Scattering Relations in Lossless Varactor Frequency Multipliers

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In recent years, the use of varactor diodes for harmonic generation has become increasingly widespread. Varactor harmonic generators come under the general class of pumped nonlinear systems, which are networks driven periodically by a pump or a local oscillator at a frequency ω_0 *and its harmonics. For such systems, a general method has been presented in this paper to obtain the scattering parameters which relate the small-signal fluctuations present at various points in the system. In particular, the scattering parameters of lossless abrupt-junction varactor harmonic generators of order* 2^n, 3^s, *and* 2^n3^s *with minimum number of idlers have been obtained. It has been shown for these multipliers that there is no amplitude-to-phase or phase-to-amplitude conversion if fluctuations are in the vicinity of the carriers. With minor modifications this theory can be extended to the study of lossy varactor harmonic generators.*

I. INTRODUCTION

The carrier voltages and currents present in a varactor frequency multiplier are perturbed by small amplitude and phase fluctuations due to a variety of causes, such as noise, synchronizing signals, etc. In some applications, these fluctuations may be due to modulations purposely applied to the carriers. An example of such applications is that in which a frequency modulated signal is multiplied in frequency to increase its modulation index. It is the purpose of this paper to study how these perturbations propagate in the circuit of a multiplier. In other words, this paper considers the problem of determining the small-signal behavior—a problem which is of basic importance in understanding the problem of stability and noise performance in high efficiency varactor multipliers.‡

‡ See Ref. 1. The problem of stability is also treated in a subsequent paper.[2] Part of the results obtained in this paper represent generalizations of some of the results presented in Refs. 1, 3, and 4.

In the earlier part of this paper, a general method has been presented in order to obtain the scattering parameters of pumped nonlinear systems which are networks driven periodically by a pump or a local oscillator at a frequency¶ ω_0 and its harmonics. Harmonic generators discussed in this paper come under this class of systems. Some of the formalisms usually used to describe the fluctuations in these systems are also briefly reviewed.

In the second part of this paper we discuss varactor multipliers in which the diode is not overdriven and is of the abrupt-junction type. The equivalent circuit of this type of multiplier consists of an ordinary linear, passive, and time-invariant circuit connected to the time-varying component of the elastance $S(t)$ of the varactor.[5] In general, it is shown that a complete solution of the small-signal behavior of such a circuit requires that $S(t)$ be known. On the other hand, it is well known that certain properties of the small-signal behavior of a multiplier do not depend at all on the particular form of $S(t)$. For instance, a general and well-known property of a multiplier of order N is that slow fluctuations in the phase of the input drive produce N times as large fluctuations in the phase of the output signal. One of the main results of this paper is that, under certain general conditions, many other properties of the multiplier are related in a simple way only to the order of multiplication N. All the small-signal characteristics of a multiplier that are of practical interest can, therefore, be readily determined without having to calculate $S(t)$.

Specifically, we consider a lossless multiplier of order $N = 2^n3^s = 2, 3, 4$ etc., which is tuned at all carrier frequencies§ and has the least number of idlers. Then, if the various small-signal fluctuations of such a multiplier are properly normalized with respect to the corresponding carriers, one finds that the small-signal terminal behavior of the elastance $S(t)$ is completely determined by N only. It is important to point out that this is exactly true only for $\omega \ll \omega_0$, where ω is the frequency of the fluctuations and ω_0 is the carrier frequency of the drive. If this inequality is not satisfied, then the small-signal behavior will also depend on ω. A consequence of these results is that the AM and PM scattering parameters of the multiplier of order $N = 2^n3^s$ considered in this paper only depend on n and s, in the vicinity of the carriers. They are given, respectively, by the two matrices

¶ In this paper the word frequency has been used exclusively for the angular frequency of a sinusoidal signal. If f is the frequency of a signal in Hz its angular frequency ω is given by $\omega = 2\pi f$ in radians/second.

§ Tuning of idlers, and input, and output circuits usually gives near optimum efficiency.[5,6,7,8]

$$\begin{bmatrix} \frac{1}{3} - \frac{(-1)^n}{3} 2^{-n} & (-1)^n 2^{-n} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & (-1)^n 3^{-s} \\ 2^n 3^s & \frac{1}{3} - \frac{(-1)^n}{3} 2^n \end{bmatrix}.$$

It is important to point out that for the above multiplier it has been assumed in deriving the results that the bias circuit is properly designed so that there are no low-frequency fluctuations of the average capacitance of the varactor diode.[6] This assumption leads to the result that there is no amplitude-to-phase and phase-to-amplitude conversion if $\omega/\omega_0 \ll 1$.

Several other results are also presented in this paper. For instance, it is shown that, if the number of idlers is minimum, then an abrupt-junction varactor multiplier of order $N = N_1 \times N_2 \times \ldots \times N_n$ is equivalent to a cascade of n multipliers of order $N_1, N_2, \ldots, N_n$. If the varactor is not overdriven, this property furnishes the basic equivalent circuit for studying the properties of most of the higher-order multipliers encountered in practice ($N = 4$, 6, 8, etc.).

Finally, it is important to point out that techniques presented in this paper are applicable to the derivation of scattering parameters of multipliers, of any order, with any arbitrary configuration of idlers, and using a varactor diode having arbitrary capacitance variation and drive level. We only assume that the elastance $S(t)$ of the diode used in the multiplier has a Fourier series.‡

II. SOME CONSIDERATIONS OF PERIODICALLY DRIVEN NONLINEAR SYSTEMS

As mentioned earlier in this paper, frequency multipliers come under the general class of nonlinear systems driven by a strong periodic carrier. It is our interest to study in this paper how small perturbations on the periodic driving of such systems are propagated, and to this end we shall give a brief introduction§ of a circuit theory which enables us to relate the perturbations at different parts of the system. The perturbations or fluctuations that we would like to analyze could be caused by desired or undesired modulation, noise, hum, or synchronizing signals. The origin of these sources of fluctuations is not relevant to our development of this theory.

Let us consider a nonlinear system. It is our assumption that the

‡ The conditions under which a periodic time function $x(t)$ has a Fourier series are well-known; and can be found in any book on Fourier series. See, for example, Ref. 9.

§ See Ref. 10 for a more detailed account of this theory.

large-signal voltages and currents at various parts within the system are, by design, periodic with some frequency ω_0. Thus, the voltage at some specific point within the network or across one of its terminal pairs, $v(t)$, is of the form

$$v(t) = \sum_{k=-\infty}^{\infty} V_k \exp(jk\omega_0 t), \tag{1}$$

where the V_k's are half-amplitude¶ Fourier coefficients, with $V_{-k} = V_k^*$. However, the actual voltage $v(t)$ may deviate from (1) because of fluctuations present in the system. Thus,

$$v(t) = \sum_{k=-\infty}^{\infty} V_k \exp(jk\omega_0 t) + \delta v(t), \tag{2}$$

where $\delta v(t)$ is small compared to $v(t)$ in (1). The circuit theory that we shall use in the rest of this paper is one which describes perturbations $\delta v(t)$ and relates them to similar perturbations of voltages and currents in other parts of the system. The perturbations are assumed to be small and they are at frequencies close to the carriers.||

The carrier voltage at some particular point in the system is of the form

$$V_k \exp(jk\omega_0 t) + V_k^* \exp(-jk\omega_0 t), \tag{3}$$

where V_k has some phase angle φ_{vk}. The actual voltage $v_k(t)$ in the vicinity of this carrier deviates from (3) because of the perturbation $\delta v_k(t)$;

$$v_k(t) = V_k \exp(jk\omega_0 t) + V_k^* \exp(-jk\omega_0 t) + \delta v_k(t). \tag{4}$$

Similar expressions can be written for currents and voltages at various places in the network. The various voltages like $v(t)$ obey Kirchhoff's voltage law, and various currents $i(t)$ defined at various points in the network obey Kirchhoff's current law. Furthermore, the carrier voltages and currents at various points in the network obey these Kirchhoff's laws, leading us to conclude that the perturbations like $\delta v(t)$ and $\delta i(t)$ also obey them.

Let us now assume that the perturbation $\delta v_k(t)$ contain frequencies that are located in a band of width $2\omega_c$ centered about frequency $k\omega_0$ where $2\omega_c < \omega_0$.[10] We can write[10] $v_k(t)$ as‡

¶ Note the use of half-amplitudes, rather than amplitudes or rms values.

|| The large signal voltage or current present in the system at frequency $\pm k\omega_0$ will be referred to from hereon as the carrier voltage or current at that frequency. In frequency multipliers carriers are at different frequencies at different parts of the system.

‡ In writing (6), it is assumed that $| v_{pk}(t) |/| V_k | \ll 1$ for all t.

$$v_k(t) = 2\,\mathrm{Re}\,[|\,V_k\,| + v_{ak}(t) - jv_{pk}(t)]\exp[j(k\omega_0 t + \varphi_{vk})] \tag{5}$$

$$\approx 2\,\mathrm{Re}\,[|\,V_k\,| + v_{ak}(t)]\exp\left\{j\left[k\omega_0 t + \varphi_{vk} - \frac{v_{pk}(t)}{|\,V_k\,|}\right]\right\}, \tag{6}$$

where $v_{ak}(t)$ and $v_{pk}(t)$ are slowly varying functions of time. The voltage $v_{ak}(t)$ can be interpreted, since it is small, as a perturbation on the amplitude $|\,V_k\,|$ of the carrier. Similarly, voltage $v_{pk}(t)$, because of (6), can be interpreted as a perturbation on the phase $k\omega_0 t + \varphi_{vk}$ of the carrier. We shall refer to $v_{ak}(t)$ as amplitude (AM) fluctuations and to $v_{pk}(t)$ as phase (PM) fluctuations. Similar AM and PM fluctuations can be defined at various points in the system.

If these AM and PM fluctuations are sinusoidal, we have§

$$v_{ak}(t) = V_{ak}\exp(j\omega t) + V_{ak}^*\exp(-j\omega t) \tag{7}$$

and

$$v_{pk}(t) = V_{pk}\exp(j\omega t) + V_{pk}^*\exp(-j\omega t). \tag{8}$$

The actual voltage $v_k(t)$ is then given by

$$v_k(t) = 2\,\mathrm{Re}\,[|\,V_k\,| + (V_{ak} - jV_{pk})\exp(j\omega t) + (V_{ak}^* - jV_{pk}^*)\exp(-j\omega t)]\exp[j(k\omega_0 t + \varphi_{vk})] \tag{9}$$

$$= 2\,\mathrm{Re}\,\{V_k\exp(jk\omega_0 t) + V_{\alpha k}\exp[j(k\omega_0 + \omega)t] + V_{\beta k}\exp[j(-k\omega_0 + \omega)t]\}, \tag{10}$$

where $V_{\alpha k}$, $V_{\beta k}$, V_{ak}, and V_{pk} are related. The relation is

$$\begin{bmatrix} V_{ak} \\ V_{pk} \end{bmatrix} = \lambda_{vk}\begin{bmatrix} V_{\alpha k} \\ V_{\beta k} \end{bmatrix}, \tag{11}$$

where the matrix λ_{vk} is a function of only the carrier phase angle φ_{vk}. The matrix λ_{vk} can be represented as

$$\lambda_{vk} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}\begin{bmatrix} \exp(-j\varphi_{vk}) & 0 \\ 0 & \exp(j\varphi_{vk}) \end{bmatrix}. \tag{12}$$

Equation (10) shows explicitly the three frequencies $k\omega_0$, $k\omega_0 + \omega$, and $-k\omega_0 + \omega$. The two sidebands here are both higher in frequency than $k\omega_0$, and $-k\omega_0$, respectively, and therefore, these representations are referred to as upper sideband $(\alpha - \beta)$ representations. We will use them along with the representations of the form (9) in the rest

§ Since the fluctuations $v_{ak}(t)$ and $v_{pk}(t)$ are band limited around dc, ω must be less than ω_c in magnitude, where $2\omega_c < \omega_0$.

of this work. Their mutual relation is given in (11). Because of (10) we shall refer to ω as the fluctuation difference frequency.

Let us now consider a pumped nonlinear system exchanging power at the carrier frequencies $\pm\omega_0$, $\pm 2\omega_0$, $\cdots$, $\pm n\omega_0$. A nonlinear system exchanging power at a number of frequencies can be considered as a multiport multifrequency system as shown in Fig. 1. In Fig. 1 the system exchanges power at n carrier frequencies and it is assumed, without loss of generality, that no two ports exchange power at the same carrier frequency. Let the perturbation voltage and current at port k be denoted by $\delta v_k(t)$ and $\delta i_k(t)$, respectively. Since δv's are small, they must be linearly related.‡ Hence, there is a relation which

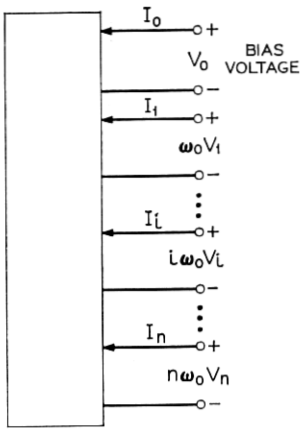

Fig. 1 — Pumped nonlinear system exchanging power at n carrier frequencies.

relates δv_k to δi's of the form

$$\delta v_k(t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} h_{kj}(t, \tau)\, \delta i_j(t - \tau)\, d\tau, \tag{13}$$

where $h_{kj}(t, \tau)$'s are functions of time t, as well as of time difference τ. Since the driving is periodic, if δi's were applied one period later, δv_k would be the same, except that it would be delayed by one period. This argument leads to the conclusion that $h_{kj}(t, \tau)$'s are periodic functions in t, with period $T_0 = 2\pi/\omega_0$ and can be expressed in a Fourier series of the form

‡ This is because only first-order terms in δv's and δi's are retained. Higher-order terms are assumed to be negligible even when first-order terms vanish.

$$h_{kj}(t, \tau) = \sum_{l=-\infty}^{\infty} (h_{kj})_l(\tau) \exp (jl\omega_0 t), \tag{14}$$

where $(h_{kj})_l(\tau)$ is a function of τ. Upon substituting (14) in (13), we find

$$\delta v_k(t) = \sum_{j=1}^{n} \sum_{l=-\infty}^{\infty} \exp (jl\omega_0 t) \int_{-\infty}^{\infty} (h_{kj})_l(\tau)\ \delta i_j(t - \tau)\ d\tau. \tag{15}$$

If $\delta v_k(t)$ is represented in the $\alpha - \beta$ form,

$$\delta v_k(t) = 2 \operatorname{Re} \{ V_{\alpha k} \exp [j(k\omega_0 + \omega)t] + V_{\beta k} \exp [j(-k\omega_0 + \omega)t] \}, \tag{16}$$

we find‡

$$\begin{bmatrix} V_{\alpha k} \\ V_{\beta k} \end{bmatrix} = \begin{bmatrix} Z_{\alpha k \alpha 1} & Z_{\alpha k \beta 1} & \cdots & Z_{\alpha k \alpha k} & Z_{\alpha k \beta k} & \cdots & Z_{\alpha k \alpha n} & Z_{\alpha k \beta n} \\ Z_{\beta k \alpha 1} & Z_{\beta k \beta 1} & \cdots & Z_{\beta k \alpha k} & Z_{\beta k \beta k} & \cdots & Z_{\beta k \alpha n} & Z_{\beta k \beta n} \end{bmatrix} \mathbf{I}, \tag{17}$$

where§

$$\mathbf{I} = \{ I_{\alpha 1}, I_{\beta 1}, \cdots, I_{\alpha k}, I_{\beta k}, \cdots, I_{\alpha n}, I_{\beta n} \}. \tag{18}$$

III. SMALL-SIGNAL ANALYSIS OF PUMPED NONLINEAR SYSTEMS

For the nonlinear systems that we shall consider in this paper, we shall assume that the total voltage $v(t)$ across the nonlinear element is related to the current $i(t)$ through it by the equation¶

$$v(t) = F\{i(t)\}, \tag{19}$$

where $F\{i(t)\}$ is a single-valued functional of $i(t)$.

Assuming that there are carrier currents flowing in the system at frequencies $\pm i\omega_0$, $0 \leqq i \leqq n$, the spot frequency terminal behavior of this system at a difference frequency ω is given according to (17) by an equation of the form||

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‡ Essentially, we are discussing impedance formalism here which relates voltages to currents through an impedance matrix. Several other kinds of formalisms like scattering matrix representation or chain matrix representation can also be used to relate other desired sets of variables.

§ A column matrix $\mathbf{a}$ is written in the form $\{a_1, a_2, \ldots, a_n\}$, the curly braces being used to identify it as a column matrix.

¶ If there are any physical sources of fluctuations (such as noise sources) in the pumped nonlinear system, (20) is to be suitably modified. For the discussion of the case in which noise sources may be present in the pumped nonlinear system, see Ref. 11.

|| It must be pointed out that this equation only relates the small-signal fluctuations present in the system and not the carrier voltages and currents.

$$\begin{bmatrix} V_{\alpha 0} \\ V_{\alpha 1} \\ V_{\beta 1} \\ \vdots \\ V_{\alpha i} \\ V_{\beta i} \\ \vdots \\ V_{\alpha n} \\ V_{\beta n} \end{bmatrix} = \begin{bmatrix} Z_{\alpha 0 \alpha 0} & Z_{\alpha 0 \alpha 1} & Z_{\alpha 0 \beta 1} & \cdots & Z_{\alpha 0 \alpha i} & Z_{\alpha 0 \beta i} & \cdots & Z_{\alpha 0 \alpha n} & Z_{\alpha 0 \beta n} \\ Z_{\alpha 1 \alpha 0} & Z_{\alpha 1 \alpha 1} & Z_{\alpha 1 \beta 1} & \cdots & Z_{\alpha 1 \alpha i} & Z_{\alpha 1 \beta i} & \cdots & Z_{\alpha 1 \alpha n} & Z_{\alpha 1 \beta n} \\ Z_{\beta 1 \alpha 0} & Z_{\beta 1 \alpha 1} & Z_{\beta 1 \beta 1} & \cdots & Z_{\beta 1 \alpha i} & Z_{\beta 1 \beta i} & \cdots & Z_{\beta 1 \alpha n} & Z_{\beta 1 \beta n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ Z_{\alpha i \alpha 0} & Z_{\alpha i \alpha 1} & Z_{\alpha i \beta 1} & \cdots & Z_{\alpha i \alpha i} & Z_{\alpha i \beta i} & \cdots & Z_{\alpha i \alpha n} & Z_{\alpha i \beta n} \\ Z_{\beta i \alpha 0} & Z_{\beta i \alpha 1} & Z_{\beta i \beta 1} & \cdots & Z_{\beta i \alpha i} & Z_{\beta i \beta i} & \cdots & Z_{\beta i \alpha n} & Z_{\beta i \beta n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ Z_{\alpha n \alpha 0} & Z_{\alpha n \alpha 1} & Z_{\alpha n \beta 1} & \cdots & Z_{\alpha n \alpha i} & Z_{\alpha n \beta i} & \cdots & Z_{\alpha n \alpha n} & Z_{\alpha n \beta n} \\ Z_{\beta n \alpha 0} & Z_{\beta n \alpha 1} & Z_{\beta n \beta 1} & \cdots & Z_{\beta n \alpha i} & Z_{\beta n \beta i} & \cdots & Z_{\beta n \alpha n} & Z_{\beta n \beta n} \end{bmatrix} \begin{bmatrix} I_{\alpha 0} \\ I_{\alpha 1} \\ I_{\beta 1} \\ \vdots \\ I_{\alpha i} \\ I_{\beta i} \\ \vdots \\ I_{\alpha n} \\ I_{\beta n} \end{bmatrix}, \quad (20)$$

where $V_{\alpha i}$ and $V_{\beta i}$ are the terminal voltages at frequencies $j\omega_0 + \omega$ and $-j\omega_0 + \omega$, respectively; and $I_{\alpha i}$ and $I_{\beta i}$ are the corresponding terminal currents. We would like to note here that $V_{\alpha 0} = V_{\beta 0}$ is the small-signal terminal voltage at the frequency ω. We shall, for brevity, write (20) as

$$(\mathbf{V}_{\alpha-\beta})_n = (\underline{Z}_{\alpha-\beta})_n(\mathbf{I}_{\alpha-\beta})_n \,. \quad (21)$$

Let us now specifically consider a varactor diode which is pumped at a frequency ω_0 and its harmonics. The varactor model that we shall use is shown in Fig. 2. It is a variable capacitance in series with a

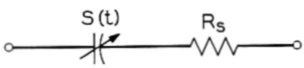

Fig. 2 — Varactor model.

constant resistance R_s.‡ The instantaneous varactor voltage $v(t)$ can be written as some function f of the charge, plus the drop across the series resistance R_s:

$$v(t) = f[q(t)] + R_s i(t), \quad (22)$$

where

$$q(t) = \int_{-\infty}^{t} i(t)\, dt. \quad (23)$$

For such a varactor, we can make use of the small-signal equations given in Ref. 5 in order to obtain the impedance matrix $\underline{Z}_{\alpha-\beta}$ in (21). If the elastance $S(t)$ of the varactor diode can be written in a Fourier series of the form

$$S(t) = \sum_{k=-\infty}^{\infty} S_k \exp(jk\omega_0 t), \quad (24)$$

‡ Mainly we shall be concerned with varactor diodes which are lossless in the succeeding sections of this paper. For a lossless varactor diode $R_s = 0$.

the matrix $(\underline{Z}_{\alpha-\beta})_n$ can be represented as§

$$(\underline{Z}_{\alpha-\beta})_n = \begin{bmatrix} \frac{S_0}{j\omega} & \frac{S_1^*}{j(\omega_0+\omega)} & \frac{S_1}{j(-\omega_0+\omega)} & \cdots & \frac{S_i^*}{j(i\omega_0+\omega)} & \frac{S_i}{j(-i\omega_0+\omega)} & \cdots & \frac{S_n^*}{j(n\omega_0+\omega)} & \frac{S_n}{j(-n\omega_0+\omega)} \\ \frac{S_1}{j\omega} & \frac{S_0}{j(\omega_0+\omega)} & \frac{S_2}{j(-\omega_0+\omega)} & \cdots & \frac{S_{i-1}^*}{j(i\omega_0+\omega)} & \frac{S_{i+1}}{j(-i\omega_0+\omega)} & \cdots & \frac{S_{n-1}^*}{j(n\omega_0+\omega)} & \frac{S_{n+1}}{j(-n\omega_0+\omega)} \\ \frac{S_1^*}{j\omega} & \frac{S_2^*}{j(\omega_0+\omega)} & \frac{S_0}{j(-\omega_0+\omega)} & \cdots & \frac{S_{i+1}^*}{j(i\omega_0+\omega)} & \frac{S_{i-1}}{j(-i\omega_0+\omega)} & \cdots & \frac{S_{n+1}^*}{j(n\omega_0+\omega)} & \frac{S_{n-1}}{j(-n\omega_0+\omega)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \frac{S_i}{j\omega} & \frac{S_{i-1}}{j(\omega_0+\omega)} & \frac{S_{i+1}}{j(-\omega_0+\omega)} & \cdots & \frac{S_0}{j(i\omega_0+\omega)} & \frac{S_{2i}}{j(-i\omega_0+\omega)} & \cdots & \frac{S_{n-i}^*}{j(n\omega_0+\omega)} & \frac{S_{n+i}}{j(-n\omega_0+\omega)} \\ \frac{S_i^*}{j\omega} & \frac{S_{i+1}^*}{j(\omega_0+\omega)} & \frac{S_{i-1}^*}{j(-\omega_0+\omega)} & \cdots & \frac{S_{2i}^*}{j(i\omega_0+\omega)} & \frac{S_0}{j(-i\omega_0+\omega)} & \cdots & \frac{S_{n+i}^*}{j(n\omega_0+\omega)} & \frac{S_{n-i}}{j(-n\omega_0+\omega)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{S_n}{j\omega} & \frac{S_{n-1}}{j(\omega_0+\omega)} & \frac{S_{n+1}}{j(-\omega_0+\omega)} & \cdots & \frac{S_{n-i}}{j(i\omega_0+\omega)} & \frac{S_{n+i}}{j(-i\omega_0+\omega)} & \cdots & \frac{S_0}{j(n\omega_0+\omega)} & \frac{S_{2n}}{j(-n\omega_0+\omega)} \\ \frac{S_n^*}{j\omega} & \frac{S_{n+1}^*}{j(\omega_0+\omega)} & \frac{S_{n-1}^*}{j(-\omega_0+\omega)} & \cdots & \frac{S_{n+i}^*}{j(i\omega_0+\omega)} & \frac{S_{n-i}^*}{j(-i\omega_0+\omega)} & \cdots & \frac{S_{2n}^*}{j(n\omega_0+\omega)} & \frac{S_0}{j(-n\omega_0+\omega)} \end{bmatrix}. \quad (25)$$

§ We have put $R_s = 0$ in order to obtain (25). If $R_s \neq 0$, $(\underline{Z}_{\alpha-\beta})_n = \{(\underline{Z}_{\alpha-\beta})_n$ in (25)$\} + R_s \underline{1}_{2n+1}$. $\underline{1}_n$ is the unit matrix of order n.

Equation (25) shows that the impedance matrix $(\underline{Z}_{\alpha-\beta})_n$ always exists for a pumped varactor diode as long as the elastance $S(t)$ is expressible in the form (24).

Let us now assume that the input carrier frequency is $l\omega_0$, $1 \leqq l \leqq n$; and that the output carrier frequency is $s\omega_0$, $1 \leqq s \leqq n$ (see Fig. 3). Let us also assume that the terminal constraints at other carrier frequencies are such that

$$\mathbf{V}' = -\underline{Z}'_{\alpha-\beta}\mathbf{I}', \tag{26}$$

where $\mathbf{V}'$ is an $\alpha - \beta$ terminal voltage column matrix given by

$$\mathbf{V}' = \begin{bmatrix} V_{\alpha 0} \\ V_{\alpha 1} \\ V_{\beta 1} \\ \vdots \\ V_{\alpha(l-1)} \\ V_{\beta(l-1)} \\ V_{\alpha(l+1)} \\ V_{\beta(l+1)} \\ \vdots \\ V_{\alpha(s-1)} \\ V_{\beta(s-1)} \\ V_{\alpha(s+1)} \\ V_{\beta(s+1)} \\ \vdots \\ V_{\alpha n} \\ V_{\beta n} \end{bmatrix}. \tag{27}$$

$\mathbf{I}'$ is the corresponding terminal current column matrix. $\underline{Z}'_{\alpha-\beta}$ is the impedance matrix determined by the terminal constraints imposed by the external circuits on the system. These terminal constraints at all carrier frequencies excluding $l\omega_0$ and $s\omega_0$ are assumed to be known. Even though the currents flowing in the varactor are not limited by the diode in the range of available frequencies, it is assumed that the external circuits are such as to offer an infinite impedance at any frequency very far from the carrier frequencies present in the multiplier. This enables us to consider the multiplier as a finite port multifrequency system.

It may now be seen that by using (25) and (26) we can obtain a relation between $V_{\alpha l}$, $V_{\beta l}$, $I_{\alpha l}$, $I_{\beta l}$, $V_{\alpha s}$, $V_{\beta s}$, $I_{\alpha s}$, and $I_{\beta s}$. In

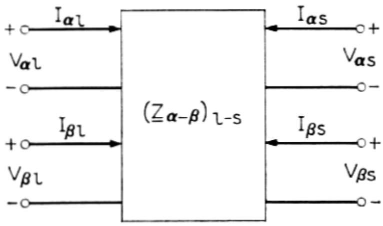

Fig. 3 — Small-signal terminal behavior of pumped nonlinear twoport.

particular we can write‡

$$\begin{bmatrix} V_{\alpha l} \\ V_{\beta l} \\ V_{\alpha s} \\ V_{\beta s} \end{bmatrix} = \begin{bmatrix} Z''_{\alpha l \alpha l} & Z''_{\alpha l \beta l} & Z''_{\alpha l \alpha s} & Z''_{\alpha l \beta s} \\ Z''_{\beta l \alpha l} & Z''_{\beta l \beta l} & Z''_{\beta l \alpha s} & Z''_{\beta l \beta s} \\ Z''_{\alpha s \alpha l} & Z''_{\alpha s \beta l} & Z''_{\alpha s \alpha s} & Z''_{\alpha s \beta s} \\ Z''_{\beta s \alpha l} & Z''_{\beta s \beta l} & Z''_{\beta s \alpha s} & Z''_{\beta s \beta s} \end{bmatrix} \begin{bmatrix} I_{\alpha l} \\ I_{\beta l} \\ I_{\alpha s} \\ I_{\beta s} \end{bmatrix} \tag{28}$$

or

$$(\mathbf{V}_{\alpha-\beta})_{l-s} = (\underline{Z}_{\alpha-\beta})_{l-s}(\mathbf{I}_{\alpha-\beta})_{l-s} . \tag{29}$$

Equation (29) relates the small-signal fluctuations existing at input and output terminals of a pumped varactor diode. In case one is interested in relating the AM and PM fluctuations at the input and output terminals of a pumped varactor diode, we make use of (11). If φ_{vl}, φ_{vs}, φ_{il}, and φ_{is} are the phase angles of carrier voltages and currents at the input and output of a pumped varactor diode we get the following equation which relates the different fluctuations:

‡ In certain cases it is possible that the matrix $(\underline{Z}_{\alpha-\beta})_{l-s}$ does not exist. Even though $(\underline{Z}_{\alpha-\beta})_{l-s}$ may not exist, in most cases of practical interest, we can always find a relation between the terminal voltages and currents at the sideband frequencies in the vicinity of input and output carriers. This will be shown to be true in the case of a tripler which is discussed elsewhere in this paper. However, the matrix $(\underline{Z}_{\alpha-\beta})_n$ always exists for a pumped varactor diode.

$$\begin{bmatrix} V_{al} \\ V_{pl} \\ V_{as} \\ V_{ps} \end{bmatrix} = (Z_{a-p})_{l-s} \begin{bmatrix} I_{al} \\ I_{pl} \\ I_{as} \\ I_{ps} \end{bmatrix}, \tag{30}$$

where

$$(Z_{a-p})_{l-s} = (\lambda_v)_{l-s}(Z_{\alpha-\beta})_{l-s}\{(\lambda_i)_{l-s}\}^{-1}, \tag{31}$$

$$(\lambda_v)_{l-s} = \left[\begin{array}{c|c} \lambda_{vl} & 0 \\ \hline 0 & \lambda_{vs} \end{array}\right], \tag{32}$$

and

$$(\lambda_i)_{l-s} = \left[\begin{array}{c|c} \lambda_{il} & 0 \\ \hline 0 & \lambda_{is} \end{array}\right]. \tag{33}$$

The matrices λ's are given as in (12).

Once we have obtained the impedance matrix representation for the pumped varactor diode other kinds of representations like scattering matrix representation or chain matrix representation could be derived for any specific application or convenience. Mutual relations between these representations are given in Refs. 11 and 12, and we shall not discuss them in this paper. Scattering matrix representation of lossless abrupt-junction varactor multipliers is extensively treated in later sections of this paper.

IV. SCATTERING PARAMETERS FOR PUMPED NONLINEAR SYSTEMS

The total voltage $v(t)$ in the vicinity of a carrier at frequency $\pm k\omega_0$ can be represented as in (5) or (6). $v_{ak}(t)$ can be interpreted as a small perturbation on the amplitude $| V_k |$ of the carrier, and $v_{pk}(t)$ as a perturbation on the phase $k\omega_0 t + \varphi_{vk}$ of the carrier. Since the device acts as a time-variant linear device to the fluctuations and since superposition holds, $v_{ak}(t)$ and $v_{pk}(t)$ can be represented as in (7) and (8).

The voltage AM and PM modulation indexes at the carrier frequency $k\omega_0$ may, therefore, be defined as

$$m_{vk} = \frac{V_{ak}}{| V_k |} \tag{34}$$

and

$$\theta_{vk} = \frac{V_{pk}}{| V_k |}. \tag{35}$$

The AM and PM indexes at the input and output of a pumped nonlinear system are, according to (30), related by an equation of the form

$$\begin{bmatrix} m_{vl} \\ \theta_{vl} \\ m_{vs} \\ \theta_{vs} \end{bmatrix} = (Z_{m-\theta})_{l-s} \begin{bmatrix} m_{il} \\ \theta_{il} \\ m_{is} \\ \theta_{is} \end{bmatrix}, \tag{36}$$

where

$$(Z_{m-\theta})_{l-s} = \begin{bmatrix} |V_l|^{-1} & & & 0 \\ & |V_l|^{-1} & & \\ & & |V_s|^{-1} & \\ 0 & & & |V_s|^{-1} \end{bmatrix} (Z_{a-p})_{l-s} \cdot \begin{bmatrix} |I_l| & & & 0 \\ & |I_l| & & \\ & & |I_s| & \\ 0 & & & |I_s| \end{bmatrix}. \tag{37}$$

It is assumed that carrier voltages at frequencies $l\omega_0$ and $s\omega_0$ are nonzero.

Until now we have exclusively used the impedance formalism to describe the properties of the pumped nonlinear system at the sideband frequencies. The choice of an appropriate formalism is particularly important in theoretical studies where important properties of the system may be obscured by complicated equations. The scattering parameters of a system are a set of quantities which can describe the performance of the system under any specified terminating conditions, just as the impedance (or admittance) quantities can, but while the scattering coefficients may not be particularly convenient for short or open-circuit computations, they may be applied in a relatively simple fashion when the network is terminated in a prescribed load impedance. Since we will be mainly interested in studying proper terminations for the system in order to realize certain desirable characteristics, scattering matrix formulation to describe AM and PM fluctuations in pumped nonlinear systems seems to be the most desirable.[1,3,4,13] Equation (36) relates the AM and PM indexes or normalized voltages and currents at the input and output of a pumped nonlinear system.‡ The incident

‡ Most of these concepts can be extended in a straightforward fashion if the pumped nonlinear system has more than two accessible ports.

and reflected AM and PM indexes (see Fig. 4) can, therefore, be written[13] as

$$(m_i)_j = \tfrac{1}{2}(m_{vj} + m_{ij}), \qquad j = l, s, \tag{38}$$

$$(m_r)_j = \tfrac{1}{2}(m_{vj} - m_{ij}), \qquad j = l, s, \tag{39}$$

$$(\theta_i)_j = \tfrac{1}{2}(\theta_{vj} + \theta_{ij}), \qquad j = l, s, \tag{40}$$

and

$$(\theta_r)_j = \tfrac{1}{2}(\theta_{vj} - \theta_{ij}), \qquad j = l, s. \tag{41}$$

Using (37) through (41), we can now obtain the following scattering matrix representation for a pumped nonlinear system:

$$\begin{bmatrix} (m_r)_l \\ (m_r)_s \\ (\theta_r)_l \\ (\theta_r)_s \end{bmatrix} = \left[\begin{array}{c|c} \underline{S}_{aa} & \underline{S}_{ap} \\ \hline \underline{S}_{pa} & \underline{S}_{pp} \end{array}\right] \begin{bmatrix} (m_i)_l \\ (m_i)_s \\ (\theta_i)_l \\ (\theta_i)_s \end{bmatrix}. \tag{42}$$

The relation between scattering matrix in (42) and impedance matrix in (37) is easily derived.[13] In our case, this is given by

$$(\underline{S})_{l-s} = \underline{1}_4 - 2\{\underline{1}_4 + (\underline{Z}'_{m-\theta})_{l-s}\}^{-1}, \tag{43}$$

where $\underline{1}_4$ is the unit matrix of order 4, and where $\underline{Z}'_{m-\theta}$ is related to $\underline{Z}_{m-\theta}$ in (37) by

$$(\underline{Z}'_{m-\theta})_{l-s} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} (\underline{Z}_{m-\theta})_{l-s} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{44}$$

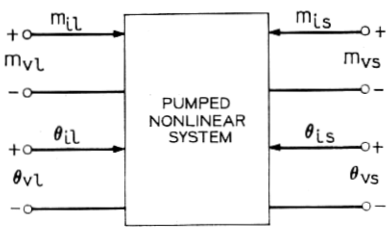

Fig. 4 — Representation of AM and PM fluctuations in a pumped nonlinear system.

The scattering matrix which relates the small-signal fluctuations of a pumped nonlinear system is, therefore, given by (43). We assume that the matrix $\underline{1}_4 + (Z'_{m-\theta})_{l-s}$ is nonsingular.

We would like to point out here that the matrices $\underline{S}_{aa}$, $\underline{S}_{ap}$, $\underline{S}_{pa}$, and $\underline{S}_{pp}$ are all square matrices of second order. For reasons which are evident from (42), the matrix $\underline{S}_{aa}$ will be referred to as AM scattering matrix, $\underline{S}_{ap}$ as the AM-PM scattering matrix, $\underline{S}_{pa}$ as the PM-AM scattering matrix, and $\underline{S}_{pp}$ as the PM scattering matrix.

V. SCATTERING MATRICES OF NOMINALLY DRIVEN LOSSLESS ABRUPT-JUNCTION VARACTOR FREQUENCY MULTIPLIERS‡

The theory developed in the preceding sections will be utilized from hereon in order to obtain the scattering parameters of nominally driven lossless abrupt-junction varactor frequency multipliers. The elastance $S(t)$ of the varactor diode as it is pumped is assumed to be given by§

$$S(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} S_k \exp(jk\omega_0 t). \tag{45}$$

In this section we shall first obtain the scattering matrix of a varactor doubler whose input and output circuits are tuned. In the later part of this section the scattering parameters of a tripler, whose input, output, and idler circuits are tuned, are also derived. The discussion of the scattering parameters of multipliers of higher order is postponed to later sections of this paper. For all the multipliers considered in this paper it is assumed that the bias circuit is properly designed so that there are no currents flowing at the sideband frequencies $\pm\omega$.[6] Even though the currents flowing in the varactor are themselves not limited by the diode in the range of available frequencies we assume that the external circuits connected to the diode are such that they allow currents to flow in the varactor if and only if the frequency spectrum of these currents is in the vicinity of input, output, and idler carrier frequencies. This enables us to consider the multiplier as a finite port multifrequency system.

‡ See also Refs. 3 and 4 for alternate derivation of some of these results.

§ The average elastance S_0 of the varactor diode can always be included with the external circuit. The assumption that $S_0 = 0$ made in this section does not, therefore, involve any loss of generality.

5.1 *Scattering Parameters of a Doubler.*

In a doubler, the only nonzero elastance coefficients are $S_{\pm 1}$, and $S_{\pm 2}$. The impedance matrix $(Z_{\alpha-\beta})_2$ in (25) is represented as (see Fig. 5)

$$\begin{bmatrix} V_{\alpha 1} \\ V_{\beta 1} \\ V_{\alpha 2} \\ V_{\beta 2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{S_2}{j(-\omega_0 + \omega)} & \frac{S_1^*}{j(2\omega_0 + \omega)} & 0 \\ \frac{S_2^*}{j(\omega_0 + \omega)} & 0 & 0 & \frac{S_1}{j(-2\omega_0 + \omega)} \\ \frac{S_1}{j(\omega_0 + \omega)} & 0 & 0 & 0 \\ 0 & \frac{S_1^*}{j(-\omega_0 + \omega)} & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\alpha 1} \\ I_{\beta 1} \\ I_{\alpha 2} \\ I_{\beta 2} \end{bmatrix}. \quad (46)$$

We shall now assume that input and output circuits are tuned which usually gives near optimum efficiency for a doubler.[5,7,8] We also assume that

$$\frac{\omega}{\omega_0} \ll 1. \quad (47)$$

With these two assumptions, we can write the following matrix equation for a doubler:

$$\begin{bmatrix} V_{\alpha 1} \\ V_{\beta 1} \\ V_{\alpha 2} \\ V_{\beta 2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{|S_2|}{\omega_0} & \frac{|S_1|}{2\omega_0} & 0 \\ \frac{|S_2|}{\omega_0} & 0 & 0 & \frac{|S_1|}{2\omega_0} \\ -\frac{|S_1|}{\omega_0} & 0 & 0 & 0 \\ 0 & -\frac{|S_1|}{\omega_0} & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\alpha 1} \\ I_{\beta 1} \\ I_{\alpha 2} \\ I_{\beta 2} \end{bmatrix}. \quad (48)$$

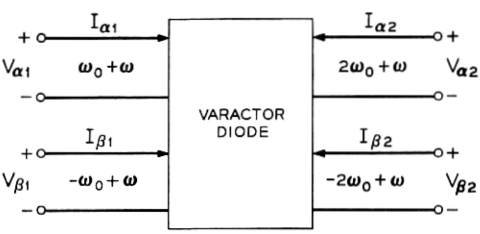

Fig. 5 — Small-signal behavior of a varactor doubler.

The phase angles‡ of carrier voltages and currents are given by[5]

$$\varphi_{V1} = 0, \tag{49}$$

$$\varphi_{I1} = 0, \tag{50}$$

$$\varphi_{V2} = \pi, \tag{51}$$

and

$$\varphi_{I2} = \pi. \tag{52}$$

From (31) the impedance matrix $(Z_{a-p})_2$ is, therefore, written as§

$$(Z_{a-p})_2 = \begin{bmatrix} \frac{|S_2|}{\omega_0} & 0 & -\frac{|S_1|}{2\omega_0} & 0 \\ 0 & -\frac{|S_2|}{\omega_0} & 0 & -\frac{|S_1|}{2\omega_0} \\ \frac{|S_1|}{\omega_0} & 0 & 0 & 0 \\ 0 & \frac{|S_1|}{\omega_0} & 0 & 0 \end{bmatrix}. \tag{53}$$

Equation (53) clearly shows that in a doubler which is properly tuned and whose bias circuit is properly designed, there is no amplitude-to-phase and phase-to-amplitude conversion.[6]

It is shown in Appendix A that

$$\frac{|V_1|}{|I_1|} = \frac{|S_2|}{\omega_0}, \tag{54}$$

$$\frac{|V_2|}{|I_2|} = \frac{|S_1|^2}{4\,|S_2|\,\omega_0}, \tag{55}$$

$$\frac{|V_1|}{|I_2|} = \frac{|S_1|}{2\omega_0}, \tag{56}$$

‡ Without loss of generality the phase angle of input carrier voltage at frequency ω_0 is assumed to be zero. φ_{I2} is the phase angle of the current through the load connected to the doubler.

§ The matrix $(Z)_{1-s}$ will be written as $(Z)_s$ in case this does not lead to any ambiguity.

and

$$\frac{| V_2 |}{| I_1 |} = \frac{| S_1 |}{2\omega_0}. \tag{57}$$

According to (37) and (53), the matrix $(\underline{Z}_{m-\theta})_2$ is represented as

$$(\underline{Z}_{m-\theta})_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}. \tag{58}$$

The scattering matrix of a doubler, whose input and output circuits are tuned is, therefore, according to (43)

$$(\underline{S})_2 = \left[\begin{array}{cc|cc} \frac{1}{2} & -\frac{1}{2} & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & -1 \\ & & 2 & 1 \end{array}\right]; \tag{59}$$

$$(\underline{S}_{aa})_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}, \tag{60}$$

$$(\underline{S}_{ap}) = \underline{0}, \tag{61}$$

$$(\underline{S}_{pa}) = \underline{0}, \tag{62}$$

and

$$(\underline{S}_{pp}) = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}. \tag{63}$$

5.2 *Scattering Parameters of a Tripler.*

In a tripler, carrier currents flow at the frequencies $\pm\omega_0$, $\pm 2\omega_0$, and $\pm 3\omega_0$. The impedance matrix $(\underline{Z}_{\alpha-\beta})_3$ is given as

$$\begin{bmatrix} V_{\alpha 1} \\ V_{\beta 1} \\ V_{\alpha 2} \\ V_{\beta 2} \\ V_{\alpha 3} \\ V_{\beta 3} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{S_2}{j(-\omega_0+\omega)} & \dfrac{S_1^*}{j(2\omega_0+\omega)} & \dfrac{S_3}{j(-2\omega_0+\omega)} & \dfrac{S_2^*}{j(3\omega_0+\omega)} & 0 \\ \dfrac{S_2^*}{j(\omega_0+\omega)} & 0 & \dfrac{S_3^*}{j(2\omega_0+\omega)} & \dfrac{S_1}{j(-2\omega_0+\omega)} & 0 & \dfrac{S_2}{j(-3\omega_0+\omega)} \\ \dfrac{S_1}{j(\omega_0+\omega)} & \dfrac{S_3}{j(-\omega_0+\omega)} & 0 & 0 & \dfrac{S_1^*}{j(3\omega_0+\omega)} & 0 \\ \dfrac{S_3^*}{j(\omega_0+\omega)} & \dfrac{S_1^*}{j(-\omega_0+\omega)} & 0 & 0 & 0 & \dfrac{S_1}{j(-3\omega_0+\omega)} \\ \dfrac{S_2}{j(\omega_0+\omega)} & 0 & \dfrac{S_1}{j(2\omega_0+\omega)} & 0 & 0 & 0 \\ 0 & \dfrac{S_2^*}{j(-\omega_0+\omega)} & 0 & \dfrac{S_1^*}{j(-2\omega_0+\omega)} & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\alpha 1} \\ I_{\beta 1} \\ I_{\alpha 2} \\ I_{\beta 2} \\ I_{\alpha 3} \\ I_{\beta 3} \end{bmatrix}. \tag{64}$$

If we now assume that

$$\frac{\omega}{\omega_0} \ll 1 \tag{65}$$

and also that input, output, and idler circuits are tuned and that[5]

$$\varphi_{V1} = 0, \tag{66}$$

$$\varphi_{I1} = 0, \tag{67}$$

$$\varphi_{V2} = 0, \tag{68}$$

$$\varphi_{I2} = 0, \tag{69}$$

$$\varphi_{V3} = \pi, \tag{70}$$

and

$$\varphi_{I3} = \pi, \tag{71}$$

we can show that the impedance matrix $(Z_{a-p})_3$ does not exist for a tripler.‡ It is also shown in Appendix A that

$$| S_3 | = | S_1 |/2. \tag{72}$$

Equations (64)-(72) show that§

$$\frac{I_{a1}}{I_{a3}} = -\frac{2}{3}, \tag{73}$$

$$\frac{V_{a1}}{V_{a3}} = \frac{3}{2}, \tag{74}$$

$$\frac{I_{p1}}{I_{p3}} = -\frac{2}{9}, \tag{75}$$

and

$$\frac{V_{p1}}{V_{p3}} = \frac{1}{2}. \tag{76}$$

Equations (73) through (76) show that even though the amplitude-phase impedance matrix may not exist for a pumped nonlinear system (such as

‡ The termination at the idler port just tunes out the average elastance of the varactor diode at the carrier frequency $2\omega_0$.

§ We note that a tripler behaves like an ideal transformer of ratio 3/2 to the amplitude components of the fluctuations. Higher order terms in ω/ω_0 are assumed to be negligible, even when first-order terms vanish. The frequency-dependence usually introduced by the external idler termination, therefore, does not appear in the scattering matrix of the tripler.

a tripler) we are able to find a relation between the different terminal variables. These relations are sufficient to obtain the scattering parameters of the network.[13] We immediately note from (73) through (76), that there is no amplitude-to-phase or phase-to-amplitude conversion in a tripler. Accordingly,

$$(\underline{S}_{ap})_3 = \underline{0} \tag{77}$$

and

$$(\underline{S}_{pa})_3 = \underline{0}. \tag{78}$$

It is shown in Appendix A that

$$\frac{|V_1|}{|V_3|} = \frac{3}{2} \tag{79}$$

and

$$\frac{|I_1|}{|I_3|} = \frac{2}{3}. \tag{80}$$

From (34), (35), (73) through (76), (79), and (80), we can show that

$$(\underline{S}_{aa})_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{81}$$

and

$$(\underline{S}_{pp})_3 = \begin{bmatrix} 0 & 3^{-1} \\ 3 & 0 \end{bmatrix}. \tag{82}$$

Equation (81) could have been written down by noting that a tripler behaves like an ideal transformer of ratio $\frac{3}{2}$ to the amplitude components of its fluctuations.

The scattering parameters of a tripler, whose input, output, and idler circuits are tuned, are therefore, given by

$$(\underline{S})_3 = \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 3^{-1} \\ & & 3 & 0 \end{array}\right]. \tag{83}$$

In order to obtain scattering parameters of multipliers of higher order with the least number of idlers we shall show that a multiplier of order 2^n is equivalent to a cascade of n doublers, a multiplier of order 3^s is equivalent to a cascade of s triplers, and that a multiplier of order $2^n 3^s$ is equivalent to a cascade of n doublers and s triplers.

VI. EQUIVALENCE OF A MULTIPLIER OF ORDER $N = N_1 \times N_2$ TO A CASCADE OF TWO MULTIPLIERS OF ORDER N_1 AND N_2‡

In this section it is shown that, if the idler configuration of a multiplier of order $N = N_1 \times N_2$ satisfies certain conditions, then the multiplier can be represented as a cascade connection of two multipliers of order N_1 and N_2. In this and in the following two sections, no restriction is placed on the type of input, output, and idler circuits. Therefore the following discussion also applies to the case of a multiplier which is lossy and not tuned.

Consider an abrupt-junction varactor multiplier of order $N = N_1 \times N_2$. Let B denote the set of all positive and negative integers which are equal in magnitude to the orders of the various harmonics present in the varactor current. Furthermore, let B_1 indicate the subset of B which consists of the elements of magnitude equal to or less than N_1, and let B_2 denote the subset of B which consists of the elements of magnitude equal to or greater than N_1. In this section it will be shown that, if B satisfies the following condition:

$$B \text{ is such that, if } (r, s, h) \text{ is a subset of } B \text{ and if } r + s + h = 0,$$

$$\text{then either } (r, s, h) \subset B_1 \text{ or } (r, s, h) \subset B_2, \tag{84}$$

then the multiplier is equivalent to a cascade connection of two multipliers of order N_1 and N_2. Notice that an abrupt-junction multiplier of order $N = N_1 \times N_2$ which satisfies (84) must have an idler at the harmonic $N_1\omega_0$. In fact, for such a multiplier this idler is necessary in order to produce harmonics of order higher than $N_1\omega_0$.[5]

Consider then a multiplier of order $N_1 \times N_2$ which satisfies (84) and let it be represented by the very general equivalent circuit shown in Fig. 6. The generator $v_g(t)$ is sinusoidal and is of frequency ω_0. $Z(\omega)$, the impedance of the external multiplier circuit as seen from the nonlinear part of the capacitance of the varactor, is assumed to be finite only in the vicinity of the input, output, and idler frequencies. Since $Z(\omega)$ includes the average elastance and the series resistance of the varactor, the nonlinear capacitor of Fig. 6 has a q-v characteristic of the type: $v = Aq^2$, in which A is a constant multiplier. Consider now

‡ The results of Sections VI, VII, and VIII represent extensions of an earlier result, the equivalence demonstrated in Ref. 1 for the case $N = 2^n$.

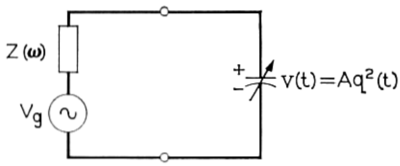

Fig. 6 — Varactor harmonic generator.

the circuit shown in Fig. 7. It will be shown that this circuit provides an alternative and complete representation of the multiplier of order $N_1 \times N_2$. The two nonlinear capacitors of Fig. 7 and that of Fig. 6 are assumed to have the same q-v characteristics. The three networks F_1, F_2, F_3 are ideal filters which have zero impedances at the carrier frequencies which satisfy respectively the relations $\omega < N_1\omega_0$, $\omega = N_1\omega_0$, $\omega > N_1\omega_0$, and also at their sidebands. Furthermore, at frequencies different from these, they have infinite impedance.

Before beginning the demonstration of the equivalence of the two circuits of Figs. 6, and 7, it may be profitable to examine briefly the behavior of the circuit of Fig. 7. The circuit of Fig. 7 represents the cascade connection of two multipliers of order N_1 and N_2. More precisely, consider the network connected on the left side of the first capacitor. For $\omega < N_1\omega_0$, it is equivalent to the network connected to the capacitor of Fig. 6. Therefore, it pumps at $\omega = \omega_0$ the first capacitor of Fig. 7 and, in addition, it provides the proper idler terminations for the generation of the harmonic $N_1\omega_0$. A current component at this harmonic is therefore generated by the first capacitor and it flows in the second loop shown in Fig. 7. The second capacitor is thus pumped at $\omega = N_1\omega_0$ by this current. Note that the network connected to its right provides the proper idler terminations for the generation of the

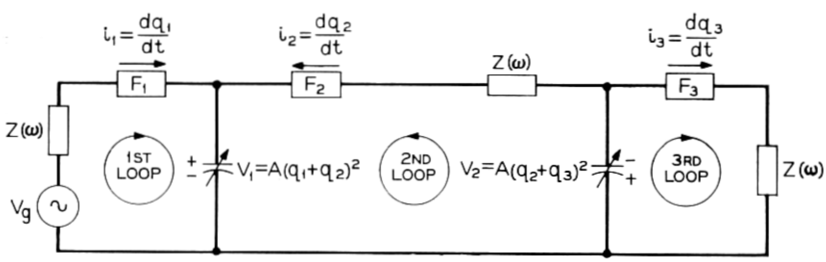

Fig. 7 — Equivalence of a multiplier of order N_1N_2 to a cascade of multipliers of orders N_1 and N_2.

output harmonic $N_1N_2\omega_0$. Therefore, the second capacitor delivers power at this harmonic to the network on its right.

Proof: First, consider the circuit of Fig. 6. The nonlinear capacitor has a q-v characteristic of the type: $v = Aq^2$. Thus, V_{-h}, the complex amplitude of $v(t)$ at the frequency $\omega = -h\omega_0$, is related to the various complex amplitudes of $q(t)$ through the relation

$$V_{-h} = A \sum_{\substack{(r,s)\subset B \\ r+s+h=0}} Q_r Q_s . \tag{85}$$

By introducing the constraint given by the linear circuit at $\omega = -h\omega_0$, one obtains

$$V_{g,-h} + jh\omega_0 Z(\omega) Q_{-h} = A \sum_{\substack{(r,s)\subset B \\ r+s+h=0}} Q_r Q_s , \qquad h \,\varepsilon\, B, \tag{86}$$

where $V_{g,-h}$ is the complex amplitude of $v_g(t)$, and is zero for $|h| \neq 1$.

Relations (86) give the equilibrium equations of the circuit of Fig. 6 and they determine the various charge amplitudes Q_1, Q_2, etc. Notice that in the summation of the righthand side of (86) one has $r+s+h=0$ and $(r, s, h) \subset B$. Therefore, from (84) one obtains the following three cases: if $|h| < N_1$, then $(r, s, h) \subset B_1$; if $|h| = N_1$, then, depending on the values of r, s, either $(r, s, h) \subset B_1$ or $(r, s, h) \subset B_2$; if $|h| > N_1$, then $(r, s, h) \subset B_2$. Accordingly, (86) can be written as

$$V_{g,-h} + jh\omega_0 Z Q_{-h} = A \sum_{\substack{(r,s)\subset B_1 \\ r+s+h=0}} Q_r Q_s , \quad \text{if} \quad |h| < N_1 . \tag{87}$$

$$jh\omega_0 Z Q_{-h} = A \{ \sum_{\substack{(r,s)\subset B_1 \\ r+s+h=0}} Q_r Q_s + \sum_{\substack{(r,s)\subset B_2 \\ r+s+h=0}} Q_r Q_s \}, \quad \text{if} \quad |h| = N_1 \tag{88}$$

$$jh\omega_0 Z Q_{-h} = A \sum_{\substack{(r,s)\subset B_2 \\ r+s+h=0}} Q_r Q_s , \quad \text{if} \quad |h| > N_1 . \tag{89}$$

Let now the circuit of Fig. 7 be examined. Consider the charges $q_1(t)$, $q_2(t)$ and $q_3(t)$ flowing through the three filters F_1, F_2, and F_3. Notice that $q_1(t) + q_2(t)$ is the total charge of the first capacitor, and that $q_2(t) + q_3(t)$ is the total charge of the second capacitor. Because of the characteristics of the three filters F_1, F_2, F_3, $q_1(t) + q_2(t)$ contains (all and) only the frequencies $r\omega_0$ for which $r \,\varepsilon\, B_1$. Similarly, $q_2(t)+q_3(t)$ contains only the frequencies for which $r \,\varepsilon\, B_2$. Now consider the total charge

$$q'(t) = q_1(t) + q_2(t) + q_3(t) \tag{90}$$

and let the symbol $(\ )'$ distinguish the complex amplitudes of $q'(t)$ from those of $q(t)$. It will be shown that $q(t) = q'(t)$; more precisely, it will be shown that the two circuits of Figs. 6 and 7 have the same equilibrium equations at the carriers.

First notice that, for $\omega = -h\omega_0$, the complex amplitude of the voltage of the first varactor of Fig. 7 is

$$A \sum_{\substack{(r,s) \subset B_1 \\ r+s+h=0}} Q'_r Q'_s \tag{91}$$

and that of the second varactor is

$$A \sum_{\substack{(r,s) \subset B_2 \\ r+s+h=0}} Q'_r Q'_s \, . \tag{92}$$

Next, notice that the equilibrium equations of the circuit of Fig. 7 for $|\omega| < N_1\omega_0$ are obtained by applying Kirchhoff's law to the first loop of Fig. 7. Similarly, for $|\omega| = N_1\omega_0$, one considers the second loop and, for $|\omega| > N_1\omega_0$, one considers the third loop. Therefore, by taking into account (91) and (92), one obtains that the equilibrium equations of the circuit of Fig. 7 are given by (87), (88), and (89), with Q_r, Q_s replaced by Q'_r, Q'_s. Therefore, $q'(t) = q(t)$.

The preceding demonstration has shown that the two circuits of Figs. 6 and 7 are equivalent at the carrier frequencies. At the various sideband frequencies, the equivalence is demonstrated in very much the same way. Since the elastance coefficients of the two circuits are equal, one finds that the sets of small-signal equations of the two circuits are equal.

VII. DISCUSSION OF THE TWO PARTICULAR CASES $N = N_1 \times 2$ AND $N = N_1 \times 3$

In this section the two particular cases $N_2 = 2$ and $N_2 = 3$ will be examined. More precisely, it will be shown that in these two cases condition (84) becomes:

If $N_2 = 2$, the two highest harmonics present in the varactor current are $N_1\omega_0$ and $2N_1\omega_0$. (93)

If $N_2 = 3$, the three highest harmonics are $N_1\omega_0$, $2N_1\omega_0$, $3N_1\omega_0$. (94)

The demonstrations are very much the same in the two cases and therefore, only the case $N_2 = 2$ will be considered.

Proof: Consider the case $N_2 = 2$ and suppose that (93) is satisfied. Then, consider the three sets B, B_1, B_2 defined in the preceding section. From (93) one has $B_2 = (-2N_1, -N_1, N_1, 2N_1)$.

Now, consider a subset (r, s, h) of B and suppose that $r + s + h = 0$.

First, notice that (r, s, h) cannot belong to both B_1 and B_2 because this would give $|r| = |s| = |h| = N_1$, which violates the hypothesis $r + s + h = 0$. It is therefore, sufficient to prove that if (r, s, h) does not belong to B_1, then it must belong to B_2.

Suppose therefore, that (r, s, h) does not belong to B_1. Then one of the three elements r, s, h has magnitude equal to $2N_1$ and, since $r + s + h = 0$, the remaining two elements have magnitude equal to N_1. Therefore, $(r, s, h) \subset B_2$.

The conclusion is that, if $(r, s, h) \subset B$ and $r + s + h = 0$, then either $(r, s, h) \subset B_1$ or $(r, s, h) \subset B_2$. This concludes the demonstration.

VIII. EQUIVALENCE OF A MULTIPLIER OF ORDER 2^n3^s TO A CHAIN OF DOUBLERS AND TRIPLERS

Consider an abrupt-junction varactor multiplier of order $N = 2^n3^s$ which has the least number of idlers. In this section it will be shown that this multiplier is equivalent to a chain of doublers and triplers. The order in which the various doublers and triplers are connected depends on the particular idler configuration. This will be clarified by the following demonstration which shows how to derive the equivalent chain of multipliers.

Proof: Since the multiplier has the least number of idlers, there are two cases:[5] either the highest idler frequency is $N\omega_0/2$, or the two highest idler frequencies are $N\omega_0/3$, $2N\omega_0/3$. In both cases, the results of the preceding sections are applicable and therefore, the multiplier can be represented by means of a cascade of two multipliers of order N_1 and N_2. Note that N_2 is 2 in the first case and 3 in the second case. Note, furthermore, that if $n + s = 2$, then either $N_1 = 2$ or $N_1 = 3$, and therefore the demonstration would end at this point.

If $n + s > 2$, on the other hand, then $N_1 > 3$ and the decomposition of the multiplier of order 2^n3^s into two multipliers of lower order can evidently be continued by the decomposition of the first of the two multipliers, the multiplier of order N_1. If this process is carried as far as possible, the final structure will be a chain of doublers and triplers.

It is important to point out that the results of this and the preceding section can be generalized in the following way:

> An abrupt-junction varactor multiplier of order $N = N_1 \times N_2 \times \cdots \times N_n$ which has the least number of idlers can be represented by a cascade of n multipliers of order N_1, N_2, etc., each with minimum number of idlers. (95)

In fact, if $n = 2$, then (95) follows directly from the equivalence demonstrated in Section VI because it can be shown that a multiplier which has the least number of idlers satisfies (84).

If $n > 2$, then the multiplier can be decomposed into multipliers of lower order as it has been done for the particular case $N = 2^n 3^s$.

IX. SCATTERING RELATIONS FOR HIGHER-ORDER LOSSLESS ABRUPT-JUNCTION VARACTOR FREQUENCY MULTIPLIERS

The scattering matrices of lossless abrupt-junction varactor frequency doubler and tripler are derived in Section V. Multipliers of order 2^n, 3^s, and $2^n 3^s$ with least number of idlers‡ are treated in this section.

9.1 *Multipliers of Order 2^n with Least Number of Idlers*

A lossless abrupt-junction varactor frequency multiplier of order 2^n with least number of idlers has been shown to be completely equivalent to a chain of n doublers. We shall assume in this section that the input, output, and all idler circuits are tuned, and that these idler terminations are lossless. The idlers are at frequencies $2^r \omega_0$, $1 \leqq r \leqq (n-1)$. The equivalence of a multiplier of order 2^n to a chain of doublers can be utilized in getting the scattering relations for multipliers of order 2^n when $n > 1$. The scattering relations when $n = 1$ are given in (59).

We can show that a multiplier of order 2^n with least number of idlers has the following scattering matrix:

$$(\underline{S})_{2^n} = \left[\begin{array}{cc:cc} \left\{\frac{1}{3} - \frac{(-1)^n}{3} 2^{-n}\right\} & (-1)^n 2^{-n} & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hdashline \multicolumn{2}{c:}{0} & 0 & (-1)^n \\ & & 2^n & \left\{\frac{1}{3} - \frac{(-1)^n}{3} 2^n\right\} \end{array}\right]. \quad (96)$$

9.2 *Multipliers of Order 3^s with Least Number of Idlers*

Multiplier of order 3^s with least number of idlers has the idler currents flowing at frequencies $2\omega_0$, $3\omega_0$, $6\omega_0$, $9\omega_0$, $\cdots$, $3^{i-1}\omega_0$, $3^i - 3^{i-1}\omega_0$, $3^i\omega_0$, $\cdots$, $3^n - 3^{n-1}\omega_0$. We have shown that such a multiplier is completely equivalent to a chain of s triplers.

‡ It is assumed that all these idler circuits are tuned and that the idler terminations are lossless. It is also assumed that input and output circuits are tuned.

The scattering parameters of a tripler are shown to be

$$(\underline{S})_3 = \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 3^{-1} \\ & & 3 & 0 \end{array}\right]. \tag{97}$$

If s such triplers are cascaded, we obtain a multiplier of order 3^s with minimum number of idlers. The scattering parameters of a cascade of s triplers can be shown to be given by‡

$$(\underline{S})_{3^s} = \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 3^{-s} \\ & & 3^s & 0 \end{array}\right]. \tag{98}$$

9.3 *Multipliers of Order 2^n3^s with Least Number of Idlers*

It has been shown that a lossless abrupt-junction harmonic generator of order 2^n3^s, n and s integers, with least number of idlers is completely equivalent to a cascade of n doublers and s triplers connected in proper order depending upon the idler configuration.

The AM and PM transfer scattering matrices[13] of a doubler and a tripler can be shown to be

$$(\underline{T}_{aa})_2 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \tag{99}$$

$$(\underline{T}_{pp})_2 = \begin{bmatrix} -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{100}$$

$$(\underline{T}_{aa})_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{101}$$

and

$$(\underline{T}_{pp})_3 = \begin{bmatrix} 3^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix}. \tag{102}$$

‡ We have assumed that input, output, and all idler circuits are tuned, and lossless.

Since AM and PM transfer scattering matrices of a tripler are scalar matrices,[14] values of $(\underline{T}_{aa})_{2^n3^s}$ and $(\underline{T}_{pp})_{2^n3^s}$ are independent of positions of the triplers in the cascaded multiplier.‡ This shows that

$$(\underline{T}_{aa})_{2^n3^s} = (\underline{T}_{aa})_{2^n} = \begin{bmatrix} (-1)^n 2^{-n} & \frac{1}{3} - \frac{(-1)^n}{3} 2^{-n} \\ 0 & 1 \end{bmatrix} \tag{103}$$

and

$$(\underline{T}_{pp})_{2^n3^s} = 3^{-s}(\underline{T}_{pp})_{2^n} = \begin{bmatrix} (-1)^n & 0 \\ \frac{(-1)^n}{3} - \frac{1}{3} 2^{-n} & 2^{-n} \end{bmatrix} 3^{-s}. \tag{104}$$

Since we also know that there is no AM to PM and PM to AM conversion in both a doubler and in a tripler it follows that

$$(\underline{S}_{ap})_{2^n3^s} = (\underline{S}_{pa})_{2^n3^s} = 0. \tag{105}$$

Using (103) through (105), we conclude that the scattering parameters of a multiplier of order 2^n3^s with minimum number of idlers are given by

$$(\underline{S})_{2^n3^s} = \left[\begin{array}{cc|cc} \frac{1}{3} - \frac{(-1)^n 2^{-n}}{3} & (-1)^n 2^{-n} & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & (-1)^n 3^{-s} \\ & & 2^n 3^s & \frac{1}{3} - \frac{(-1)^n 2^n}{3} \end{array}\right]; \tag{106}$$

or

$$(\underline{S}_{aa})_{2^n3^s} = \begin{bmatrix} \frac{1}{3} - \frac{(-1)^n}{3} 2^{-n} & (-1)^n 2^{-n} \\ 1 & 0 \end{bmatrix} \tag{107}$$

and

$$(\underline{S}_{pp})_{2^n3^s} = \begin{bmatrix} 0 & (-1)^n 3^{-s} \\ 2^n 3^s & \frac{1}{3} - \frac{(-1)^n}{3} 2^n \end{bmatrix}. \tag{108}$$

‡ Matrix product $\underline{A}\underline{B}$ is not, in general, commutative.[14]

Equations (106) and (107) show that the scattering parameters of a multiplier of order 2^n3^s are independent of the idler configuration of the multiplier. For example, 1-2-3-6-12 and 1-2-4-6-12 multipliers have the same scattering matrix. This result arises because of the scalar character of AM and PM transfer scattering matrices of a tripler and is not true in general.‡

X. RESULTS AND CONCLUSIONS

A general method to obtain the scattering parameters of a pumped nonlinear system when the system is subjected to small band limited fluctuations has been presented.

For a lossless abrupt-junction varactor frequency multiplier of order 2^n which has minimum number of idlers and whose input, output, and idler circuits are tuned, it is shown that the scattering matrix $\underset{\sim}{S}$ is given by

$$(\underset{\sim}{S})_{2^n} = \left[\begin{array}{cc|cc} \frac{1}{3} - \frac{(-1)^n}{3} 2^{-n} & (-1)^n 2^{-n} & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & (-1)^n \\ & & 2^n & \frac{1}{3} - \frac{(-1)^n}{3} 2^n \end{array}\right]. \tag{96}$$

Such a multiplier has also been shown to be completely equivalent to a cascade of n doublers.

For a lossless abrupt-junction varactor harmonic generator of order 3^s with minimum number of idlers and whose input, output, and idler circuits are all tuned it is shown that the scattering matrix $\underset{\sim}{S}$ can be represented as

$$(\underset{\sim}{S})_{3^s} = \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 3^{-s} \\ & & 3^s & 0 \end{array}\right]. \tag{98}$$

A multiplier of order 3^s has been shown to be equivalent to a cascade of s triplers.

However, for a lossless abrupt-junction varactor multiplier of order 2^n3^s with minimum number of idlers it has been shown that this multi-

‡ The transfer scattering matrix $(\underset{\sim}{T})_{N_1N_2}$ is not, in general, equal to $(\underset{\sim}{T})_{N_2N_1}$.

plier is equivalent to a cascade of n doublers and s triplers, and that the scattering matrix $\underset{\sim}{S}$ can be written as

$$(\underset{\sim}{S})_{2^n3^s} = \left[\begin{array}{cc|cc} \frac{1}{3} - \frac{(-1)^n}{3}\, 2^{-n} & (-1)^n 2^{-n} & & \\ & & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline & & 0 & (-1)^n 3^{-s} \\ \multicolumn{2}{c|}{0} & & \\ & & 2^n 3^s & \frac{1}{3} - \frac{(-1)^n}{3}\, 2^n \end{array}\right]. \tag{106}$$

For lossless abrupt-junction varactor multipliers of order 2^n, 3^s, and 2^n3^s, n and s integers, with minimum number of idlers, one of the general results is also that if $\omega/\omega_0 \ll 1$, there is no amplitude to phase or phase to amplitude conversion or equivalently

$$\underset{\sim}{S}_{ap} = \underset{\sim}{S}_{pa} = 0. \tag{109}$$

The scattering matrices, of lossless abrupt-junction varactor multipliers of order different from those treated in this paper can be obtained by straightforward application of the methods presented in this paper. We, however, feel that most of the lossless abrupt-junction varactor multipliers commonly encountered in practice are covered in this paper. If the junction characteristic of the varactor diode is far from being abrupt or if the junction is overdriven, the same general methods can be applied in order to get the general scattering matrix which relates the fluctuations at different parts of the system. If the bias circuit is poorly designed so that there are currents flowing in the system at frequencies $\pm\omega$, the techniques developed in this paper are still applicable.

At present very little is known about the stability of driven systems like harmonic generators. The results derived in this paper can be made use of in studying the stability of such systems and, in particular, in obtaining the restrictions imposed by the condition of stability on the available circuit configurations. This theory also enables us to derive an expression for the output signal of a pumped nonlinear system having noise sources at several locations in the circuit. A complete analysis of the noise performance of the systems like harmonic generators can be carried out once we know the general scattering parameters of the system. All these and other related results are reserved for a future publication.

APPENDIX *A*

Large-Signal Analysis of Abrupt-Junction Varactor Doubler and Tripler

The large-signal equations of a varactor harmonic generator are given in Ref. 5. The varactor considered in this paper is a lossless varactor whose average elastance S_0 is considered to be a part of the external circuit for the sake of convenience. Let $S(t)$ be the elastance of the varactor as pumped. For an abrupt-junction varactor diode we also note[5] that

$$\frac{jk\omega_0 S_k}{I_k} = \frac{j(k-1)\omega_0 S_{k-1}}{I_{k-1}} = \cdots = \frac{j2\omega_0 S_2}{I_2} = \frac{j\omega_0 S_1}{I_1}, \quad k \text{ an integer.} \tag{110}$$

The large-signal equations for a doubler can be written as

$$V_1 = \frac{S_2 I_1^* + S_1^* I_2}{j\omega_0} \tag{111}$$

and

$$V_2 = \frac{S_1 I_1}{j2\omega_0}. \tag{112}$$

It can be shown[5] that the time origin can be chosen so that I_1, and I_2 are both real. From (110) through (112), we can now write

$$\left| \frac{V_1}{I_1} \right| = \frac{| S_2 |}{\omega_0}, \tag{113}$$

$$\left| \frac{V_2}{I_2} \right| = \frac{| S_1 |^2}{4 \, | S_2 | \, \omega_0}, \tag{114}$$

$$\left| \frac{V_1}{I_2} \right| = \frac{| S_1 |}{2\omega_0}, \tag{115}$$

and

$$\left| \frac{V_2}{I_1} \right| = \frac{| S_1 |}{2\omega_0}. \tag{116}$$

The large-signal equations for a tripler can be written as

$$V_1 = \frac{S_3 I_2^* + S_2 I_1^* + S_1^* I_2 + S_2^* I_3}{j\omega_0}, \tag{117}$$

$$V_2 = \frac{S_3 I_1^* + S_1 I_1 + S_1^* I_3}{j2\omega_0}, \tag{118}$$

and

$$V_3 = \frac{S_1 I_2 + S_2 I_1}{j3\omega_0}. \tag{119}$$

It can again be shown[5] that if we choose I_1 to be real and positive I_2 and I_3 are real. Let us assume that the idler termination is tuned and is lossless. From (118) we can write

$$| S_3 | = | S_1 |/2. \tag{120}$$

According to (110), (117), (119), and (120), we also have

$$\frac{| I_1 |}{| I_3 |} = \frac{2}{3} \tag{121}$$

and

$$\left| \frac{V_1}{V_3} \right| = \frac{3}{2}. \tag{122}$$

REFERENCES

1. Dragone, C., Phase and Amplitude Modulation in High-Efficiency Varactor Frequency Multipliers—General Scattering Properties, B.S.T.J., *46,* No. 4, April, 1967, pp. 775–796.
2. Prabhu, V. K., Stability Considerations in Lossless Varactor Frequency Multipliers, to be published.
3. Dragone, C., Phase and Amplitude Modulation in High Efficiency Varactor Frequency Multipliers of Order $N = 2^n$—Stability and Noise, B.S.T.J., *46,* No. 4, April, 1967, pp. 797–834.
4. Dragone, C., AM and PM Scattering Properties of a Lossless Multiplier of Order $N = 2^n$, Proc. IEEE, *54,* No. 12; December, 1966.
5. Penfield, Jr., P. and Rafuse, R. P., *Varactor Applications,* The M.I.T. Press, Cambridge, Mass.; 1962.
6. Prabhu, V., Noise Performance of Abrupt-Junction Varactor Frequency Multipliers, Proc. IEEE, *54,* No. 2, February, 1966, pp. 285–287.
7. Davis, J. A., The Forward-Driven Varactor Frequency Doubler, S.M. Thesis, Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Mass., 1963.
8. Burckhardt, C. B., Analysis of Varactor Frequency Multipliers for Arbitrary Capacitance Variation and Drive Level, B.S.T.J., *44,* No. 4, April, 1965, pp. 675–692.
9. Hardy, G. H. and Rogosinski, W. W., *Fourier Series,* Cambridge University Press, London, U.K., 1950.
10. Penfield, Jr., P., Circuit Theory of Periodically Driven Nonlinear Systems, Proc. IEEE, *54,* No. 2, February, 1966, pp. 266–280.
11. Prabhu, V. K., Representation of Noise Sources in Pumped Nonlinear Systems, to be published.
12. Haus, H. A. and Adler, R. B., *Circuit Theory of Linear Noisy Networks,* The Technology Press, Cambridge, Mass., and John Wiley and Sons, Inc., New York, N. Y., 1959.
13. Carlin, H. J. and Giordano, A. B., *Network Theory,* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.
14. Hohn, F. E., *Elementary Matrix Algebra,* The MacMillan Co., New York, N. Y., 1963.