Axis-Crossing Intervals of Sine Wave Plus Noise

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I. INTRODUCTION

Let $I(t, a)$ denote the stationary random process consisting of a sinusoidal signal of amplitude $\sqrt{2a}$ and angular frequency q plus Gaussian noise, $I_N(t)$, of zero mean and unit variance. Thus,

$$I(t, a) = \sqrt{2a} \cos (qt + \theta_0) + I_N(t). \qquad (1)$$

θ_0 denotes a random phase angle which is distributed uniformly in the interval $(-\pi, \pi)$. "a" denotes the signal-to-noise power ratio. When $a = 0$ Rice[1] presented some theoretical results which are very useful for studying statistical properties of the axis-crossing intervals and the axis-crossing points of $I(t, 0)$ at an arbitrary level I. The axis-crossing intervals and the axis-crossing points of $I(t, a)$ are defined in Fig. 1. In recent work Cobb[2] presented some theoretical results concerning the zero-crossing intervals, the axis-crossing intervals defined by the level $I = 0$, of $I(t, a)$. Some experimental and theoretical results concerning the zero-crossing intervals of $I(t, a)$ were reported by Rainal.[3] For the case when the power spectral density of $I_N(t)$ is narrow-band and symmetrical about the sine wave frequency, Blachman[4] presented some

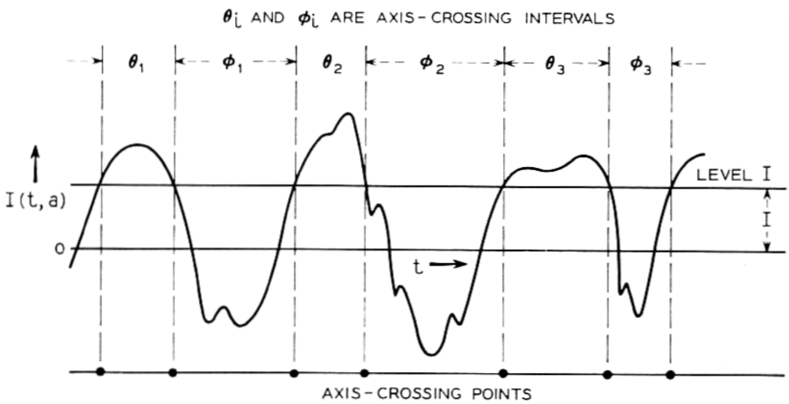

Fig. 1 — The level I defines the axis-crossing points and the axis-crossing intervals of $I(t, a) = \sqrt{2a} \cos (qt + \theta_0) + I_N(t)$.

theoretical results concerning the zero-crossing points, the axis-crossing points defined by the level $I = 0$, of $I(t, a)$.

The purpose of this brief is to present some theoretical results which are useful for studying statistical properties of the axis-crossing intervals and the axis-crossing points of $I(t, a)$ at an arbitrary level I. These results stem from a straightforward extension of Rice's[1] analysis.

II. THEORETICAL RESULTS

Using a notation consistent with Refs. 5 and 6 we define the following probability functions at an arbitrary level I and arbitrary signal-to-noise power ratio "a":

(i) $Q_2^+(\tau, I, a)d\tau$, the conditional probability that a downward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

(ii) $Q_2^-(\tau, I, a)d\tau$, the conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward axis-crossing at t.

(iii) $[U_2(\tau, I, a) - Q_2(\tau, I, a)]d\tau$, the conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

This latter conditional probability is also equal to the conditional probability that a downward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward axis-crossing at t.

The reader should refer to Rice[1] for the definition of all notation which is not defined in this brief. When $a \geqq 0$, Rice's[1] (38) becomes

$$Q_2^+(\tau, I, a) = -[2\pi N_I]^{-1} \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dI_1' \int_{-\infty}^{0} dI_2' I_1' I_2' p(I, I_1', I_2', I), \quad (2)$$

where N_I = Rice's[7] equation (2.6) or (2.7)

$$p(I, I_1', I_2', I) = (2\pi)^{-2} M^{-\frac{1}{2}}$$

$$\cdot \exp\left\{-\frac{1}{2M}[M_{22}(I_1'^2 + I_2'^2) + 2M_{22}r_1 I_1' I_2' + 2D_1 I_1' + 2E_1 I_2' + F_1]\right\}$$

$$r_1 = \frac{M_{23}}{M_{22}} \qquad Q = \sqrt{2a}$$

$$D_1 = M_{12}[I - Q\cos\theta] + M_{13}[Q\cos(q\tau + \theta) - I] + M_{22}Qq\sin\theta + M_{23}Qq\sin(q\tau + \theta)$$

$$E_1 = M_{12}[Q \cos (q\tau + \theta) - I] + M_{13}[I - Q \cos \theta] + M_{22}Qq \sin (q\tau + \theta) + M_{23}Qq \sin \theta$$

$$\begin{aligned} F_1 = {} & M_{11}\{2I^2 - 2QI[\cos \theta + \cos (q\tau + \theta)] + Q^2[\cos^2 \theta + \cos^2 (q\tau + \theta)]\} \\ & + 2M_{12}Qq\{[I - Q \cos \theta] \sin \theta + [Q \cos (q\tau + \theta) - I] \sin (q\tau + \theta)\} \\ & + 2M_{13}Qq\{[I - Q \cos \theta] \sin (q\tau + \theta) + [Q \cos (q\tau + \theta) - I] \sin \theta\} \\ & + 2M_{14}\{I[I - Q \cos \theta] + Q[Q \cos \theta - I] \cos (q\tau + \theta)\} \\ & + M_{22}(Qq)^2[\sin^2 \theta + \sin^2 (q\tau + \theta)] + 2M_{23}(Qq)^2 \sin \theta \sin (q\tau + \theta). \end{aligned}$$

The M's are given in Rice's[1] Appendix I with

$$m(\tau) = \int_0^\infty W(f) \cos 2\pi f\tau \, df, \tag{3}$$

where $W(f)$ = one-sided power spectral density of $I_N(t)$. When $I = 0$, $N_I Q_2^+(\tau, I, a)$ is equivalent to (9) of Cobb's[2] recent work.

Equation (2) can be put in a form analogous to Rice's[1] equation (47):

$$Q_2^+(\tau, I, a) = [4\pi^2 N_I]^{-1} M_{22}(1 - m^2)^{-\frac{1}{2}} \cdot \int_{-\pi}^{\pi} \exp(-G_1/2M) J(r_1, h_2, k_2) \, d\theta, \tag{4}$$

where

$$J(r_1, h_2, k_2) \equiv \frac{1}{2\pi\sqrt{1 - r_1^2}} \int_{h_2}^\infty dx \int_{k_2}^\infty dy (x - h_2)(y - k_2) e^z$$

$$z = -\frac{x^2 + y^2 - 2r_1 xy}{2(1 - r_1^2)}$$

$$h_2 = M_{22}^{-1}[1 - r_1^2]^{-1}[D_1 - r_1 E_1]\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}}$$

$$k_2 = -M_{22}^{-1}[1 - r_1^2]^{-1}[E_1 - r_1 D_1]\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}}$$

$$G_1 = M_{22}^{-1}[1 - r_1^2]^{-1}[2r_1 D_1 E_1 - D_1^2 - E_1^2] + F_1 .$$

$Q_2^-(\tau, I, a)$ is obtained from (2) by changing the signs of the ∞'s in the limits of integration. We find that $Q_2^-(\tau, I, a)$ is equal to the right-hand side of (4) with h_2, k_2 replaced by $-h_2$, $-k_2$.

$[U_2(\tau, I, a) - Q_2(\tau, I, a)]$ is obtained from (2) by changing the lower limit of integration of I_2' to $+\infty$. We find that $[U_2(\tau, I, a) - Q_2(\tau, I, a)]$

is equal to the right-hand side of (4) with the function $J(r_1, h_2, k_2)$ replaced by the function $J_1(r_1, h_2, k_2)$, where

$$J_1(r_1, h_2, k_2) \equiv \frac{1}{2\pi\sqrt{1 - r_1^2}} \int_{h_2}^{-\infty} dx \int_{k_2}^{\infty} dy(x - h_2)(y - k_2)e^z. \quad (5)$$

The functions $J(r_1, h_2, k_2)$ and $J_1(r_1, h_2, k_2)$ are expressed in terms of Karl Pearson's well-known tabulated function (d/N) in Ref. 5.

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