Characteristics of Superconductor Strip Transmission Lines with Periodic Load

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The characteristic impedance and propagation constant of a thin film superconducting strip transmission line has been derived by use of London's two fluid model. It is shown that this line at moderate frequency has negligible attenuation and dispersion. A periodically loaded cross film cryotron circuit is also analyzed. The attenuation, phase constant, and characteristic impedance of this loaded line is given and related to the parameters of the unloaded line by the factor K which is the ratio of the gate separation to the gate width.

I. INTRODUCTION

This paper presents a study of the high-frequency performance of thin film superconducting transmission circuits. Particular attention is given to transmission lines between cryotron elements and between substrates each carrying many cryotrons.

These interconnections are microstrip lines with very low characteristic impedances. Since the separation between the transmitting strip and the ground plane is very small in comparison to the width of the strip, edge effects can be neglected. This simplifies the analysis and gives an easy understanding of the propagation phenomena.

II. MICROSTRIP TRANSMISSION LINE WITH SUPERCONDUCTING STRIP

Due to the finite conductivity of the strip and ground plane in a non-superconductive microstrip transmission line, the phase characteristic has some dispersion and the attenuation is frequency dependent.[1] If the strip and ground plane are superconducting, the phase dispersion and the attenuation will disappear at frequencies below 1 GHz. This was shown by Swihart[2] using Maxwell's and London's equation. Using a very simple but not rigorous approach, the characteristic impedance and propagation constant are presented to give an understanding of this type of transmission line. The inductance, capacitance, and conductance in the dielectric region are the same as for non-superconducting strip line and have the following values:[1]

$$l^e = \frac{\mu_e h}{b} \tag{1}$$

$$c = \frac{\epsilon_e b}{h} \tag{2}$$

$$g = \frac{\sigma_e b}{h}, \tag{3}$$

provided $b/h \ll 1$,
where

l^e is the inductance per unit length,
c is the capacitance per unit length,
g is the conductance per unit length,
b is the width of the strip,
h is the distance between the strip and the ground plane, and
μ_e, ϵ_e, and σ_e are, respectively, the permeability, permitivity, and conductance of the dielectric material between the strip and ground plane.

The internal impedance of the conducting strip and ground plane of the superconducting line are different. These are found by the following manipulation. Assume that a superconducting strip of thickness $x = d$ and infinite width y, has a current flowing in the z-direction as shown in Fig. 1. The superconducting strip is immersed in a uniform dielectric material. The current is uniformly distributed along the y direction. London's equation that is based on a two fluid model[3] includes the following relations:

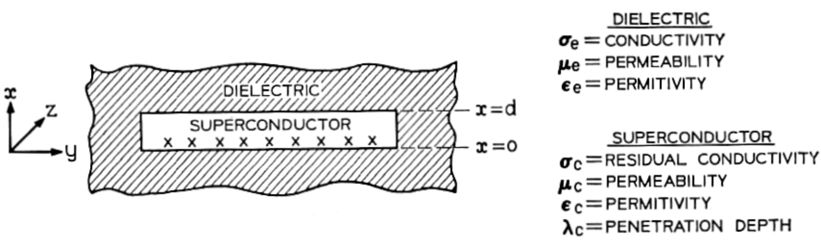

Fig. 1 — Current flowing in a superconductor strip.

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s \tag{4}$$

$$\nabla \times \lambda_c^2 \mathbf{J}_s = \mathbf{H} \tag{5}$$

$$\mu_c \frac{d}{dt} (\lambda_c^2 \mathbf{J}_s) = \mathbf{E}, \tag{6}$$

where

$\mathbf{J}_s$ is the superconducting current,
$\mathbf{J}_n$ is the normal conducting current,
$\mathbf{J}$ is the total current,
$\mathbf{H}$ and $\mathbf{E}$ are, respectively, magnetic and electric fields,
λ_c is the penetration depth of the superconductor, and
μ_c is the permeability of the superconductor.

Maxwell's equations applicable to both the superconducting and dielectric regions are as follows:

$$\nabla \cdot \mathbf{D} = \rho \qquad \mathbf{D} = \epsilon \mathbf{E} \tag{7}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{B} = \mu \mathbf{H} \tag{8}$$

$$\nabla \times \mathbf{E} = -\mu \frac{d\mathbf{H}}{dt} \tag{9}$$

$$\nabla \times \mathbf{H} = \epsilon \left(\frac{d\mathbf{E}}{dt} \right) + \mathbf{J}. \tag{10}$$

These equations are based on MKS units.

From these equations, in the superconductor region, replacing J by $J_n + J_s$ and then using (6) and (7) through (10), we have a general expression of London's equation.

$$\nabla^2 \mathbf{E}_c = \frac{1}{\lambda_c^2} \mathbf{E}_c + \sigma_c \mu_c \frac{d\mathbf{E}_c}{dt} + \epsilon_c \mu_c \frac{d^2 E_c}{dt^2}, \tag{11}$$

where σ_c is the residual normal conductivity. This is the conductivity measured just above critical temperature. E_c, μ_c, and ϵ_c are, respectively, the E-field, permeability, and permittivity in the superconductor region.

The three terms on the right side of this equation are contributions of superconducting current, normal current, and the displacement current, respectively. In the superconductor region the displacement current term can be neglected. Hence, (11) becomes

$$\nabla^2 \mathbf{E}_c = \frac{1}{\lambda_c^2} \mathbf{E}_c + \sigma_c \mu_c \frac{d\mathbf{E}_c}{dt}. \tag{12}$$

In the dielectric region, we have

$$\nabla^2 \mathbf{E}_e = \sigma_e \mu_e \frac{d\mathbf{E}_e}{dt} + \epsilon_e \mu_e \frac{d^2 \mathbf{E}_e}{dt^2}. \tag{13}$$

Use the coordinates defined by Fig. 1 and assuming the fields to be sinusoidal with respect to time,

$$\frac{d^2 E_{zc}}{dx^2} = \left(\frac{1}{\lambda_c^2} + j\omega\sigma_c\mu_c\right) E_{zc} \tag{14}$$

$$\frac{d^2 E_{ze}}{dx^2} = (-\omega^2 \epsilon_e \mu_e + j\omega\sigma_e\mu_e) E_{ze} . \tag{15}$$

The solutions of the above equations are, respectively,

$$E_{zc} = A_1 \cosh k_1 x + {}_z A_2 \sinh k_1 x \qquad 0 \leqq x \leqq d$$

$$E_{ze} = B \exp(-k_2 x) \qquad d \leqq x,$$

where k, and k_2 are defined as

$$k_1 = \frac{1}{\lambda_c} (1 + j\omega\sigma_c\mu_c\lambda_c^2)^{\frac{1}{2}} \tag{16}$$

$$k_2 = j\omega\sqrt{\mu_e\epsilon_e}\left(1 - j\frac{\sigma_e}{\omega\epsilon_e}\right)^{\frac{1}{2}}. \tag{17}$$

In the superconducting region, we retain two solutions in order to match the boundary condition at $x = d$, while in dielectric region we retain only one solution because we assume that the dielectric material is uniform and extends to infinity, and there is no reflection wave.

The H-fields in both regions are, respectively,

$$j\omega\mu_c H_{yc} = -A_1 k_1 \sinh k_1 x - A_2 k_1 \cosh k_1 x$$

$$j\omega\mu_e H_{ye} = B k_2 \exp(-k_2 x).$$

At the boundary $x = d$

$$E_{zc} = E_{ze}, \qquad H_{yc} = H_{ye} .$$

Hence,

$$A_2/A_1 = -\frac{k_1/k_2 \sinh k_1 d + \cosh k_1 d}{\sinh k_1 d + k_1/k_2 \cosh k_1 d} \quad \text{for} \quad \mu_c = \mu_e . \tag{18}$$

At

$$x = 0$$

$$E_{z_c} = A_1$$

$$H_{y_c} = -\frac{1}{j\omega\mu_c} A_2 k_1 .$$

By use of Ampere's Law the current inside the superconductor can be found. For unit width, it is

$$H_{y_c}\big|_{x=0} - H_{y_c}\big|_{x=d} = \frac{1}{j\omega\mu_c}(-A_2k_1 + A_1k_1 \sinh k_1d + A_2k_1 \cosh k_1d). \tag{19}$$

The internal impedance of a conductor is defined as[4]

$$Z^i = \frac{E_0}{I} \text{ ohm/m}^2, \tag{20}$$

where E_0 is the surface E-field and I is the total current in the conductor for unit width. Hence,

$$Z^i = \frac{j\omega\mu_c}{k_1} \frac{k_2/k_1 + \coth k_1d}{1 + k_2/k_1 \tanh k_1d/2}. \tag{21}$$

The classic skin-effect depth of a normal conductor is defined as[4]

$$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}}.$$

From this and (16), for the superconducting region

$$k_1 = \frac{1}{\lambda_c}\left(1 + 2j\frac{\lambda_c^2}{\delta^2}\right)^{\frac{1}{2}}.$$

For several common superconducting materials, the classic skin-effect penetration depth δ are listed as follows:

	Transition[5] temperature	Conductivity[5] at transition temperature	δ at $f = 10^8$ Hz	δ at $f = 10^9$ Hz
Lead	7.22°K	0.52×10^{10} mho/m	0.7×10^4Å	0.216×10^4Å
Tin	3.74	0.896×10^{10}	0.53×10^4	0.167×10^4
Tantalum	4.38	0.806×10^9	1.77×10^4	0.56×10^4
Indium	3.374–3.432	0.36×10^9	2.65×10^4	0.84×10^4

The pentration depth λ_c is in the vicinity of thousand Å (for lead, it is 500Å, while for tin it is 1500 Å). Hence, in general, at a frequency

10^8 Hz $\lambda_c/\delta \approx 1/10$. This is assumed at a temperature T which is at least 0.1°K below transition temperature. Under this condition, for a good approximation

$$k_1 \approx \frac{1}{\lambda_c}.$$

The $\sigma_e/\omega\epsilon_e$ term in (17) is defined as the loss tangent of dielectric material. For most dielectric material at room temperature, it is approximately 10^{-3} to 10^{-4}.[6] For SiO, it is 10^{-2} at 1500 Hz. There is no available data at helium temperature. However, for reasonable approximation we can say that

$$\frac{\sigma_e}{\omega\epsilon_e} \ll 1.$$

Hence,

$$k_2 \approx j\omega\sqrt{\mu_e\epsilon_e}$$

$$\approx j\frac{2\pi}{\Lambda_e},$$

where Λ_e is the wave length in the dielectric. For SiO its relative permitivity is approximately 5.[7] Hence, $\Lambda_e \approx 74.5$ cm at 10^8 Hz.

The $|k_2|/|k_1|$ ratio is then

$$|k_2|/|k_1| \approx \frac{\lambda_c}{\Lambda_e} \approx 10^{-7}.$$

Under this assumption, (21) can be approximated as

$$Z^i = j\omega\mu_c\lambda_c \coth d/\lambda_c \text{ ohm/m}^2. \tag{22}$$

Next let us assume that a superconducting strip transmission line is formed by two strips immersed in a dielectric material as shown in Fig. 2. For this line, its series impedance is the sum of the Z^i and the inductance in the dielectric; hence, its series impedance and parallel admittance are, respectively

$$Z = j\omega\mu_c\frac{h}{b}\left(1 + \frac{\lambda_{c_1}}{h}\coth d_1/\lambda_{c_1} + \frac{\lambda_{c_2}}{h}\coth d_2/\lambda_{c_2}\right) \tag{23}$$

$$y = j\omega\epsilon_e\frac{b}{h}. \tag{24}$$

If the dielectric loss and classic skin-effect loss is not negligible, then

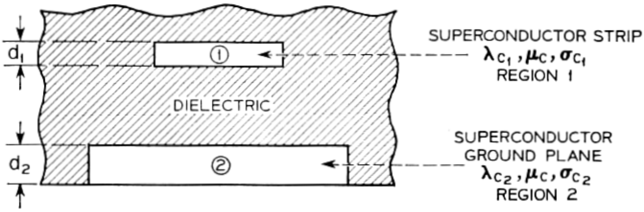

Fig. 2 — Superconductor Strip Transmission Line.

$$Z = j\omega\mu_c \frac{h}{b}\left[1 + \frac{\lambda_{c_1}}{h}\left(1 + 2j\frac{\lambda_{c_1}^2}{\delta_1^2}\right)^{-\frac{1}{2}} \coth k_1 d + \frac{\lambda_{c_2}}{h}\left(1 + 2j\frac{\lambda_{c_2}^2}{\delta_2^2}\right)^{-\frac{1}{2}} \coth k_2 d\right] \tag{23a}$$

$$y = j\omega\epsilon_e \frac{b}{h}\left(1 - j\frac{\sigma_e}{\omega\epsilon_e}\right). \tag{24a}$$

The characteristic impedance and propagation constant of this line can be found by use of the following relations:

$$Z_c = \sqrt{Z/y}$$

$$\gamma_c = \sqrt{Zy}.$$

Hence, by use of (23) and (24), we get

$$Z_c = \sqrt{\frac{\mu_c}{\epsilon_e}\frac{h}{b}}\left(1 + \frac{\lambda_{c_1}}{h}\coth d_1/\lambda_{c_1} + \frac{\lambda_{c_2}}{h}\coth d_2/\lambda_{c_2}\right)^{\frac{1}{2}} \tag{25}$$

$$\gamma_c = j\omega\sqrt{\mu_c\epsilon_e}\left(1 + \frac{\lambda_{c_1}}{h}\coth d_1/\lambda_{c_1} + \frac{\lambda_{c_2}}{h}\coth d_2/\lambda_{c_2}\right)^{\frac{1}{2}}. \tag{26}$$

Set $\gamma_c = j\beta_c$.

Then the phase velocity of this strip line is

$$V_p = \frac{\omega}{\beta_c}$$
$$= \frac{1}{\sqrt{\mu_c\epsilon_e}}\left(1 + \frac{\lambda_{c_1}}{h}\coth d_1/\lambda_{c_1} + \frac{\lambda_{c_2}}{h}\coth d_2/\lambda_{c_2}\right)^{-\frac{1}{2}} \text{ m/sec.}$$

And its delay time is

$$\tau_c = \sqrt{\mu_c\epsilon_e}\left(1 + \frac{\lambda_{c_1}}{h}\coth d_1/\lambda_{c_1} + \frac{\lambda_{c_2}}{h}\coth d_2/\lambda_{c_2}\right)^{\frac{1}{2}} \text{ sec/m.}$$

While by use of (23a) and (24a) we get

$$Z_c = \sqrt{\frac{\mu_c}{\epsilon_e}}\frac{h}{b} \cdot \frac{\left[1 + \frac{\lambda_{c_1}}{h}(1 + 2j\lambda_{c_1}^2/\delta_1^2)^{-\frac{1}{2}} \coth k_1 d + \frac{\lambda_{c_2}}{h}(1 + 2j\lambda_{c_2}^2/\delta_2^2)^{-\frac{1}{2}} \coth k_2 d\right]^{\frac{1}{2}}}{(1 - j\sigma_e/\omega\epsilon_e)^{\frac{1}{2}}} \tag{25a}$$

$$\gamma_c = j\omega\sqrt{\mu_c\epsilon_e}\,(1 - j\sigma_e/\omega\epsilon_e)^{\frac{1}{2}}\left[1 + \frac{\lambda_{c_1}}{h}(1 + 2j\lambda_{c_1}^2/\delta_1^2)^{-\frac{1}{2}} \coth k_1 d + \frac{\lambda_{c_1}}{h}(1 + 2j\lambda_{c_2}^2/\delta_2)^{-\frac{1}{2}} \coth k_2 d\right]^{\frac{1}{2}}. \tag{26a}$$

At frequencies below 1 GHz with the temperature at least 0.1°K below transition temperature, (25) and (26) give fairly accurate results, providing the dielectric loss is negligible. Then the characteristic impedance is a real number with negligible frequency dependence. The propagation constant is directly proportional to frequency; hence, its group velocity and phase velocity are the same and there is no attenuation.

Fig. 3 shows the characteristic impedance and delay time of some superconducting strip lines with various dielectric thickness (h).

It is shown that the thickness (d_1) of the strip line film has little effect on the characteristic impedance and delay time. However, the characteristic impedance changes proportionally with dielectric thickness (h) while the delay time τ_c decreases nonlinearly by only 20 percent for an order of magnitude change in h.

III. A SUPERCONDUCTING STRIP LINE WITH PERIODIC STRUCTURE

The cross film cryotron consists of a control strip and a gate strip crossing and perpendicular to each other. In a memory circuit or in a tree-type selective circuit, a single control strip usually crosses many gate strips. At each intersection there exists coupling between the control and gate strips to form a periodic structure. The characteristic impedance and propagation constant of the control line are functions of the periodic loading. (Refer to Fig. 4.)

The control line and gate line are assumed to be terminated with their respective characteristic impedances Z_c and Z_g. The control line is also assumed to be uniform without discontinuity except for the

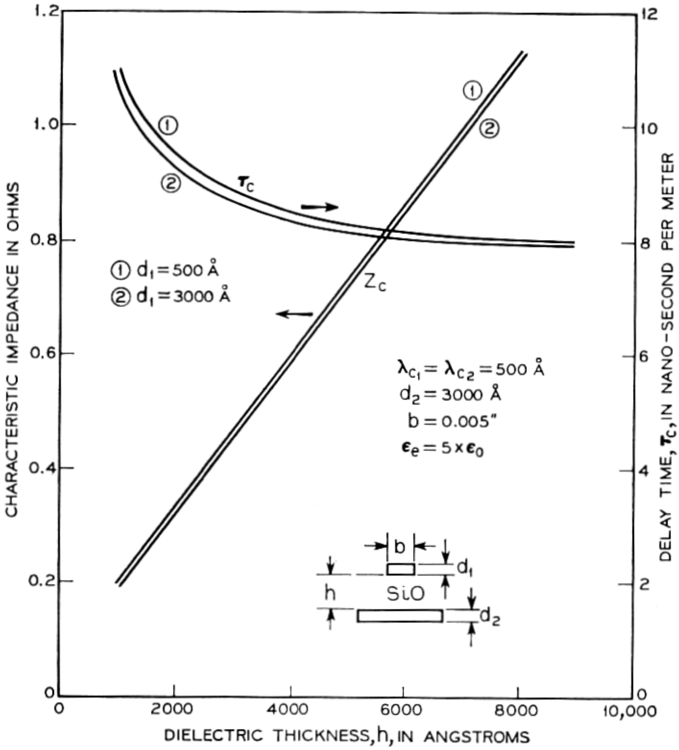

Fig. 3 — Relation between characteristic impedance delay time, and dielectric thickness for superconducting thin film strip line.

periodic loading of the couplings to the gates. The equivalent circuit is shown in Fig. 5.

In Fig. 5, γ_c is the propagation constant of the control line, Y_c is the coupling admittance between gate and control line. The characteristic impedance and propagation constant of this periodically-loaded line is as follows (see Appendix A):

$$Z_0 = Z_c\left[1 - \frac{YZ_c}{\sinh 2\gamma_c l + YZ_c \cosh^2 \gamma_c l}\right]^{\frac{1}{2}} \tag{27}$$

$$\gamma_0 = \cosh^{-1}\left[\cosh 2\gamma_c l + YZ_c/2 \sinh 2\gamma_c l\right], \tag{28}$$

where

$$Y = \frac{Y_c}{1 + \frac{1}{2}Y_c Z_g} \qquad l = \tfrac{1}{2}(d_g + W_g).$$

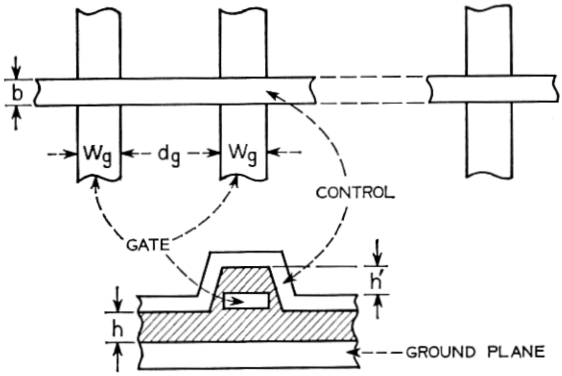

Fig. 4 — Periodic gate crossing of a cryotron circuit.

If the ratio of the control line width b to the distance h_g between gate and controls lines are large $(b/h) > 10'$ this capacitance to a very good approximation is[8]

$$C = \epsilon_e \frac{A}{h'}, \tag{29a}$$

where A is the intersection area of the control and gate lines, and ϵ_e is the permittivity of the insulation material.

Since the magnetic fields in control and gate lines is assumed orthogonal to each other; therefore, there is no magnetic coupling, and

$$Y_c = j\omega C.$$

If we want to take into account of the dielectric loss of the insulation material, Y_c becomes

$$Y_c = j\omega\epsilon_e \frac{A}{h'} \left(1 - j\frac{\sigma_e}{\omega\epsilon_e}\right), \tag{29b}$$

where σ_e is the conductivity of the insulation material. For SiO, $\sigma_e/\omega\epsilon_e$ is approximately 10^{-2}.[7] Hence, this term can be neglected in this case.

By use of this result, we find

$$Y = \frac{j\omega C}{1 + \frac{1}{2}j\omega C Z_g}. \tag{30}$$

For a typical cryotron, the width of the gate is approximately 20 milli-inches and the width of control line is 5 milli-inches. The insulation material SiO has a relative permittivity of 5. If h' is 5000 Å, then

$$C = \frac{1}{36\pi} \times 10^{-9} \times 5 \frac{20 \times 5 \times 2.54^2 \times 10^{-10}}{5 \times 10^{-7}}.$$

$$C \approx 5.7 \times 10^{-12} \text{ farads.}$$

If the gate is terminated by its characteristic impedance, then Z_g is approximately one ohm for an ordinary cryotron. Hence,

$$\omega C Z_g \approx 6 \times 10^{-3}$$

at a frequency of 100 MHz. Therefore, (30) can be simplified as

$$Y \approx j\omega C(1 - \tfrac{1}{2} j\omega C Z_g). \tag{31}$$

We have shown in Section II that the propagation constant of a superconducting strip line is an imaginary number; therefore, we set

$$\gamma_c = j\beta_c .$$

Hence,

$$Z_0 = Z_c\left[1 - \frac{\omega C Z_c(1 - \frac{1}{2} j\omega C Z_g)}{\sin 2\beta_c l + \omega C Z_c(1 - j\frac{1}{2}\omega C Z_g) \cos^2 \beta_c l}\right]^{\frac{1}{2}}. \tag{32}$$

For typical cryotron circuits, the spacing between gates (d_g) is about equal to the gate width (W_g) for maximum compactness. At a frequency of 10^8 Hz,

$$\beta_c l \approx 10^{-3} \text{ radians.}$$

Hence,

$$\sin 2\,\beta_c l \approx 2\,\beta_c l$$

$$\cos^2 2\,\beta_c l \approx 1.$$

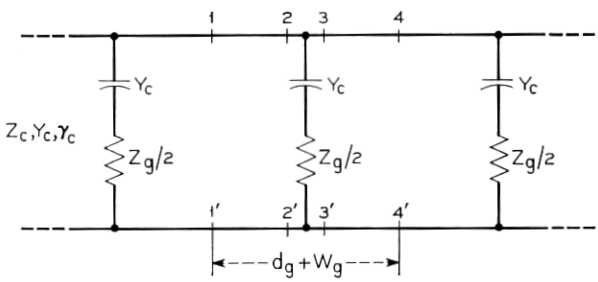

Fig. 5 — Equivalent circuit of a periodic loaded cryotron.

Neglecting the $(\omega C)^2 Z_g Z_c$ term, we find

$$Z_0 = Z_c\left[1 - \frac{1}{1 + \dfrac{2\beta_c l}{\omega C Z_c}}\right]^{\frac{1}{2}}. \tag{33}$$

By use of (25), (26), and (29a), we find

$$\frac{2\beta_c l}{\omega C Z_c} = \frac{h'(d_g + W_g)}{h W_g}. \tag{34}$$

If $h' = h$, then

$$Z_0 = Z_c\left(\frac{W_g + d_g}{2W_g + d_g}\right)^{\frac{1}{2}}. \tag{35a}$$

Setting

$$d_g = K W_g\ ,$$

$$Z_0 = Z_c\left(\frac{1 + K}{2 + K}\right)^{\frac{1}{2}}. \tag{35b}$$

For maximum package density, the gates are placed as close as possible, and thus, d_g and consequently K are made as small as possible. However, to avoid interference between adjacent gates, it is usual to set the distance between gates at least equal to their width. For this condition, $Z_0 = 0.815\ Z_c$. As K becomes larger, Z_0 approaches Z_c in value.

In the next step, the propagation constant of the periodic-loaded line is determined.

First substituting the following condition in (28):

$$\gamma_c = j\beta_c$$

$$\sinh 2\gamma_c l \approx 2j\beta_c l$$

$$\cosh 2\gamma_c l \approx 1 - \frac{(2\beta_c l)^2}{2}$$

and set

$$\gamma_0 = \alpha_0 + j\beta_0\ .$$

Equation (28) can be rewritten as

$$\cosh(\alpha_0 + j\beta_0) = 1 - \frac{(2\beta_c l)^2}{2} - \omega C Z_c \beta_c l + j\tfrac{1}{2}(\omega C)^2 Z_c Z_g \beta_c l.$$

It is noticed that the real part on the right side of this equation is very close to but less than 1. The imaginary part is very small. Hence, we conclude that α_0 and β_0 must be a small quantity. The real and imaginary parts of this equation become, respectively,

$$\cosh \alpha_0 \cos \beta_0 = 1 - \frac{(2\beta_c l)^2}{2} - \omega C Z_c \beta_c l \tag{36a}$$

$$\sinh \alpha_0 \sin \beta_0 = \tfrac{1}{2}(\omega C)^2 Z_c Z_g \beta_c l. \tag{36b}$$

Using the approximate relationship

$$\cosh \alpha_0 \approx 1 + \frac{\alpha_0^2}{2}$$

$$\cos \beta_0 \approx 1 - \frac{\beta_0^2}{2}$$

$$\sinh \alpha_0 \approx \alpha_0$$

$$\sin \beta_0 \approx \beta_0$$

$$Z_c \approx Z_g$$

and defining the following constants [see (34)]:

$$\frac{\omega C Z_c}{2\beta_c l} = \frac{W_g}{d_g + W_g} = R \leqq 1 \tag{37}$$

$$2\beta_c l = \theta_c ,$$

where θ_c is the phase shift between gate crossings along the control line without loading, (36a) and (36b) can be rewritten, respectively, as

$$\left(1 + \frac{\alpha_0^2}{2}\right)\left(1 - \frac{\beta_0^2}{2}\right) = 1 - \frac{\theta_c^2}{2} - \tfrac{1}{2}R\theta_c^2$$

$$\alpha_0\beta_0 = \tfrac{1}{4}R^2\theta_c^3 .$$

Neglecting the $\alpha_0^2\beta_0^2$ term, we find

$$\beta_0^2 - \alpha_0^2 = \theta_c^2 + R\theta_c^2$$

$$\alpha_0\beta_0 = \tfrac{1}{4}R^2\theta_c^3 .$$

Hence,

$$\beta_0 = \frac{1}{\sqrt{2}}\,\theta_c\left\{(1 + R)\left[1 \pm \left(1 + \frac{\frac{1}{4}(R^2\theta_c)^2}{(1 + R)^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}.$$

Since

$$\tfrac{1}{4}(R^2\theta_c)^2 \ll (1 + R)^2,$$

β_0 becomes

$$\beta_0 = \theta_c(1 + R)^{\frac{1}{2}}\left(1 + \frac{1}{16}\frac{R^4\theta_c^2}{(1 + R)^2}\right)^{\frac{1}{2}}. \tag{38}$$

Further neglecting $R^4\theta_c^2/(1 + R)^2$, then we obtain

$$\beta_0 = \theta_c(1 + R)^{\frac{1}{2}}$$

$$\alpha_0 = \tfrac{1}{4}\theta_c^2 \frac{R^2}{(1 + R)^{\frac{3}{2}}}.$$

Replacing R by (37), we obtain

$$\beta_0 = \theta_c\left(\frac{d_g + 2W_g}{d_g + W_g}\right)^{\frac{1}{2}} \tag{39a}$$

$$\alpha_0 = \tfrac{1}{4}\theta_c^2 \frac{W_g^2}{(d_g + W_g)^{\frac{3}{2}}(d_g + 2W_g)^{\frac{1}{2}}}. \tag{39b}$$

Using the relation $d = KW_g$, then

$$\beta_0 = \theta_c\left(\frac{2 + K}{1 + K}\right)^{\frac{1}{2}} = \text{phase constant} \tag{40a}$$

$$\alpha_0 = \tfrac{1}{4}\theta_c^2 \frac{1}{(K + 1)^{\frac{3}{2}}(K + 2)^{\frac{1}{2}}} = \text{attenuation constant.} \tag{40b}$$

Equations (40a) and (40b) are phase constant and attenuation per period.* If there are n gates crossing in one meter length of control line and if they are equally spaced, then the phase constant and attenuation per meter is

$$\beta = \beta_c(d_g + W_g)n\left(\frac{2 + K}{1 + K}\right)^{\frac{1}{2}} \text{ rad/m}$$

$$\alpha = \tfrac{1}{4}\beta_c^2(d_g + W_g)^2 n \frac{1}{(1 + K)^{\frac{3}{2}}(2 + K)^{\frac{1}{2}}} \text{ neper/m.}$$

Replacing β_c by τ_c and realizing that $n = 1/d_g + W_g$, we find

$$\beta = \tau_c\omega\left(\frac{2 + K}{1 + K}\right)^{\frac{1}{2}} \text{ radians/m} \tag{41a}$$

$$\alpha = \tfrac{1}{4}\tau_c^2\omega^2 W_g \frac{1}{(K + 1)^{\frac{3}{2}}(K + 2)^{\frac{1}{2}}} \text{ neper/m.} \tag{41b}$$

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* One period is the distance $(d_g + W_g)$ between cryotron gate crossing of the strip line.

The delay time per meter of the loaded line is

$$\tau_0 = \tau_c \left(\frac{2 + K}{1 + K}\right)^{\frac{1}{2}} \text{ sec/m.} \tag{42}$$

For a typical cryotron having control width 0.005 inch and gate width $W_g = 0.02$ inch, SiO as dielectric of a thickness 5000 Å, and $K = 1$, then its attenuation at 10^8 Hz becomes

$$\alpha = 2.56 \times 10^{-3} \text{ neper/m}$$

or

$$= 2.12 \times 10^{-2} \text{ dB/m.}$$

With this configuration, on a 3″ × 3″ substrate from one side to the other side of the substrate, 75 cryotron can be laid down. The attenuation will be only about 5×10^{-4} dB. This is extremely small. For larger K, this attenuation will still be less. Hence, for our purpose, it can be neglected. The delay time for this particular example is approximately 10 nanosecond per meter. For the same substrate, the time for a pulse to travel from one side to the other side of the substrate is approximately one nanosecond.

Fig. 6 shows the relation between characteristic impedance Z_0, delay time τ_0 and the ratio (K) of gate separation (d_g) to gate width (W_g). From Fig. 6 it is shown that the characteristic impedance Z_0 of loaded line is less than the characteristic impedance Z_c of an unloaded line at smaller K (at $K = 1$, $Z_0 \approx 0.82\ Z_c$). As K becomes larger, ($K > 10$) Z_0/Z_c ratio approaches unity. The delay time τ_0 per meter of a loaded line is larger than the delay time of an unloaded line (at $K = 1$ $\tau_0 = 1.22\ \tau_c$). However, their ratio also approaches unity as K becomes larger.

IV. CONCLUSION

Section II derives the characteristic impedance and propagation constant for typical thin film superconducting strip lines used for interconnecting cryotron elements. The characteristic impedance is shown to be a real number with negligible frequency dependence. The propagation constant is shown to be directly proportional to frequency and hence the group and phase velocity is identical. It is found that the transmission performance of these lines are, for practical purposes, independent of film thickness when in excess of 500 Å. Fig. 3 shows the characteristic impedance (Z_c) and delay time τ_c for a film strip

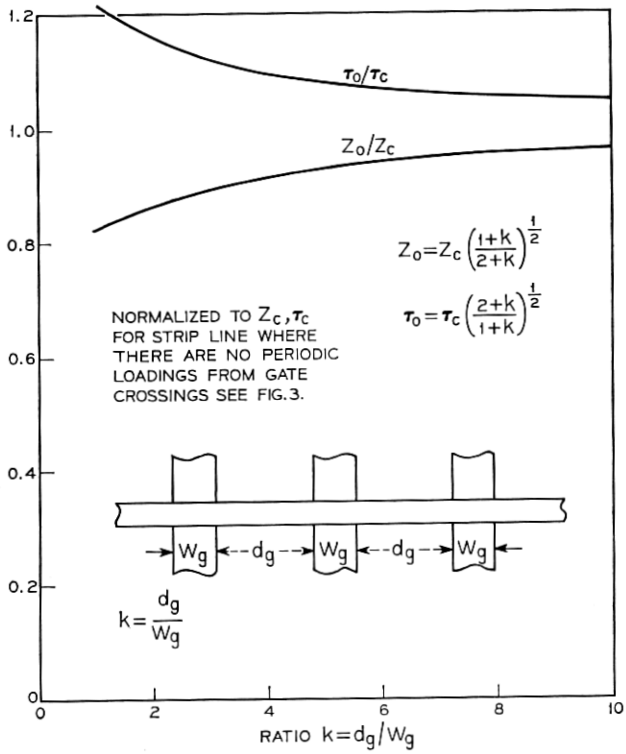

Fig. 6 — Relation between characteristic impedance (Z_0) delay ( ), and ratio (k) of gates separation (d) and gate width (W_g) of superconducting strip line with periodic gate crossing.

of 5 milli-inches width on SiO dielectric. For other widths, Z_c is inversely proportional to width and τ_c is independent of width. For example, at a dielectric thickness of 5000 Å and film width of 5 milli-inches, $Z_c = 0.725$ ohms and $\tau_c = 8.2 \times 10^{-9}$ sec/meter.

Section III derives the characteristic impedance (Z_0) and propagation constant for thin film superconducting strip line used as a common control which crosses a series of cryotron gates periodically spaced. This is related to the transmission characteristics (Z_c, τ_c) for the strip line if they did not cross cryotron gates. Fig. 6 shows this relation with plots of characteristic impedance and delay time per meter versus the ratio (K) of distance between gate crossing with gate width. For high packing density $K = 1$, it is found that the characteristic impedance is reduced almost 20 percent due to the loading of the gate crossings.

The delay time τ_0 is increased 20 percent. The attenuation constant (α_0) is extremely small, hence it can be neglected for practical purposes.

APPENDIX A

Referring to Fig. 5 from point 1–1′ to point 2–2′ the control line has characteristic impedance Z_c and propagation constant γ_c. The voltage and current equations in terms of the Z_c and γ_c are[9,10,11,12]

$$\begin{vmatrix} V_1 \\ I_1 \end{vmatrix} = \begin{vmatrix} \cosh \gamma_c l & Z_c \sinh \gamma_c l \\ \frac{1}{Z_c} \sinh \gamma_c l & \cosh \gamma_c l \end{vmatrix} \begin{vmatrix} V_2 \\ I_2 \end{vmatrix}, \tag{43}$$

where

$$l = \tfrac{1}{2}(d_g + W_g).$$

From point 2–2′ to point 3–3′, the voltage and current relations are

$$\begin{vmatrix} V_2 \\ I_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ Y & 1 \end{vmatrix} \begin{vmatrix} V_3 \\ I_3 \end{vmatrix}, \tag{44}$$

where

$$Y = \frac{Y_c}{1 + \frac{1}{2} Y_c Z_g}.$$

If the gate lines are terminated in their characteristic impedance then Z_g is the characteristic impedance and a real number independent of frequency. Otherwise, Z_g might be a complex and be frequency dependent.

Similarly, from point 3–3′ to 4–4′ we have

$$\begin{vmatrix} V_3 \\ I_3 \end{vmatrix} = \begin{vmatrix} \cosh \gamma_c l & Z_c \sinh \gamma_c l \\ \frac{1}{Z_c} \sinh \gamma_c l & \cosh \gamma_c l \end{vmatrix} \begin{vmatrix} V_4 \\ I_4 \end{vmatrix}. \tag{45}$$

Hence, for each period, we have

$$\begin{vmatrix} V_1 \\ I_1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} V_4 \\ I_4 \end{vmatrix},$$

where

$$A = \cosh 2\gamma_c l + YZ_c/2 \sinh 2\gamma_c l,$$

$$A = D,$$

and

$$B = Z_c \sinh 2\gamma_c l + YZ_c^2 \sinh^2 \gamma_c l,$$

$$C = \frac{1}{Z_c} \sinh 2\gamma_c l + Y \cosh^2 \gamma_c l.$$

Since $A = D$ this is a symmetrical circuit, accordingly its characteristic impedance Z_0 and propagation constant γ_0 are

$$\begin{aligned} Z_0 &= \sqrt{B/C} \\ &= Z_c\left[1 - \frac{YZ_c}{\sinh 2\gamma_c l + YZ_c \cosh^2 \gamma_c l}\right]^{\frac{1}{2}} \end{aligned} \tag{46}$$

$$\begin{aligned} \gamma_0 &= \cosh^{-1} A \\ &= \cosh^{-1} [\cosh 2\gamma_c l + YZ_c/2 \sinh 2\gamma_c l]. \end{aligned} \tag{47}$$

LIST OF SYMBOLS USED

l^e = Inductance
C = Capacitance
g = Conductance
$W_g d_1 b, h, d_g$ = Geometric Parameters
μ_e = Permeability of dielectric material
ϵ_e = Permittivity of dielectric material
σ_e = Conductivity of dielectric material
μ_c = Permeability of superconductor
ϵ_c = Permittivity of superconductor
σ_c = Residual normal conductivity of superconductor
J = Current Density
E = Electric Field
H = Magnetic Field
ω = Angular velocity
λ_c = Penetration depth of superconductor
δ = Classic skin-effect depth
Z = Impedance
Z_c , Z_0 = Characteristic Impedance
γ_c = Progapation constant
θ_c = Phase constant
Λ_e = Wave length
τ = Delay time

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