\, \delta(s - t) \tilde{b}(t, k)$$

$$= \frac{2E_m}{N_0} B(j, m) \sigma B(m, k) + B(j, k),$$

where $B(j, m) = \int ds\, b^*(s, j)\tilde{b}(s, m)$ is a cross-correlation matrix. Hence,

$$L^{jk}(m) = (E_m/E_1)B(j, m)\sigma B(m, k) + (N_0/2E_1)B(j, k).$$

Similarly, it is found that $\langle Z\tilde{Z}\rangle_m$ is the $2P \times 2P$ zero matrix.

For future computations, it is convenient to write

$$Q = \begin{pmatrix} Q^{11} & 0 \\ 0 & Q^{22} \end{pmatrix},$$

where

$$Q^{11} = B^{-1}(1)\{I + (N_0/2E_1)[B(1)\sigma]^{-1}\}^{-1},$$

$$Q^{22} = -(E_2/E_1)B^{-1}(2)\{(E_2/E_1)I + (N_0/2E_1)[B(2)\sigma]^{-1}\}^{-1}.$$

2.3 *The Characteristic-Function Method*

To obtain the distribution, consider the conditional characteristic function

$$\varphi_m(t) = \langle \exp\,(itZ^\dagger QZ)\rangle_m .$$

It is well known, e.g., Ref. 4, that

$$\varphi_m(t) = \det{}^{-1}\,[I - itL(m)Q] = \prod_k\,[1 - it\lambda_k(m)]^{-1},$$

where $\{\lambda_k(m)\}$ is the set of eigenvalues of the matrix $L(m)Q$. The eigenvalues are real, since $L(m)Q$ is similar to the Hermitian matrix $L^{\frac{1}{2}}(m)QL^{\frac{1}{2}}(m)$.

The distribution function can now be obtained from the characteristic function. As a preliminary, it is noted that the characteristic function $(1 - it\lambda)^{-n}$ corresponds to one of two distribution functions, according to the sign of λ. When λ is positive, the distribution function is

$$\int_{-\infty}^{y} dx\; U(x)\,\frac{x^{n-1}\exp\,(-x/\lambda)}{\lambda^n(n-1)!} = \begin{cases} I(y/\lambda, n-1) & (y > 0), \\ 0 & (y < 0), \end{cases}$$

$$= U(y)I(y/\lambda, n - 1),$$

where $U(x)$ is the unit step function (unity for $x > 0$, zero for $x < 0$, one-half for $x = 0$) and where

$$I(y, n) = \frac{1}{n!}\int_0^y dx\, x^n e^{-x} = 1 - e^{-y}\sum_{k=0}^{n}\frac{y^k}{k!}$$

is the incomplete gamma function. Similarly, when λ is negative, the distribution function is

$$\int_{-\infty}^{y} dx\, U(-x)\frac{(-x)^{n-1}\exp(x\,|\,\lambda\,|^{-1})}{|\,\lambda\,|^n (n-1)!}$$

$$= \begin{cases} 1 & (y > 0), \\ 1 - I(-y\,|\,\lambda\,|^{-1}, n-1) & (y < 0), \end{cases}$$

$$= 1 - U(-y)I(y/\lambda, n-1).$$

To obtain the distribution function of $Z^\dagger QZ$, the characteristic function is expanded into its partial fractions. Each term will be proportional to $(1 - it\lambda)^{-n}$ for some n, and corresponds to a term in the expansion of the distribution function. For example, when all eigenvalues are distinct, the expansion of the characteristic function is

$$\varphi_m(t) = \sum_k \frac{d_k(m)}{1 - it\lambda_k(m)},$$

where

$$d_k(m) = \prod_{j \neq k}\left(1 - \frac{\lambda_j(m)}{\lambda_k(m)}\right)^{-1}.$$

The expansion of the distribution function $F_m(x)$ is then

$$\sum_{\{k:\lambda_k(m)>0\}} d_k(m)U(x)I(y/\lambda_k(m), 0)$$

$$+ \sum_{\{k:\lambda_k(m)<0\}} d_k(m)[1 - U(-x)I(y/\lambda_k(m), 0]$$

In the case of a degenerate spectrum, an eigenvalue λ with multiplicity r contributes the sum $\sum_{n=1}^{r} A_n(1 - it\lambda)^{-n}$ to the expansion of the characteristic function, and the corresponding part of the distribution function involves $I(\cdot, n)$ for $n = 0, 1, 2, \cdots, r - 1$.

It should be observed that the general approach of summing distribution functions corresponding to partial fractions is fully equivalent to inverting the characteristic function by contour integration, the approach used by Turin[5] for a similar problem. (When all poles are simple, the expansion coefficients $\{d_k(m)\}$ are residues of the poles.)

III. UPPER AND LOWER BOUNDS ON THE ERROR PROBABILITY

3.1 *Error-probability Bounds from Degenerate-spectrum Variables*

Exact computation of error probability involves considerable numerical work in computing eigenvalues followed by evaluation of cumbersome formulas. Moreover, an often inordinately large number of independent parameters must be specified. To simplify this situation, we consider bounds on the spectrum of $L(m)Q$ rather than the spectrum itself. With a technique suggested in Ref. 6, we can obtain error-probability bounds. Although we do not obtain the error probability itself, the error-probability bounds apply to not just one channel but rather to any channel for which the spectral bounds are met.

Observe that the characteristic function is precisely specified by the spectrum of $L(m)Q$. This spectrum is the same as the spectrum of I diag $[\lambda_1(m), \cdots, \lambda_{2P}(m)]$, where I plays the role of a covariance matrix and the diagonal matrix plays the role of a matrix of a Hermitian form. Hence, the distribution of $Z^\dagger QZ$ is the same as the distribution of

$$q(m) = \sum_{k=1}^{2P} \lambda_k(m) \mid z_k \mid^2,$$

where $\{z_k\}$ are complex zero-mean Gaussian variates with covariance matrix $\langle z_j z_k^* \rangle = \delta_{jk}$, $\langle z_j z_k \rangle$ being zero.

Suppose bounds on the eigenvalues are available. That is to say, suppose it is known that the positive eigenvalues satisfy

$$\underline{\mu} \leqq \lambda_k(m) \leqq \bar{\mu}, \tag{1a}$$

and that the negative eigenvalues satisfy

$$-\bar{\nu} \leqq \lambda_k(m) \leqq -\underline{\nu}, \tag{1b}$$

where the μ's and ν's are positive numbers that depend on m. Then, a lower bound on $q(m)$ is the degenerate-spectrum random variable $\underline{q}(m)$, defined by

$$\underline{q}(m) = \underline{\mu} \sum_{k=1}^{P} \mid z_k \mid^2 - \bar{\nu} \sum_{k=P+1}^{2P} \mid z_k \mid^2.$$

Note that we have used the fact that the number of positive eigenvalues and the number of negative eigenvalues are the same, see Appendix A. Similarly, an upper bound on $q(m)$ is provided by the random variable

$$\bar{q}(m) = \bar{\mu} \sum_{k=1}^{P} \mid z_k \mid^2 - \underline{\nu} \sum_{k=P+1}^{2P} \mid z_k \mid^2.$$

Since $\underline{q}(m) \leqq q(m) \leqq \bar{q}(m)$, it follows that

$$\Pr\{\bar{q}(m) \leqq y\} \leqq F_m(y) = \Pr\{q(m) \leqq y\} \leqq \Pr\{\underline{q}(m) \leqq y\}.$$

Evaluation of these bounds requires the distribution function $G(y; P, \alpha)$ of the degenerate-spectrum random variable

$$\sum_{k=1}^{P} | z_k |^2 - \alpha \sum_{k=P+1}^{2P} | z_k |^2,$$

which is the difference of two chi-square variables each with an even number of degrees of freedom. The bounds become

$$G[(\bar{\mu})^{-1}y; P, \underline{\nu}(\bar{\mu})^{-1}] \leqq F_m(y) \leqq G[(\underline{\mu})^{-1}y; P, \bar{\nu}(\underline{\mu})^{-1}],$$

where we use $y = (N_0/2E)\theta$ and reiterate that the μ's and ν's depend on m.

It is anticipated that these bounds are sharp when the spectrum is narrow, the spread of the positive spectrum being much less than any positive eigenvalue and similarly for the negative spectrum. Also, when θ itself is not precisely known, but bounds $\underline{\theta} \leqq \theta \leqq \bar{\theta}$ are available, the distribution function is bounded by

$$G[(\bar{\mu})^{-1}\underline{y}; P, \underline{\nu}(\bar{\mu})^{-1}] \leqq F_m(y) \leqq G[(\underline{\mu})^{-1}\bar{y}; P, \bar{\nu}(\underline{\mu})^{-1}], \tag{2}$$

where $\underline{y} = (N_0/2E)\underline{\theta}$ and $\bar{y} = (N_0/2E)\bar{\theta}$.

3.2 *Distribution of a Degenerate-Spectrum Variable*

It will be demonstrated that $G(y; P, \alpha)$ equals

$$\left(\frac{\alpha}{1+\alpha}\right)^P \sum_{k=0}^{P-1} \binom{P-1+k}{k} \left(\frac{1}{1+\alpha}\right)^k \left[1 - I\left(\frac{|y|}{\alpha}, P-1-k\right)\right] \tag{3a}$$

when $y < 0$, and equals

$$\sum_{k=0}^{P-1} \binom{P-1+k}{k} \left(\frac{1}{1+\alpha}\right)^k \left[\left(\frac{\alpha}{1+\alpha}\right)^P + \frac{\alpha^k}{(1+\alpha)^P} I(y, P-1-k)\right] \tag{3b}$$

when $y > 0$.

Before doing so, note that when $y < 0$, the parameter α serves as a scale size for y in the argument of $I(x, n)$, but that this is not true when $y > 0$. Nevertheless, α does act as a scale size in the following way. A power-series expansion of $I(x, n)$ yields

$$\frac{\alpha^k}{(1+\alpha)^P} I(y, P-1-k)$$
$$= \left(\frac{\alpha}{1+\alpha}\right)^P \left(\frac{y}{\alpha}\right)^{P-k} \frac{1}{(P-k)!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{P-k}{P-k+n} y^n,$$

and when $\alpha \ll 1$, the factor $(y/\alpha)^{P-k}$ determines the small-y behavior. Also, this result exhibits $[\alpha/(1 + \alpha)]^P$ as a factor for the case $y > 0$, in agreement with the expression for $y < 0$.

To find $G(y; P, \alpha)$, we consider its characteristic function

$$(1 - it)^{-P}(1 + it\alpha)^{-P}.$$

Let the partial-fraction expansion of this characteristic function be

$$\sum_{m=0}^{P-1} A_{P-m}(1 - it)^{-(P-m)} + \sum_{n=0}^{P-1} B_{P-n}(1 + it\alpha)^{-(P-n)}.$$

To evaluate A_{P-m}, multiply by $(1 - it)^P$ and let $1 - it = \tau$ to obtain

$$(1 + \alpha - \alpha\tau)^{-P} = \sum_{m=0}^{P-1} A_{P-m}\tau^m + \tau^P \sum_{n=0}^{P-1} B_{P-n}(1 + \alpha - \alpha\tau)^{-(P-n)}.$$

Since the second sum is analytic at $\tau = 0$, we have exhibited the Taylor expansion with remainder. But

$$\begin{aligned}(1 + \alpha - \alpha\tau)^{-P} &= (1 + \alpha)^{-P}\left(1 - \frac{\alpha}{1 + \alpha}\tau\right)^{-P} \\ &= \left(\frac{1}{1 + \alpha}\right)^P \sum_{k=0}^{\infty} \binom{P + k - 1}{k}\left(\frac{\alpha}{1 + \alpha}\right)^k \tau^k,\end{aligned}$$

where we have used (7) on page 2 of Ref. 7. Hence,

$$A_{P-m} = \left(\frac{1}{1 + \alpha}\right)^P \binom{P + m - 1}{m}\left(\frac{\alpha}{1 + \alpha}\right)^m.$$

Similarly, to obtain B_{P-n}, multiply by $(1 + it\alpha)^P$ and let $1 + it\alpha = \tau$ to obtain

$$\left(1 + \frac{1}{\alpha} - \frac{\tau}{\alpha}\right)^{-P} = \sum_{n=0}^{P-1} B_{P-n}\tau^n + \tau^P \sum_{m=0}^{P-1} A_{P-m}\left(1 + \frac{1}{\alpha} - \frac{\tau}{\alpha}\right)^{-(P-m)}.$$

Reasoning as before, it is seen that

$$B_{P-n} = \left(\frac{\alpha}{1 + \alpha}\right)^P \binom{P + n - 1}{n}\left(\frac{1}{1 + \alpha}\right)^n.$$

Collecting these results, it is seen that the characteristic function is

$$\sum_{k=0}^{P-1} \binom{P + k - 1}{k}\left(\frac{1}{1 + \alpha}\right)^k\left[\left(\frac{\alpha}{1 + \alpha}\right)^P(1 + it\alpha)^{-(P-k)} + \frac{\alpha^k}{(1 + \alpha)^P}(1 - it)^{-(P-k)}\right].$$

This immediately establishes the distribution function $G(y; P, \alpha)$.

IV. WIDELY ORTHOGONAL SIGNALS

4.1 *Matrices for the Two Hypotheses*

We consider the special case in which the signals are widely orthogonal, $B(1, 2) = B(2, 1) = O_{P\times P}$. That is to say, all time-doppler shifted versions of one signal are orthogonal to all such versions of the other signal, a situation that would prevail in frequency-shift keying with widely separated frequencies. In this case,

$$L^{ik}(m) = \delta_{jk}\left[\frac{E_m}{E_1} B(j, m)\sigma B(m, k) + \frac{N_0}{2E_1} B(j, k)\right].$$

The "diagonal" form of the covariance matrix $L(m)$ and of the matrix Q implies that the spectrum of $L(m)Q$ comprises the specturm of $L^{11}(m)Q^{11}$ together with the spectrum of $L^{22}(m)Q^{22}$. This can be seen by employing the formulas of Schur (Ref. 8, pp. 45–46) to reduce the determinantal equation $\det [L(m)Q - \lambda I] = 0$ from order $2P$ to order P. For $m = 1$,

$$L^{11}(1)Q^{11} = B(1)\sigma,$$

$$L^{22}(1)Q^{22} = -(N_0/2E_1)(E_2/E_1)\{(E_2/E_1)I + (N_0/2E_1)[B(2)\sigma]^{-1}\}^{-1}.$$

For $m = 2$,

$$L^{11}(2)Q^{11} = (N_0/2E_1)\{I + (N_0/2E_1)[B(1)\sigma]^{-1}\}^{-1},$$

$$L^{22}(2)Q^{22} = -(E_2/E_1)B(2)\sigma.$$

It should be observed that the spectra of the above matrices are simply related to the spectra of $B(1)\sigma$ and of $B(2)\sigma$. When $E_2 = E_1 = E$, the spectrum of $L^{22}(1)Q^{22}$ is $\{-(N_0/2E)(1 + (N_0/2E)\delta_k^{-1})^{-1}\}$, where $\{\delta_k\}$ is the spectrum of $B(2)\sigma$. Similarly, the spectrum of $L^{11}(2)Q^{11}$ is $\{(N_0/2E)(1 + (N_0/2E)\omega_k^{-1})^{-1}\}$, where $\{\omega_k\}$ is the spectrum of $B(1)\sigma$.

Second, it should be observed that when $E_2 = E_1 = E$, the forms of the matrices for the cases $m = 1$ and $m = 2$ are the same, with the roles of positive and negative matrices interchanged. To compute error probability for $m = 1$, we use the distribution function $F_1(x)$; for $m = 2$, we use the conjugate distribution $1 - F_2(x)$ which can be expressed as $P\{-Z^\dagger QZ < -x \mid 2\}$, the distribution function of the negative of the original variable evaluated at $-x$. Introduction of this random variable for the case $m = 2$ reverses the roles of positive and negative matrices, the net effect being that for both $m = 1$ and $m = 2$ the positive and negative matrices have the same forms.

4.2 *Bounds on Spectra, θ, and Error Probability*

It is clear that spectral bounds on $L(1)Q$ can be obtained from spectral bounds on $B(m)\sigma$, $m = 1, 2$, and similarly for $L(2)Q$. Consider the bounds on $L(1)Q$ when $E_1 = E_2 = E$. The positive spectrum is bounded as follows:

$$\underline{\mu} = \underline{\omega} \leqq \min \omega_k \leqq \lambda(1) \leqq \max \omega_k \leqq \bar{\omega} = \bar{\mu},$$

and the negative spectrum is bounded as follows:

$$-\bar{\nu} = -(N_0/2E)[1 + (N_0/2E)(\bar{\delta})^{-1}]^{-1} \leqq \lambda(1)$$
$$\lambda(1) \leqq -(N_0/2E)[1 + (N_0/2E)(\underline{\delta})^{-1}]^{-1} = -\underline{\nu},$$

where $\underline{\delta} \leqq \min \delta_k \leqq \max \delta_k \leqq \bar{\delta}$.

Moreover, bounds on θ can also be obtained. When $E_1 = E_2 = E$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$ (equilikely signals),

$$y = (N_0/2E)\theta = (N_0/2E) \log \frac{\det [B(1)\sigma + (N_0/2E)I]}{\det [B(2)\sigma + (N_0/2E)I]}.$$

Since a determinant is the product of the eigenvalues of the matrix, we have

$$(N_0/2E)\theta = (N_0/2E) \log \prod_{k=1}^{P} \frac{\omega_k + (N_0/2E)}{\delta_k + (N_0/2E)}.$$

Thus, an upper bound is

$$(N_0/2E)\bar{\theta} = (N_0/2E)P \log \frac{\bar{\omega} + (N_0/2E)}{\underline{\delta} + (N_0/2E)},$$

and a lower bound is

$$(N_0/2E)\underline{\theta} = (N_0/2E)P \log \frac{\underline{\omega} + (N_0/2E)}{\bar{\delta} + (N_0/2E)}.$$

Recall that the distribution function $F_1[(N_0/2E)\theta]$ is bounded from above by $G[(\underline{\mu})^{-1}(N_0/2E)\bar{\theta};\ P,\ \bar{\nu}(\underline{\mu})^{-1}]$. Further, suppose that the spectra of $B(1)\sigma$ and $B(2)\sigma$ are narrow about the nominal value $(1/P) \operatorname{tr} B(m)\sigma = (1/P) \operatorname{tr} \sigma = (1/P)$. We can put

$$\bar{\omega} = \bar{\delta} = \frac{1 + \beta}{P}, \qquad \underline{\omega} = \underline{\delta} = \frac{1 - \beta}{P}, \tag{4}$$

where β is the fractional spectral half width. Then, the parameters required to compute the upper bound on the distribution function are

$$(\underline{\mu})^{-1}(N_0/2E)\bar{\theta} = \frac{1}{1-\beta}(N_0P/2E)P \log \frac{1+\beta+(N_0P/2E)}{1-\beta+(N_0P/2E)}, \quad (5a)$$

$$\bar{\nu}(\underline{\mu})^{-1} = \frac{1}{1-\beta}(N_0P/2E)\left[1+\frac{1}{1+\beta}(N_0P/2E)\right]^{-1}. \quad (5b)$$

Similarly, the distribution function $F_1[(N_0/2E)\theta]$ is bounded from below by $G[(\bar{\mu})^{-1}(N_0/2E)\underline{\theta}; P, \underline{\nu}(\bar{\mu})^{-1}]$. The parameters required for this bound are

$$(\bar{\mu})^{-1}(N_0/2E)\underline{\theta} = \frac{1}{1+\beta}(N_0P/2E)P \log \frac{1-\beta+(N_0P/2E)}{1+\beta+(N_0P/2E)}, \quad (5c)$$

$$\underline{\nu}(\bar{\mu})^{-1} = \frac{1}{1+\beta}(N_0P/2E)\left[1+\frac{1}{1-\beta}(N_0P/2E)\right]^{-1}. \quad (5d)$$

Having considered the case $m = 1$, the bounds for the case $m = 2$ are apparent. Considering the random variable $-Z^{\dagger}QZ$ with θ assumed known, the positive and negative spectral bounds are precisely the same as for the case $m = 1$, and the upper bound is

$$G[-(\underline{\mu})^{-1}(N_0/2E)\theta; P, \bar{\nu}(\underline{\mu})^{-1}]$$

whereas the lower bound is $G[-(\bar{\mu})^{-1}(N_0/2E)\theta; P, \underline{\nu}(\bar{\mu})^{-1}]$. But θ is unknown, and the upper bound is given by replacing $-\theta$ by $\bar{\theta}$, and the same result is obtained as previously; similarly, the lower bound is given by replacing $-\theta$ by $\underline{\theta}$. In short, the bounds apply to both cases, $m = 1$ and 2.

The numerical values of these bounds are given in Figs. 1 to 3 as functions of $2E/N_0P$ (the signal-to-noise ratio per path) for various fixed values of β (the fractional spectral half-width) and P (the number of paths). The curves are nested with respect to values of the fractional spectral half-width β; an increase of β always yields an increase of the upper bound and a decrease of the lower bound. A measure of the sharpness of the bounds (given a nominal value of error probability P_e) is provided by the difference of the upper-bound and lower-bound values of $2E/N_0P$ (in dB) for given values of β and P. For $P_e = 10^{-4}$ and $P = 4$, the sharpness is $1\frac{1}{4}$ dB for $\beta = 0.05$ and $2\frac{1}{4}$ dB for $\beta = 0.1$. This measure of sharpness appears to be relatively insensitive to the value of P. An alternate measure would be the difference in error probability for a given value of $2E/N_0P$, and this measure is indeed markedly sensitive to P.

In the region of the curves corresponding to high signal-to-noise ratio, there is an improvement in error probability associated with

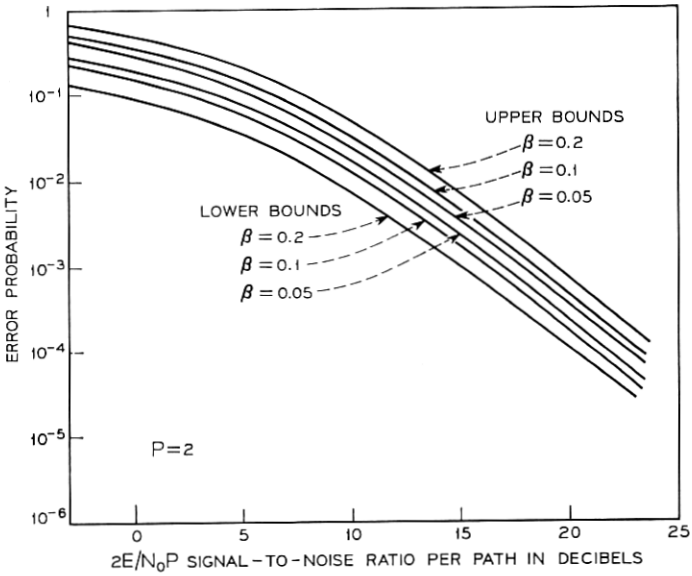

Fig. 1 — Error-probability bounds for widely-orthogonal signaling, $P = 2$.

larger P; the curves become straight lines since P_e becomes proportional to $(2E/N_0P)^{-P}$. However, this improvement is in part attributable to choosing $2E/N_0P$, the average per-path signal-to-noise ratio, as the abscissa rather than $2E/N_0$, the total signal-to-noise ratio. To obtain plots vs $2E/N_0$, one moves the $P = 2^n$ curves to the right by $3n$ dB; then, the improvement with increased P is less dramatic in this region of high signal-to-noise ratio.

4.3 *Computing Spectral Bounds*

It has been observed that bounds on the error probability for the case of widely orthogonal signals can be obtained from bounds on the spectrum of $B(m)\sigma$, $m = 1, 2$. We now give several easily computed formulas for these bounds.

Recall that $B(m)$ is defined to be $\int dt\, b^*(t, m)\tilde{b}(t, m)$, a matrix of scalar products or a Gram matrix. In general, this is uninformative, since a matrix is a Gram matrix if and only if the matrix is positive semidefinite. However, we will shortly use the fact that in our case the diagonal entries of $B(m)$ are unity because of the normalization. Next, note that $B(m)\sigma$ is similar to $\sigma^{\frac{1}{2}}B(m)\sigma^{\frac{1}{2}}$, a hermitian matrix which has real roots (since σ is a real diagonal matrix with positive entries, the matrices $\sigma^{\frac{1}{2}}$ and $\sigma^{-\frac{1}{2}}$ exist; then $\sigma^{\frac{1}{2}}[B(m)\sigma]\sigma^{-\frac{1}{2}} = \sigma^{\frac{1}{2}}B(m)\sigma^{\frac{1}{2}})$. When $B(m)$ is diagonal or nearly so, the roots of $B(m)\sigma$ should be close to the entries of σ; this is justified by the following theorem.[9] The characteristic roots of any matrix A lie in the closed region of the z-plane consisting of all the disks $\{z: | z - A_{ii} | \leqq \sum_{j \neq i} | A_{ij} |, i = 1, 2, \cdots, P\}$. In our case, the region must be on the real line, and we obtain a set of not necessarily nonoverlapping intervals centered about $\{\sigma_i\}$, the half-widths being $\{\sum_{j \neq i} | B_{ij}(m) | \sigma_j\}$ when we take $A = B(m)\sigma$. The spectral bounds are then the rightmost right-end point

$$\max_i \, [A_{ii} + \sum_{j \neq i} | A_{ij} |],$$

and the leftmost left-end point

$$\min_i \, [A_{ii} - \sum_{j \neq i} | A_{ij} |]$$

(when it is positive).

A family of spectral bounds is obtainable from this theorem by applying it to $B(m)\sigma$ and to matrices similar to $B(m)\sigma$, e.g., $\sigma^{\frac{1}{2}}B(m)\sigma^{\frac{1}{2}}$, $\sigma B(m)$, and more generally $\sigma^{\alpha}B(m)\sigma^{1-\alpha}$, $0 \leqq \alpha \leqq 1$. Thus, we have the family

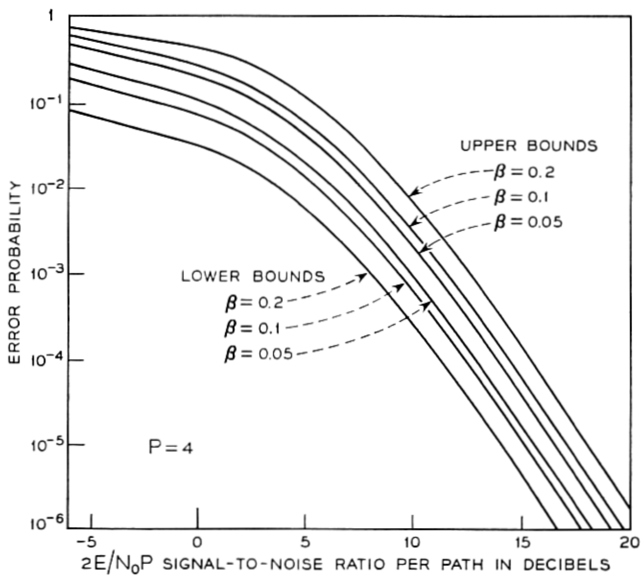

Fig. 2 — Error-probability bounds for widely-orthogonal signaling, $P = 4$.

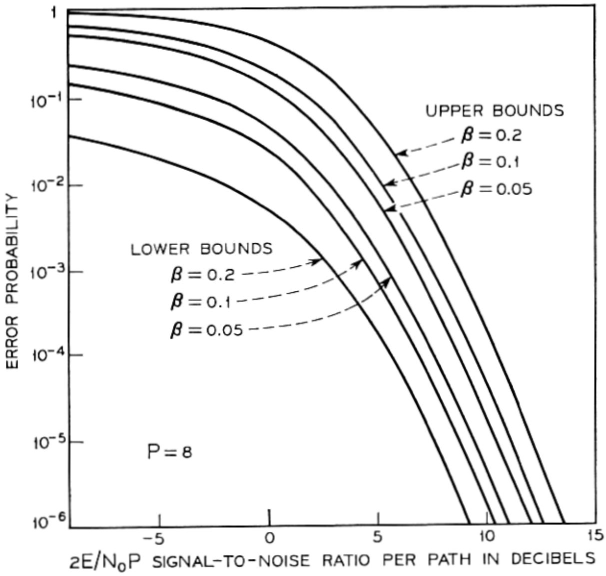

Fig. 3 — Error-probability bounds for widely-orthogonal signaling, $P = 8$.

of upper spectral bounds

$$\max_i \{\sigma_i + \sum_{j \neq i} \sigma_i^{\alpha} \mid B_{ij}(m) \mid \sigma_j^{1-\alpha}\}, \qquad 0 \leqq \alpha \leqq 1. \tag{6}$$

The question arises: which is the smallest upper bound? It is not true in general that a bound is attained for the value of i that maximizes σ_i, but suppose this is the case when $\alpha = 0$. That is to say, suppose $\sigma_i = \max_k \sigma_k$ and that

$$\sigma_i\left[1 + \sum_{j \neq i} \mid B_{ij}(m) \mid \frac{\sigma_j}{\sigma_i}\right] = \max_k \sigma_k\left[1 + \sum_{j \neq k} \mid B_{kj}(m) \mid \frac{\sigma_j}{\sigma_k}\right].$$

Then it follows that this is the smallest bound in the family, for $\sigma_j/\sigma_i \leqq 1$ implies that

$$\sum_{j \neq i} \mid B_{ij}(m) \mid \frac{\sigma_j}{\sigma_i} \leqq \sum_{j \neq i} \mid B_{ij}(m) \mid \left(\frac{\sigma_j}{\sigma_i}\right)^{1-\alpha},$$

and hence

$$\sigma_i\left[1 + \sum_{j \neq i} \mid B_{ij}(m) \mid \frac{\sigma_j}{\sigma_i}\right] \leqq \sigma_i\left[1 + \sum_{j \neq i} \mid B_{ij}(m) \mid \left(\frac{\sigma_j}{\sigma_i}\right)^{1-\alpha}\right]$$
$$\leqq \max_k \left\{\sigma_k\left[1 + \sum_{j \neq k} \mid B_{kj}(m) \mid \left(\frac{\sigma_j}{\sigma_k}\right)^{1-\alpha}\right]\right\}.$$

Similarly, we have the family of lower spectral bounds

$$\min_i \{\sigma_i - \sum_{j \neq i} \sigma_i^{\alpha} \mid B_{ij}(m) \mid \sigma_j^{1-\alpha}\}, \qquad 0 \leqq \alpha \leqq 1. \tag{7}$$

The largest lower bound is obtained when $\alpha = 1$ provided that $\sigma_i = \min_k \sigma_k$ and that

$$\sigma_i[1 - \sum_{j \neq i} \mid B_{ij}(m) \mid] = \min_k \{\sigma_k[1 - \sum_{j \neq k} \mid B_{kj}(m) \mid]\}.$$

To see this, observe that $\sigma_j/\sigma_i \geqq 1$ implies

$$\sum_{j \neq i} \mid B_{ij}(m) \mid \left(\frac{\sigma_j}{\sigma_i}\right)^{1-\alpha} \geqq \sum_{j \neq i} \mid B_{ij}(m) \mid,$$

and hence

$$\sigma_i[1 - \sum_{j \neq i} \mid B_{ij}(m) \mid] \geqq \sigma_i\left[1 - \sum_{j \neq i} \mid B_{ij}(m) \mid \left(\frac{\sigma_j}{\sigma_i}\right)^{1-\alpha}\right]$$
$$\geqq \min_k \left\{\sigma_k\left[1 - \sum_{j \neq k} \mid B_{kj}(m) \mid \left(\frac{\sigma_j}{\sigma_k}\right)^{1-\alpha}\right]\right\}.$$

It should be noted that less sharp bounds are easily obtained. For example, the matrix $\sigma B(m)$ yields the upper bound

$$\max_i \{\sigma_i[1 + \sum_{j \neq i} \mid B_{ij}(m) \mid]\} \leqq \max_i \sigma_i \max_i [1 + \sum_{j \neq i} \mid B_{ij}(m) \mid],$$

and the right-hand side is easily computed. The corresponding lower bound is

$$\min_i \{\sigma_i[1 - \sum_{j \neq i} \mid B_{ij}(m) \mid]\} \geqq [\min_i \sigma_i][1 - \max_i \sum_{j \neq i} \mid B_{ij}(m) \mid].$$

These less sharp bounds are easier to compute than those obtained in a similar fashion from $B(m)\sigma$ or from $\sigma^{\alpha}B(m)\sigma^{1-\alpha}$.

Also, it should be noted that sharper bounds can be obtained by employing a sharper theorem of matrix theory:[9] The characteristic roots of any matrix A lie in the closed region of the z-plane consisting of all the ovals $\mid z - A_{ii} \mid \mid z - A_{jj} \mid \leqq (\sum_{k \neq i} A_{ik})(\sum_{k \neq j} A_{jk})$, $i \neq j$. We do not pursue these bounds, but note that simple formulas are obtained only when all paths have equal strength, $\sigma_i = 1/P$.

It is now clear that when $B(m)$ is essentially diagonal, with $\sum_{j \neq i} \mid B_{ij}(m) \mid \ll 1$ for all i, the path gains σ_i are good nominal values for the characteristic roots of $B(m)\sigma$. If, moreover, these path gains are equal, or approximately equal, then the upper and lower spectral bounds are close to one another. When this narrow-spectrum condition prevails, the positive and negative portions of the spectrum of $L(m)Q$ are also narrow, and the bounds on error probability are sharp.

V. OTHER SPECIAL CASES

5.1 *Well-resolved Paths and the Theory of Diversity*

We consider the case in which the signals are resolvable, $B(1) = B(2) = I$, i.e., the paths are well separated in time and frequency so that any time-Doppler shifted version of a signal is orthogonal to any other version of itself. Moreover, we also assume that $B(1, 2) = \int dt\, b^*(t, 1)\tilde{b}(t, 2)$ becomes a diagonal matrix, $B(1, 2) = \rho I$ where $\rho = \int dt\, x_1^*(t)x_2(t)$, i.e., the paths are sufficiently separated so that any version of one signal is orthogonal to all but the same-path version of the other signal.

It is then easily seen that the covariance matrix is comprised of diagonal submatrices. For $m = 1$,

$$L^{11}(1) = \sigma + (N_0/2E_1)I \qquad L^{12}(1) = \rho[\sigma + (N_0/2E_1)I]$$

$$L^{21}(1) = \rho^*[\sigma + (N_0/2E_1)I] \qquad L^{22}(1) = |\rho|^2\sigma + (N_0/2E_1)I.$$

For $m = 2$, assuming $E_2 > 0$,

$$L^{11}(2) = (E_2/E_1)[|\rho|^2\sigma + (N_0/2E_2)I]$$

$$L^{12}(2) = \rho(E_2/E_1)[\sigma + (N_0/2E_2)I]$$

$$L^{21}(2) = \rho^*(E_2/E_1)[\sigma + (N_0/2E_2)I]$$

$$L^{22}(2) = (E_2/E_1)[\sigma + (N_0/2E_2)I].$$

Moreover, the matrix Q is diagonal, being related to

$$(2E_m/N_0)H(m) = (2E_m/N_0)[(2E_m/N_0)I + \sigma^{-1}]^{-1} = \sigma[\sigma + (N_0/2E_m)I]^{-1}.$$

It then follows that $L(m)Q$ is comprised of diagonal submatrices. To find the spectrum, the order of the determinantal equation can be reduced from $2P$ to P. Then the argument of the determinant is quadratic in λ. For the case $E_1 = E_2$, a method of Turin [(22)–(23) in Ref. 5] can be used relating the λ_k to the eigenvalues (elements) of σ.

The above example brings the present analysis in contact with the theory of diversity combining, see e.g., Ref. 3, Sec. 7.4. Turin,[5] for example, considered the case in which separate waveforms are available and the fading is nonindependent in general. In our analysis, only one waveform is in general available. But in the case of well-separated paths, we may assume P separate signal waveforms have been observed. However, these separate waveforms must fade independently in keeping with our general discrete-path model, and the on-diagonal component matrices of $L(m)$, *viz.*, $L^{11}(m) = (N_0/2E_1)\langle Z_1 Z_1^\dagger \rangle_m$ and $L^{22}(m) = (N_0/2E_1)\langle Z_2 Z_2^\dagger \rangle_m$, are themselves diagonal matrices. It is still entirely possible that $L^{12}(m)$, the off-diagonal component matrix of $L(m)$, is not a diagonal matrix; e.g., when $x_2(t)$ is a delayed version of $x_1(t)$, then time overlap may preclude $B(1, 2)$ being diagonal even though $B(1)$ and $B(2)$ are diagonal. But when we assume that $B(1, 2)$ is also diagonal, then we obtain the form for $L(m)$ exhibited above. It can be observed that this is precisely the result Turin obtained for the case of optimum diversity combining, where his not necessarily diagonal A becomes our diagaonal σ. When $B(1, 2)$ is not diagonal, then our results do not specialize to the form given by Turin, a reflection of the fact that the multipath channel is not in general fully equivalent to a diversity channel.

5.2 *On-Off Keying*

Another example is the case of on-off keying in which $E_2 = 0$. The test statistic $Z^\dagger Q Z$ becomes

$$[(N_0/2E_1)^{\frac{1}{2}} Z(1)]^\dagger (2E_1/N_0) H(1) [(N_0/2E_1)^{\frac{1}{2}} Z(1)], \quad \text{since} \quad Q^{22} = 0.$$

Thus, the distribution is determined by the spectrum of the matrix $L^{11}(m)Q^{11}$, where

$$L^{11}(m) = \delta_{m1} B(1, m)\sigma B(m, 1) + (N_0/2E_1)B(1),$$
$$Q^{11} = B^{-1}(1)\{I + (N_0/2E_1)[B(1)\sigma]^{-1}\}^{-1}.$$

Observe that we no longer have the difference of positive-definite forms, the test statistic now being a positive random variable. The threshold $(N_0/2E_1)\theta(2)$ is

$$(N_0/2E_1) \log \det \left[\left(\frac{2E_1}{N_0} \right) B(1)\sigma + I \right]$$

which is positive since the eigenvalues of $(2E_1/N_0)B(1)\sigma + I$ are greater than unity.

Assuming that the spectrum of $L^{11}(m)(2E_1/N_0)H(1)$ lies in the interval $(\underline{\mu}, \bar{\mu})$, where $\underline{\mu}$ and $\bar{\mu}$ are functions of m, the bounds on the distribution function are

$$G[(\bar{\mu})^{-1}(N_0/2E)\theta; P, 0] \leqq F_m\left[\left(\frac{N_0}{2E} \right) \theta \right] \leqq G[(\underline{\mu})^{-1}(N_0/2E)\theta; P, 0].$$

Recall that $G(x; P, 0)$ is related to the incomplete gamma function,

$$G(x; P, 0) = I(x, P - 1).$$

The spectral bounds must exhibit two forms of $(2E_1/N_0)$-dependence. When $m = 1$, $L^{11}(1)Q^{11} = B(1)\sigma$, and bounds on $B(1)\sigma$ become $\underline{\mu}$ and $\bar{\mu}$. When $m = 2$, $L^{11}(2)Q^{11} = (N_0/2E_1)\{I + (N_0/2E_1)[B(1)\sigma]^{-1}\}^{-1}$, so that

$$\underline{\mu} = (N_0/2E_1)[1 + (N_0/2E_1)(\underline{\omega})^{-1}]^{-1},$$

$$\bar{\mu} = (N_0/2E_1)[1 + (N_0/2E_1)(\bar{\omega})^{-1}]^{-1},$$

where the spectrum of $B(1)\sigma$ is confined to $(\underline{\omega}, \bar{\omega})$.

Collecting our results, when $m = 1$,

$$F_1[(N_0/2E)\theta] \leqq I\{(\underline{\omega})^{-1}(N_0/2E)P \log [(2E/N_0)\bar{\omega} + 1]; P - 1\}$$

$$F_1[(N_0/2E)\theta] \geqq I\{(\bar{\omega})^{-1}(N_0/2E)P \log [(2E/N_0)\underline{\omega} + 1]; P - 1\}.$$

Similarly, when $m = 2$

$$F_2[(N_0/2E)\theta] \leqq I\{[1 + (N_0/2E)(\underline{\omega})^{-1}]P \log [(2E/N_0)\bar{\omega} + 1]; P - 1\}$$

$$F_2[(N_0/2E)\theta] \geqq I\{[1 + (N_0/2E)(\bar{\omega})^{-1}]P \log [(2E/N_0)\underline{\omega} + 1]; P - 1\}.$$

These results permit the computation of error-probability-bound curves that would be universal in the same sense as the curves for widely-orthogonal signaling, i.e., the curves would apply to any element of the set of channels for which the spectral bounds are met.

VI. CHERNOFF BOUNDS

6.1 *General Case*

Up to this point, consideration of spectral bounds has lead to error-probability bounds which are sharp when the spectrum comprises narrow positive and negative portions. These bounds are easy to employ when $B(1, 2) = 0$ and $B(1)$, $B(2)$ are nearly diagonal matrices. But in more general cases, the estimation of spectral bounds may be difficult and bounds may be poor approximations of eigenvalues. We turn to another technique of bounding error probability which does not explicitly require spectral bounds.

Consider the error probability when hypothesis $m = 2$ is true, $P_e(2) = \Pr\{Z^\dagger QZ > (N_0/2E)\theta \mid 2\}$. Recall that the unit step function $U(x)$ is unity for $x > 0$, zero for $x < 0$, and one-half for $x = 0$. Then

$$P_e(2) = \Pr \{U[Z^\dagger QZ - (N_0/2E)\theta] = 1 \mid 2\}$$
$$= \mathcal{E}_2\{U[Z^\dagger QZ - (N_0/2E)\theta]\},$$

where $\mathcal{E}_2$ denotes expectation under hypothesis $m = 2$. But since $U(x) \leqq \exp(\mu_2 x)$ for any $\mu_2 > 0$, we have

$$P_e(2) \leqq \mathcal{E}_2\{\exp \mu_2[Z^\dagger QZ - (N_0/2E)\theta]\}.$$

This average can readily be computed, since $Z^\dagger QZ$ has the same distribution as $\sum \lambda_k(2) \mid z_k \mid^2$, where $\{\lambda_k(2)\}$ is the spectrum of $L(2)Q$. Since $\mathcal{E}z_k = 0$, $\mathcal{E}z_j z_k^* = \delta_{jk}$, and $\mathcal{E}z_j z_k = 0$, the Gaussian variables $\{\text{Re } z_j\}$, $\{\text{Im } z_j\}$ are independent with zero mean and variance equal to $\frac{1}{2}$. Thus, $P_e(2)$ is bounded from above by

$$\exp[-\mu_2(N_0/2E)\theta]\left[\prod_{k=1}^{2P} \mathcal{E} \exp(\mu_2\lambda_k(2) \mid \text{Re } z_k \mid^2)\right]^2,$$

where the outer square appears because the product involving $\{\text{Im } z_k\}$ has been suppressed. But a standard calculation shows,

$$\mathcal{E} \exp(\mu_2\lambda_k(2) \mid \text{Re } z_k \mid^2) = [1 - \mu_2\lambda_k(2)]^{-\frac{1}{2}}, \quad \text{when} \quad \mu_2\lambda_k(2) < 1,$$

and our bound is

$$\exp[-\mu_2(N_0/2E)\theta] \prod_{k=1}^{2P} [1 - \mu_2\lambda_k(2)]^{-1}.$$

Thus,

$$P_e(2) \leqq \exp[-\mu_2(N_0/2E)\theta] \det{}^{-1} [I - \mu_2 L(2)Q], \tag{8}$$

which holds for all μ_2 such that $0 < \mu_2 < [\max \lambda_k(2)]^{-1}$.

The above procedure is adopted from the technique due to Chernoff (see Ref. 3, Sec. 2.5 and 7.4). Here, we do not have identically distributed variables; indeed, half are positive and half are negative random variables.

To find the best value of μ_2, we write the bound as

$$\exp\left\{-\mu_2 \frac{N_0}{2E}\theta - \ln \prod_{k=1}^{2P} [1 - \mu_2\lambda_k(2)]\right\}$$

and differentiate the argument of the exponential. A necessary condition for an extremum is that the derivative be zero, and this yields

$$(N_0/2E)\theta = \sum_{k=1}^{2P} \frac{\lambda_k(2)}{1 - \mu_2\lambda_k(2)}$$
$$= \sum_{k=1}^{2P} \frac{1}{\lambda_{k(2)}^{-1} - \mu_2} = \text{tr}\{[(L(2)Q)^{-1} - \mu_2 I]^{-1}\}.$$

If the value of μ_2 that satisfies this equation lies within the allowable interval $[0, \max^{-1} \lambda_k(2)]$, then this value of μ_2 minimizes the upper bound. A minimum occurs because the second derivative of the argument of the exponential is positive, being

$$\sum_{k=1}^{2P} \left[\frac{\lambda_k(2)}{1 - \mu_2 \lambda_k(2)} \right]^2 .$$

In a similar fashion, the error probability for $m = 1$ can be overbounded.

$$P_e(1) = \Pr \{ -Z^{\dagger} Q Z > -(N_0/2E)\theta \mid 1 \}$$
$$\leqq \mathcal{E}_1 \{ \exp \mu_1 [-Z^{\dagger} Q Z + (N_0/2E)\theta] \}$$
$$P_e(1) \leqq \exp [\mu_1 (N_0/2E)\theta] \det^{-1} [I + \mu_1 L(1) Q].$$

The best value of μ_1 satisfies

$$(N_0/2E)\theta = \operatorname{tr} \{ L(1) Q [I + \mu_1 L(1) Q]^{-1} \},$$

provided this value lies in the allowable interval $[0, \max^{-1}(-\lambda_k(1))]$.

6.2 *Widely-orthogonal Signals*

Consider the case in which the signals are widely orthogonal, $B(1, 2) = 0$, but have equal energy, $E_1 = E_2 = E$, and are equilikely, $\alpha_1 = \alpha_2 = \frac{1}{2}$. The overbound on $P_e(1)$ is obtained from the spectrum of $L(1)Q$ which comprises the spectrum of $L^{11}(1)Q^{11}$ together with the spectrum of $L^{22}(1)Q^{22}$. Thus,

$$P_e(1) \leqq \exp [\mu_1 (N_0/2E)\theta] \det^{-1} [I + \mu_1 L^{11}(1) Q^{11}] \det^{-1} [I + \mu_1 L^{22}(1) Q^{22}].$$

But the matrices used here were related in Paragraph 4.1 to $B(1)\sigma$ and $B(2)\sigma$, and our bound becomes

$$\exp [\mu_1 (N_0/2E)\theta] \det^{-1} [I + \mu_1 B(1)\sigma] \cdot \det^{-1} \{ I - \mu_1 (N_0/2E)[I + (N_0/2E)(B(2)\sigma)^{-1}]^{-1} \}.$$

After some manipulation, this bound becomes

$$\left(\frac{E}{2N_0}\right)^{-P} \left\{ \exp \left[\mu_1 \left(\frac{N_0}{2E}\right) \theta \right] \right\} \cdot \frac{\det \left[B(2)\sigma + \left(\frac{N_0}{2E}\right) I \right]}{\det \left[2\mu_1 \left(\frac{N_0}{2E}\right) B(1)\sigma + \left(\frac{N_0}{E}\right) I \right] \det \left\{ 2 \left[1 - \mu_1 \left(\frac{N_0}{2E}\right) \right] B(2)\sigma + \left(\frac{N_0}{E}\right) I \right\}},$$

where

$$\exp[\mu_1(N_0/2E)\theta] = \left\{\frac{\det[B(1)\sigma + (N_0/2E)I]}{\det[B(2)\sigma + (N_0/2E)I]}\right\}^{\mu_1(N_0/2E)}.$$

The maximum allowable value of μ_1 is determined by the largest eigenvalue of $L^{22}(1)Q^{22}$ which in turn is determined by the largest eigenvalue of $B(2)\sigma$:

$$0 < \mu_1 < \frac{2E}{N_0} + \max^{-1}(\delta_k),$$

where $\{\delta_k\}$ is the spectrum of $B(2)$.

The best value of μ_1 is found from the relation

$$\begin{aligned}(N_0/2E)\theta &= \sum \frac{\lambda_k(1)}{1 + \mu_1\lambda_k(1)} \\ &= \operatorname{tr}\{L^{11}(1)Q^{11}[I + \mu_1 L^{11}(1)Q^{11}]^{-1}\} \\ &\qquad + \operatorname{tr}\{L^{22}(1)Q^{11}[I + \mu_1 L^{22}(1)Q^{11}]^{-1}\},\end{aligned}$$

where we again have exploited the decomposition of the spectrum of $L(1)Q$. After some manipulation, we find

$$\begin{aligned}(N_0/2E)\theta &= \operatorname{tr}\{B(1)\sigma[I + \mu_1 B(1)\sigma]^{-1}\} \\ &\qquad - \operatorname{tr}\left\{B(2)\sigma\left[I + \left(\frac{2E}{N_0} - \mu_1\right)B(2)\sigma\right]^{-1}\right\}.\end{aligned}$$

An approximate solution can be obtained for the case of high signal-to-noise ratio. Let $\mu_1 = \bar{\mu}_1(2E/N_0)$; the relation becomes

$$(N_0/2E)\theta = \sum \frac{\omega_k}{1 + (2E/N_0)\bar{\mu}_1\omega_k} - \sum \frac{\delta_k}{1 + (2E/N_0)(1 - \bar{\mu}_1)\delta_k}.$$

Suppose $\bar{\mu}_1(2E/N_0)\omega_k \gg 1$ and $(1 - \bar{\mu}_1)(2E/N_0)\delta_k \gg 1$. Then the right side becomes approximately

$$\frac{P}{(2E/N_0)\bar{\mu}_1} - \frac{P}{(2E/N_0)(1 - \bar{\mu}_1)}.$$

Equating this to $(N_0/2E)\theta$ and solving the resulting quadratic for the root applicable for the case $\theta = 0$ yields

$$\bar{\mu}_1 = \left[\left(1 + \frac{\theta}{2P}\right) + \left(1 + \frac{\theta^2}{4P^2}\right)^{\frac{1}{2}}\right]^{-1}.$$

When $\theta/2P$ is small, this value of $\bar{\mu}_1$ is approximately

$$\frac{1}{2}\left[1 - \frac{\theta}{4P}\right],$$

and the corresponding value of μ_1 is $(E/N_0)[1 - (\theta/P)]$ which is approximately at the midpoint of the allowable interval.

In a similar fashion, the overbound on $P_e(2)$ is

$$\exp\,[-\mu_2(N_0/2E)\theta]\,\det{}^{-1}\,[I - \mu_2 L^{11}(2)Q^{11}]\,\det{}^{-1}\,[I - \mu_2 L^{22}(2)Q^{22}],$$

which becomes

$$\exp\,[-\mu_2(N_0/2E)\theta]\,\det{}^{-1}\,\{I - \mu_2(N_0/2E)[I + (N_0/2E)(B(1)\sigma)^{-1}]^{-1}\}\cdot\det{}^{-1}\,[I + \mu_2 B(2)\sigma],$$

or

$$\left(\frac{E}{2N_0}\right)^{-P}\left\{\exp\left(-\mu_2\frac{N_0}{2E}\,\theta\right)\right\}\cdot\frac{\det\left[B(1)\sigma + \left(\frac{N_0}{2E}\right)I\right]}{\det\left\{2\left[1 - \mu_2\left(\frac{N_0}{2E}\right)B(1)\sigma + \left(\frac{N_0}{E}\right)I\right\}\det\left[2\mu_2\left(\frac{N_0}{2E}\right)B(2)\sigma + \left(\frac{N_0}{E}\right)I\right]},$$

where

$$\exp\,[-\mu_2(N_0/2E)\theta] = \left\{\frac{\det\,[B(2)\sigma + (N_0/2E)I]}{\det\,[B(1)\sigma + (N_0/2E)I]}\right\}^{\mu_2(N_0/2E)}.$$

The maximum allowable value of μ_2 is determined by the largest eigenvalue of $L^{11}(2)Q^{11}$ in turn determined by the largest eigenvalue of $B(1)\sigma$:

$$0 < \mu_2 < \frac{2E}{N_0} + \max{}^{-1}\,(\omega_k),$$

where $\{\omega_k\}$ is the spectrum of $B(1)\sigma$. The best value of μ_2 satisfies

$$(N_0/2E)\theta = \operatorname{tr}\,\{L^{11}(2)Q^{11}[I - \mu_2 L^{11}(2)Q^{11}]^{-1}\} + \operatorname{tr}\,\{L^{22}(2)Q^{22}[I - \mu_2 L^{22}(2)Q^{22}]^{-1}\}$$

$$(N_0/2E)\theta = \operatorname{tr}\left\{B(1)\sigma\left[\left(\frac{2E}{N_0} - \mu_2\right)B(1)\sigma + I\right]^{-1}\right\} - \operatorname{tr}\,\{B(2)\sigma[I + \mu_2 B(2)\sigma]^{-1}\}.$$

Let $\mu_2 = \bar{\mu}_2(2E/N_0)$ and suppose $\bar{\mu}_2(2E/N_0)\delta_k \gg 1$, $(1-\bar{\mu}_2)(2E/N_0)\omega_k \gg 1$.

Then the right side becomes

$$\frac{P}{(2E/N_0)(1 - \bar{\mu}_2)} - \frac{P}{(2E/N_0)\bar{\mu}_2},$$

and the approximation of the best value of $\bar{\mu}_2$ is

$$\bar{\mu}_2 = \left[\left(1 - \frac{\theta}{2P}\right) + \left(1 + \frac{\theta^2}{4P^2}\right)^{\frac{1}{2}}\right]^{-1}$$

which is approximately $\frac{1}{2}[1 - (\theta/4P)]$ when $(\theta/2P) \ll 1$.

The foregoing results can be specialized to the case in which the paths are resolvable, $B(1) = B(2) = I$. Then $\theta = 0$, and it is easily seen that the best value of $\bar{\mu}_m$ is $\frac{1}{2}$. Both overbounds become

$$(E/2N_0)^{-P} \frac{\det [\sigma + (N_0/2E)I]}{\det^2 [\sigma + (N_0/E)I]},$$

and this agrees with equation 7.134 in Ref. 3.

It should be noted that $\bar{\mu}_m = \frac{1}{2}$ is always an allowed value of $\bar{\mu}_m$. For the case of resolvable paths, it is the best value, and whenever $\theta/P \ll 1$ and $2E/N_0$ is sufficiently large, it is close to the best value. Using $\bar{\mu}_m = \frac{1}{2}$, we can obtain an overbound for both error probabilities, i.e., for $P_e(m)$, $m = 1, 2$. This overbound is

$$(E/2N_0)^{-P} \exp\left(\tfrac{1}{2} \mid \theta \mid\right) \frac{\det [B(3 - m)\sigma + (N_0/2E)I]}{\det [B(1)\sigma + (N_0/E)I] \det [B(2)\sigma + (N_0/E)I]}. \tag{9a}$$

The factor $\exp (\frac{1}{2} \mid \theta \mid)$ can also be written in terms of determinants. When $\det [B(1)\sigma + (N_0/2E)I]$ is larger than $\det [B(2)\sigma + (N_0/2E)I]$, we have

$$\exp\left(\tfrac{1}{2} \mid \theta \mid\right) = \left\{\frac{\det [B(1)\sigma + (N_0/2E)I]}{\det [B(2)\sigma + (N_0/2E)I]}\right\}^{\frac{1}{2}}, \tag{9b}$$

and when the reverse inequality holds, $\exp(\frac{1}{2} \mid \theta \mid)$ is the reciprocal of the above.

For the case in which the spectrum of $B(m)\sigma$ lies in the interval $(1-\beta/P,\ 1+\beta/P)$ where $\beta < 1$, the overbound can be further overbounded. The factor involving determinants is less than

$$\left\{\frac{P\left[1 + \beta + \left(\frac{N_0 P}{2E}\right)\right]}{\left[1 - \beta + 2\left(\frac{N_0 P}{2E}\right)\right]^2}\right\}^P,$$

and $|\theta|/2$ is less than

$$\frac{P}{2}\log\frac{1+\beta+(N_0P/2E)}{1-\beta+(N_0P/2E)}.$$

It follows that the Chernoff bound is less than

$$(N_0P/2E)^P\left\{\frac{4[1+\beta+(N_0P/2E)]^{\frac{3}{2}}}{[1-\beta+(N_0P/2E)]^{\frac{1}{2}}[1-\beta+2(N_0P/2E)]^2}\right\}^P. \qquad (10)$$

Numerical values of this bound are given in Fig. 4, and it has the same general character as the spectral-related bounds. Rather than sharpness given a nominal value of error probability P_e, we consider the sensitivity measured by the change in $2E/N_0P$ (in dB) vs β; for $P_e = 10^{-4}$ and $P = 4$, the sensitivity is 2 dB for $\beta = 0.1$. The sensitivity does not markedly increase with an increase in P, in agreement with the behavior of the sharpness of the previous bounds.

Comparison of the Chernoff bound with the previous bounds is conveniently done for the case $\beta = 0$ (cf. Sec. 7.4 of Ref. 3). The Chernoff bound does not specify a signal-to-noise ratio (required to achieve a nominal P_e) excessively greater than the previous value; for $P = 4$, less than 2.2 dB difference is observed. This excess does decrease with increasing P. Moreover, it is entirely conceivable that in a broad-spectrum case with a large number of paths, an exact value of the Chernoff bound would be better than the spectral-bound result. Of course, our inexact (overbounded) Chernoff bound is poor in the

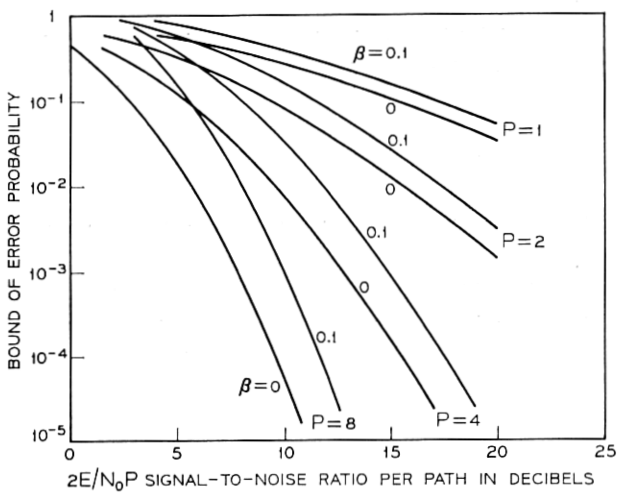

Fig. 4 — Overbounded Chernoff bounds for widely-orthogonal signaling.

broad-spectrum case, but a Chernoff bound using the proper values of the determinants should be good for two reasons. (*i*) Such a bound reflects the precise values of the eigenvalues of the matrix $L(m)Q$. (*ii*) When P is large, the probability density function is bell shaped with the probability "mass" being concentrated near the mean and most of the tail mass being at the leading portion of the tail; then the tail mass can be weighted by the exponential function with little error. On the other hand, the spectral-bound approach suffers in the broad-band case since the spectral bounds are not meaningful approximations of all the eigenvalues.

VII. DISCUSSION

Having observed that exact computation of error probability is cumbersome and depends upon an often inordinately large number of parameters, we considered error-probability bounds (2) that are universal in the sense that they apply to any one of a set of channels satisfying spectral bounds (1). Our bounds employ (3), the distribution function of the difference of chi-square variables. For the special case of widely orthogonal signals, we obtained bounds employing parameters (5) in terms of the spectral width β, see (4), of the matrices $B(m)\sigma$. Plots of these bounds showed that sharpness measured in dB change of $2E/N_0P$ with respect to β for a fixed value of error probability is not sensitive to the value of P. We presented a technique for obtaining spectral bounds for $B(m)\sigma$ when it is nearly diagonal, representative results being (6) and (7). This technique can also be applied to $L(m)Q$ for the more general case in which the signals are not widely orthogonal.

The case of resolvable signals ($B(m) = I$) made contact with the theory of diversity; we found that for the multipath channel to be a diversity channel, $B(1, 2)$ must also be a diagonal matrix. Of course, the previous results also were in contact with diversity theory. With $B(1, 2) = 0$ (a diagonal matrix) but $B(m)$ not necessarily diagonal, our results generalize those of diversity theory in the following sense. The special case $\beta = 0$ corresponds to a diversity channel with equal link gains, but the general case $\beta \neq 0$ can arise in the nondiversity situation when the matrix $B(m)$ is not diagonal. (If $B(m)$ were diagonal, $B(m) = I$ and the diversity case prevails.)

We then turned to the Chernoff bound (8) which does not explicitly employ spectral bounds. The overbounded form (10) for the case of widely orthogonal signals was poorer than the previous bound when $\beta = 0$. Nevertheless, there is promise that in a broad-spectrum case, the original form (9) would be better than the spectral-related bounds. A further advantage is that once the determinants are evaluated, perhaps on an electronic computer, the error-probability bound is immediately obtained. In contrast, the spectral-related bounds require a certain amount of computation involving incomplete gamma functions even after spectral bounds are obtained.

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APPENDIX A

Here we show that the number of positive eigenvalues of LQ equals the number of negative eigenvalues.* Recall that L is positive definite and that Q can be written in the partitioned form

$$Q = \begin{bmatrix} Q^{11} & 0 \\ 0 & Q^{22} \end{bmatrix},$$

where Q^{11} and $-Q^{22}$ are positive definite. Clearly, the number of positive eigenvalues of Q equals the number of negative eigenvalues. We can construct a family of positive definite matrices L_t, $0 \leqq t \leqq 1$, such that $L_0 = I$, $L_1 = L$, and L_t is continuous in t. For example, let $L_t = (1 - t)I + tL$; L_t has positive eigenvalues $\{(1 - t) + t\gamma_k\}$, where $\{\gamma_k\}$ are the eigenvalues of L. Now the eigenvalues of L_tQ are real, for L_tQ is similar to the Hermitian matrix $L_t^{\frac{1}{2}}QL_t^{\frac{1}{2}} = L_t^{-\frac{1}{2}}(L_tQ)L_t^{\frac{1}{2}}$, where $L_t^{\frac{1}{2}}$ and $L_t^{-\frac{1}{2}}$ exist since L_t is positive definite, Moreover, the eigenvalues of L_tQ are continuous in t, since L_t is continuous in t. But L_tQ never has a zero eigenvalue, for L_t is positive definite and $(L_tQ)^{-1} = Q^{-1}L_t^{-1}$ always exists. Since the eigenvalues are real, continuous in t, and never zero, it follows that no positive eigenvalue of L_0Q can become negative as t varies on [0, 1], and no negative eigenvalue of L_0Q can become positive. The conclusion is established.

APPENDIX B

This appendix presents another derivation of the distribution function of $\sum_1^P |z_k|^2 - \alpha \sum_{P+1}^{2P} |z_k|^2$. This derivation makes contact with

* We are indebted to B. H. Bharucha for the conception of this proof.

the special functions that have appeared in analyses of diversity channels; also, this derivation appears to admit generalization to the case $z_k = \text{Re}\, z_k$ with $\langle z_j z_k \rangle = \delta_{jk}$. (An odd number of variables in the real case corresponds to half-integer P in the complex case.)

The density function of $\sum_1^P |z_k|^2$ is

$$f(x) = \begin{cases} \dfrac{x^{P-1}e^{-x}}{(P-1)!} & (x > 0), \\ 0 & (x < 0), \end{cases}$$

and the density function of $-\alpha \sum_{P+1}^{2P} |z_k|^2$ is

$$g(x) = \begin{cases} 0 & (x > 0), \\ \dfrac{(-x)^{P-1}e^{x/\alpha}}{\alpha^P(P-1)!} & (x < 0). \end{cases}$$

The density of the sum is the convolution of the densities,

$$h(x) = \int_{\max(0,x)}^{\infty} dy\, f(y)g(x-y),$$

where the first argument of max $(\cdot\,,\,\cdot)$ arises from the truncated form of f and the second argument arises from the truncated form of g. It follows that

$$h(x) = \frac{\exp(x/\alpha)}{\alpha^P(P-1)!(P-1)!} \int_{\max(0,x)}^{\infty} dy(y-x)^{P-1} \exp\left[-\left(\frac{1}{\alpha}+1\right)y\right].$$

For the case $x > 0$, the lower limit is x. For the case $x < 0$, the integral can be cast into the form of the integral for the case $x > 0$ by a change of variable. The result differs only in the exponential factor, i.e.,

$$h(x) = \frac{\exp(-x)}{\alpha^P(P-1)!(P-1)!} \cdot \int_{|x|}^{\infty} dy(y-|x|)^{P-1}y^{P-1} \exp\left[-\left(\frac{1}{\alpha}+1\right)y\right], \qquad x < 0.$$

The integral can be evaluated with the aid of relation (12) on page 202, Vol. II of Ref. 10, and the common result for the cases $x < 0$ and $x > 0$ is

$$h(x) = \frac{|x|^{P-\frac{1}{2}} \exp\left[\left(\frac{1}{\alpha}-1\right)\frac{x}{2}\right]}{\sqrt{\pi\alpha}\,(1+\alpha)^{P-\frac{1}{2}}(P-1)!} K_{P-\frac{1}{2}}\left(\frac{1+\alpha}{\alpha}\frac{|x|}{2}\right),$$

where $K_{P-\frac{1}{2}}(z)$ is the modified Bessel function of the third kind.

The above expression for the density is valid for all P, noninteger as well as integer. But in our application, P is an integer; a relation on page 80 of Ref. 11 yields

$$K_{P-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \sum_{k=0}^{P-1} \frac{(P-1+k)!}{k!(P-1-k)!(2z)^k}.$$

The density is then

$$h(x) = \left(\frac{\alpha}{1+\alpha}\right)^P \exp\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{x}{2} - \frac{1+\alpha}{\alpha}\frac{|x|}{2}\right]$$
$$\cdot \sum_{k=0}^{P-1} \frac{(P-1+k)!}{(P-1)!k!(P-1-k)!} \left(\frac{1}{1+\alpha}\right)^k \frac{1}{\alpha}\left(\frac{|x|}{\alpha}\right)^{P-1-k}.$$

When $x < 0$, the exponential becomes $\exp(x/\alpha)$, and when $x > 0$, it becomes $\exp(-x)$.

Observe that when $\alpha = 1$, the density is symmetric. When $\alpha < 1$, the factor $\exp[(1 - \alpha/\alpha)x/2]$ shifts the mass to the right. When $\alpha \to 0$, it can be shown that $h(x) \to f(x)$.

To obtain the distribution function $G(y; P, \alpha)$, consider first the case $y < 0$. Since $\int_{-\infty}^{y} dx\, h(x)$ equals $\int_{|y|}^{\infty} dx\, h(-x)$, the following integral arises in each term of the sum,

$$\int_{|y|}^{\infty} \frac{dx}{\alpha} e^{-x/\alpha}\left(\frac{x}{\alpha}\right)^{P-1-k} = (P-1-k)![1 - I(|y|/\alpha, P-1-k)].$$

The case $y > 0$ is treated by considering $\int_{-\infty}^{0} dx\, h(x) + \int_{0}^{y} dx\, h(x)$. The integral that arises is just $(P-1-k)!I(y, P-1-k)$. These steps establish our final result, quoted above.

Our result could also have been obtained from the Fourier transform of the characteristic function $(1 - it)^{-P}(1 + it\alpha)^{-P}$. The Fourier transform of $(\alpha + it)^{-2\mu}(\beta - it)^{-2\nu}$ is given by relation (12) on page 119, Vol. *I* of Ref. 10 in terms of Whittaker functions that reduce to Bessel functions for the case $\mu = \nu = P/2$ in view of relation (14) on page 265, Ref. 12. The density function can thus be obtained.

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