Unified Matrix Theory of Lumped and Distributed Directional Couplers

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(Manuscript received September 12, 1967)

This paper presents the theory of directional couplers using matrix formulation for lumped, distributed, or any linear time-invariant black-box. The starting point is the matrix theory of uniform multiple coupled transmission lines in terms of which are brought in the concepts of characteristic impedance matrix, propagation matrix, reflection matrix, and scattering matrix. Next, we use the results obtained with the theory of multiple coupled transmission lines to derive expressions for the loading impedance and voltage ratios for distributed directional couplers. We do this using the spectral properties of the matrices. Then we generalize the concepts of characteristic impedance matrix and propagation matrix for a class of black-boxes with the aid of the A, B, C, D transmission matrix. We give conditions for the loading impedance and expressions for the voltage ratios, using the spectral theory of the transmission matrix. We discuss the physical significance of the directional coupler effect at all frequencies in a vector-matrix framework and analyse in detail some lumped directional couplers. Finally we discuss hybrid (lumped and distributed) directional couplers.

I. INTRODUCTION

The directional coupler is an important device in many transmission systems. The theory for the electrical design of certain types of distributed-parameter directional couplers is well established through the contributions of a number of researchers.[1–7]

The purpose of this paper is to present the theory of transmission-line symmetric directional couplers in matrix form and then extend that theory to arbitrary lossless reciprocal circuits. There are many notational and conceptual advantages in using a matrix formulation which give further insight for the synthesis of different types of directional couplers.

The starting point is to examine the coupled-line directional coupler as a particular application of multiple coupled transmission line theory and then to use the concepts of characteristic impedance matrix and propagation matrix to examine the whole problem, rather than one line at a time or one mode at a time. By using a matrix approach the fundamental properties of the modes become more evident and thus physical intuition and mathematical reasoning blend to give a clearer picture of the situation. For instance, when the lines are no longer identical it is the eigenvectors of the matrices which provide the information of how to extend the mode concept.

II. MULTIPLE COUPLED TRANSMISSION

2.1 *Propagation and Characteristic Impedance Matrices*

Several researchers[8–10] have analyzed the behavior of a set of multiple coupled transmission lines. For reference in subsequent sections of this paper we present a brief account of this theory. For simplicity the discussion is restricted to two identical lines operating in the TEM mode. Fig. 1 shows schematically a differential section of two lines and a ground plane. The circuit of which Fig. 1 is a differential section obeys the following vector differential equations* in the steady state

$$\frac{d\mathbf{V}}{dx} = -\mathbf{ZI}, \tag{1}$$

$$\frac{d\mathbf{I}}{dx} = -\mathbf{YV}, \tag{2}$$

where

$$\mathbf{V} = \begin{bmatrix} V_1(x) \\ V_2(x) \end{bmatrix}, \qquad \mathbf{I} = \begin{bmatrix} I_1(x) \\ I_2(x) \end{bmatrix},$$

$$\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} Y_1 + Y_m & -Y_m \\ -Y_m & Y_2 + Y_m \end{bmatrix}.$$

By solving for **I** in (1) and substituting its value in (2), or solving for **V** in (2) and substituting its value in (1) the following are obtained:

$$\frac{d^2\mathbf{V}}{dx^2} = \mathbf{ZYV}, \tag{3}$$

* The matrices **V**, **I**, **Z**, **Y** are all functions of frequency. For simplicity in the notation this dependency is not explicitly indicated.

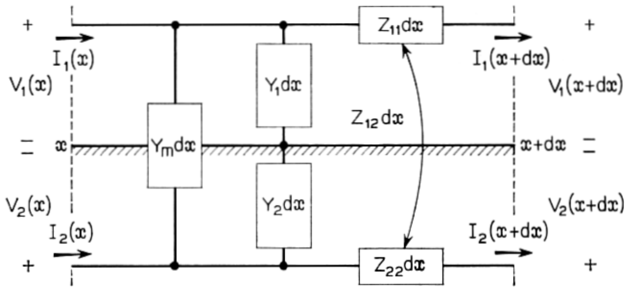

Fig. 1 — Differential section of two coupled lines above a ground plane.

$$\frac{d^2\mathbf{I}}{dx^2} = \mathbf{YZI}. \tag{4}$$

Define the matrices $\mathbf{\Gamma}$, $\mathbf{Y}_0$ and $\mathbf{Z}_0$ as follows:

$$\mathbf{\Gamma} \triangleq \sqrt{\mathbf{ZY}}, \tag{5}$$

$$\mathbf{Y}_0 \triangleq \mathbf{Z}^{-1}\mathbf{\Gamma}, \tag{6}$$

$$\mathbf{Z}_0 \triangleq \mathbf{Y}_0^{-1}. \tag{7}$$

In terms of the matrices defined, the solution of (3) and (4) is*

$$\mathbf{V}(x) = e^{-\mathbf{\Gamma}x}\mathbf{v}_+ + e^{\mathbf{\Gamma}x}\mathbf{v}_- , \tag{8}$$

$$\mathbf{I}(x) = \mathbf{Y}_0(e^{-\mathbf{\Gamma}x}\mathbf{v}_+ - e^{\mathbf{\Gamma}x}\mathbf{v}_-), \tag{9}$$

where the 2-vectors $\mathbf{v}_+$ and $\mathbf{v}_-$ are arbitrary constants (dependent on frequency) which depend on the boundary conditions. Their interpretation is very similar to the one of single line theory, $\mathbf{v}_+$ may be called the forward wave and $\mathbf{v}_-$ the reflected wave.

Equations (3), (8) and (9) have the same form as the single transmission line equations. For this reason the matrices $\mathbf{\Gamma}$, $\mathbf{Y}_0$, $\mathbf{Z}_0$, are called propagation matrix, characteristic admittance matrix, and characteristic impedance matrix, respectively. Many properties of the single transmission line hold for the multiple case. In particular, if the set of lines is terminated in a network whose open circuit impedance matrix is equal to the characteristic impedance matrix of the set of lines, a vector of incident voltage waves traveling down the lines will experience no reflection. The manner in which the voltages and currents

* In evaluating $\sqrt{\mathbf{ZY}}$ to calculate $\mathbf{\Gamma}$ the convention is made to associate with $\mathbf{\Gamma}$ eigenvalues whose real part are positive and with $-\mathbf{\Gamma}$ the ones with negative real part.

in the lines interact is best understood by examining the matrix functions $e^{-\mathbf{\Gamma} x}$ and $e^{\mathbf{\Gamma} x}$. (See Appendix.)

2.2 *Reflection and Scattering Matrices*

In single transmission line theory the voltage reflection coefficient Γ_R at a discontinuity is defined as

$$v_- = \Gamma_R v_+ , \tag{10}$$

where v_+ is the incident voltage and v_- is the reflected voltage. The expression for Γ_R in terms of the characteristic admittance Y_0 of the line and the input impedance Z_L at the discontinuity is

$$\Gamma_R = \frac{Z_L Y_0 - 1}{Z_L Y_0 + 1}. \tag{11}$$

The concept of reflection coefficient can be generalized very simply for the case of a multiple set of lines. Consider Fig. 2 depicting a pair of coupled lines of length l and matrices $\mathbf{\Gamma}$, and $\mathbf{Y}_0$ terminated at $x = 0$ in a device of open circuit impedance matrix $\mathbf{Z}_L$. According to (8) and (9) for $x = 0$

$$\mathbf{V}(0) = \mathbf{v}_+ + \mathbf{v}_- , \tag{12}$$

$$\mathbf{I}(0) = \mathbf{Y}_0(\mathbf{v}_+ - \mathbf{v}_-). \tag{13}$$

The box marked $\mathbf{Z}_L$ obeys

$$\mathbf{V}(0) = \mathbf{Z}_L \mathbf{I}(0). \tag{14}$$

Substitution of $\mathbf{V}(0)$ and $\mathbf{I}(0)$ from (12) and (13) into (14) gives

$$\mathbf{v}_+ + \mathbf{v}_- = \mathbf{Z}_L \mathbf{Y}_0(\mathbf{v}_+ - \mathbf{v}_-), \tag{14b}$$

from which solving for $\mathbf{v}_-$

$$\mathbf{v}_- = (\mathbf{Z}_L \mathbf{Y}_0 + \mathbf{I})^{-1}(\mathbf{Z}_L \mathbf{Y}_0 - \mathbf{I})\mathbf{v}_+ . \tag{15}$$

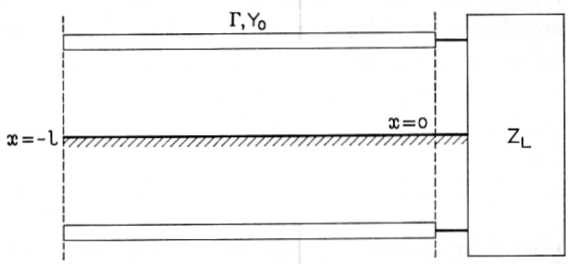

Fig. 2 — Pair of coupled lines terminated in a circuit of impedance matrix $\mathbf{Z}_L$.

Thus, defining the *reflection matrix* $\mathbf{\Gamma}_R$ as a generalization of (10) according to

$$\mathbf{v}_- = \mathbf{\Gamma}_R \mathbf{v}_+ , \tag{16}$$

it follows from (15) that

$$\mathbf{\Gamma}_R = (\mathbf{Z}_L\mathbf{Y}_0 + \mathbf{I})^{-1}(\mathbf{Z}_L\mathbf{Y}_0 - \mathbf{I}), \tag{17}$$

which is the generalization to multiple lines of (11).

The *scattering matrix* $\mathbf{S}$ is a reflection matrix in which the incident and reflected voltages are normalized.* It is defined by

$$\hat{\mathbf{v}}_- = \mathbf{S}\hat{\mathbf{v}}_+ ; \tag{18}$$

where

$$\hat{\mathbf{v}}_+ = \mathbf{Y}_0^{\frac{1}{2}}\mathbf{v}_+ , \qquad \hat{\mathbf{v}}_- = \mathbf{Y}_0^{\frac{1}{2}}\mathbf{v}_- , \tag{19}$$

$$\hat{\mathbf{V}} = \mathbf{Y}_0^{\frac{1}{2}}\mathbf{V}, \qquad \hat{\mathbf{I}} = \mathbf{Z}_0^{\frac{1}{2}}\mathbf{I}. \tag{20}$$

From (14b) it follows that

$$\mathbf{Z}_0^{\frac{1}{2}}(\hat{\mathbf{v}}_+ + \hat{\mathbf{v}}_-) = \mathbf{Z}_L\mathbf{Y}_0[\mathbf{Z}_0^{\frac{1}{2}}(\mathbf{v}_+ - \mathbf{v}_-)]. \tag{21}$$

Defining the normalized load impedance matrix $\hat{\mathbf{Z}}_L$ according to

$$\hat{\mathbf{Z}}_L = \mathbf{Y}_0^{\frac{1}{2}}\mathbf{Z}_L\mathbf{Y}_0^{\frac{1}{2}} , \tag{22}$$

equation (21) may be rearranged as follows:

$$\hat{\mathbf{v}}_- = (\hat{\mathbf{Z}}_L + \mathbf{I})^{-1}(\hat{\mathbf{Z}}_L - \mathbf{I})\hat{\mathbf{v}}_+ . \tag{23}$$

Comparison of (23) and (18) gives

$$\mathbf{S} = (\hat{\mathbf{Z}}_L + \mathbf{I})^{-1}(\hat{\mathbf{Z}}_L - \mathbf{I}). \tag{24}$$

From either (17) or (24) it is clear that if the load has an open circuit impedance matrix equal to $\mathbf{Z}_0$ of the lines the reflected wave is zero since both the reflection matrix $\mathbf{\Gamma}_R$ and the scattering matrix $\mathbf{S}$ vanish.

The complex power in terms of the normalized variables is given by

$$P = \hat{\mathbf{V}}^+\hat{\mathbf{V}} = \hat{\mathbf{I}}^+\hat{\mathbf{I}}. \tag{25}$$

(The superscript + denotes transposed conjugate of a matrix.) For a lossless device the incident power must be equal to the reflected power,

* In the literature on the scattering matrix the variables are normalized with respect to a diagonal matrix (usually a real matrix). Here the matrix may be complex and not necessarily diagonal. This derivation provides the physical interpretation for a scattering matrix normalized with respect to a complex nondiagonal matrix.

hence, from (18)

$$P = \hat{\mathbf{v}}_-^+ \hat{\mathbf{v}}_- = \hat{\mathbf{v}}_+^+ \mathbf{S}^+ \mathbf{S} \hat{\mathbf{v}}_+ = \hat{\mathbf{v}}_+^+ \hat{\mathbf{v}}_+ , \tag{26}$$

which means that

$$\mathbf{S}^+ \mathbf{S} = \mathbf{I}, \tag{27}$$

that is, the scattering matrix is unitary.

2.3 *Terminal Matrices of a Set of Lines*

Viewed from its terminal behavior the two coupled lines and ground of Fig. 3 can be studied as a four port which may be characterized by, among others, an impedance matrix $\boldsymbol{\zeta}$ or a transmission $\mathbf{E}$ matrix. (The $\mathbf{E}$ matrix is the extension to $2N$ ports of the concept of the A, B, C, D parameter matrix of two ports. See Ref. 11.)

The $\boldsymbol{\zeta}$ matrix may be written in partitioned form as follows:

$$\boldsymbol{\zeta} = \left[\begin{array}{c|c} \coth (\boldsymbol{\Gamma} \cdot l) \mathbf{Z}_0 & (\operatorname{csch} [\boldsymbol{\Gamma} \cdot l] \mathbf{Z}_0)' \\ \operatorname{csch} (\boldsymbol{\Gamma} \cdot l) \mathbf{Z}_0 & \coth (\boldsymbol{\Gamma} \cdot l) \mathbf{Z}_0 \end{array}\right]. \tag{28}$$

(The prime indicates the transpose matrix.) The $\mathbf{E}$ matrix, written in partitioned form, is

$$\mathbf{E} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \left[\begin{array}{c|c} \cosh \boldsymbol{\Gamma} \cdot l & \sinh (\boldsymbol{\Gamma} \cdot l) \mathbf{Z}_0 \\ \hline \mathbf{Y}_0 \sinh \boldsymbol{\Gamma} \cdot l & \cosh \boldsymbol{\Gamma}' l \end{array}\right]. \tag{29}$$

If the two lines in Fig. 3 are identical then $\boldsymbol{\Gamma}$ is symmetric and hence $\boldsymbol{\Gamma}' = \boldsymbol{\Gamma}$ so that in (29), $\mathbf{A} = \mathbf{D}$. Furthermore, for this case all the matrices $\mathbf{Z}_0$, $\mathbf{Y}_0$, $\boldsymbol{\Gamma}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and their analytic functions commute and therefore may be treated unambiguously as ordinary numbers. For instance (29) may be written

$$\mathbf{E} = \left[\begin{array}{c|c} \cosh \boldsymbol{\Gamma} \cdot l & \mathbf{Z}_0 \sinh \boldsymbol{\Gamma} \cdot l \\ \hline \dfrac{1}{\mathbf{Z}_0} \sinh \boldsymbol{\Gamma} \cdot l & \cosh \boldsymbol{\Gamma} l \end{array}\right]. \tag{30}$$

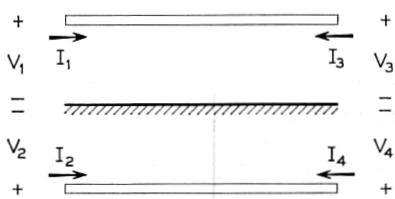

Fig. 3 — Two coupled lines above a ground plane forming a four-port.

The analogy between the **E** matrix of (30) and the A, B, C, D parameter matrix of a single transmission line considered as a two port is now complete.

III. SYMMETRICAL TRANSMISSION DIRECTIONAL COUPLERS

3.1 *Derivation of Matching Impedances*

A symmetric matched directional coupler is a four port device whose scattering matrix is of the form of **S** of (31) when the ports are adequately numbered. β and δ are, in general, frequency dependent. It is well known in microwave circuits that, given any lossless reciprocal 4-port, if all ports are matched, then the device is a directional coupler.[12] By properly numbering the ports it may be assumed that the directional coupler has the following scattering matrix

$$\mathbf{S} = \begin{bmatrix} 0 & \beta & \delta & 0 \\ \beta & 0 & 0 & \delta \\ \delta & 0 & 0 & \beta \\ 0 & \delta & \beta & 0 \end{bmatrix}. \qquad (31)$$

For the case of two identical lossless coupled lines equally loaded at the four ports all that is necessary to obtain a directional coupler is to match one of the ports. This results from the great symmetry.

Fig. 4 shows two identical lines loaded at ports 2, 3, and 4 with equal resistances, R. To calculate the driving point impedance at port 1 the following procedure can be used. Consider Fig. 5. The equation

$$\mathbf{Z} = \frac{R\mathbf{A} + \mathbf{B}}{R\mathbf{C} + \mathbf{D}} = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix} \qquad (32)$$

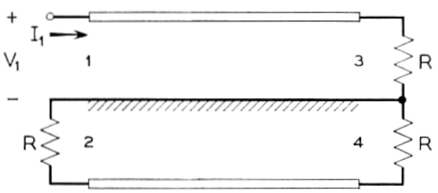

Fig. 4 — Two-terminal circuit formed by a pair of coupled lines above a ground plane loaded with resistances at three ports.

is the matrix version of the well known formula for calculating the driving point impedance at one port of a two-port when the other port is loaded. In (32) $\mathbf{Z}$ is the open circuit impedance matrix of the two-port at the left in Fig. 5, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are the matrices given by (29).

Once the $\mathbf{Z}$ of (32) is known the resistor R may be connected to port 2 as shown in Fig. 6 and the driving point impedance at port 1 calculated by a second application of (32) although now, instead of matrices, A, B, C, D are scalars. The matrices in (32) are all of the form

$$\mathbf{K} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}, \tag{33}$$

hence the methods of the appendix are applicable. Using (30) and (32)

$$\mathbf{Z} = \mathbf{R}_1 p + \mathbf{R}_2 q = \left[\begin{array}{c|c} \frac{1}{2}(p+q) & \frac{1}{2}(p-q) \\ \frac{1}{2}(p-q) & \frac{1}{2}(p+q) \end{array}\right], \tag{34}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ form the spectral set of $\mathbf{Z}$ and are

$$\mathbf{R}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad \mathbf{R}_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{35}$$

and p and q are the eigenvalues of $\mathbf{Z}$ and are

$$p = R\,\frac{\cosh \gamma^+ l + \dfrac{Z_0^+}{R}\sinh \gamma^+ l}{\dfrac{R}{Z_0^-}\sinh \gamma^+ l + \cosh \gamma^+ l}, \tag{36}$$

$$q = R\,\frac{\cosh \gamma^- l + \dfrac{Z_0^-}{R}\sinh \gamma^- l}{\dfrac{R}{Z_0^-}\sinh \gamma^- l + \cosh \gamma^- l}. \tag{37}$$

The symbols γ^+, γ^- denote the eigenvalues of $\mathbf{\Gamma}$; and Z_0^+, Z_0^- the eigenvalues of $\mathbf{Z}_0$.

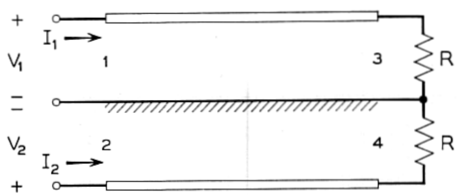

Fig. 5 — Intermediate circuit for calculating the driving point impedance at port 1 of the circuit in Fig. 4.

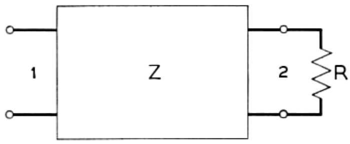

Fig. 6.—Second intermediate circuit for calculating the driving point impedance at port 1 of the circuit in Fig. 4.

To transform the $\mathbf{Z}_{ij}$ of (32) to **A**, **B**, **C**, **D** matrices the well known formulas:

$$\mathbf{A} = \mathbf{Z}_{11}\mathbf{Z}_{21}^{-1} , \tag{38}$$

$$\mathbf{B} = \mathbf{Z}_{11}\mathbf{Z}_{21}^{-1}\mathbf{Z}_{22} - \mathbf{Z}_{12} , \tag{39}$$

$$\mathbf{C} = \mathbf{Z}_{21}^{-1} , \tag{40}$$

$$\mathbf{D} = \mathbf{Z}_{21}^{-1}\mathbf{Z}_{22} , \tag{41}$$

which hold for matrices (and hence for scalars considering them as 1×1 matrices) may be used. The second application of (32) with the aid of (38)–(41) gives

$$z = \frac{(p + q)R + 2pq}{2R + p + q}. \tag{42}$$

For port 1 to be matched z must equal R, hence, the condition for directional coupler effect is

$$R = \frac{(p + q)R + 2pq}{2R + p + q} ,$$

which reduces to

$$R = \sqrt{pq}. \tag{43}$$

When the values given by (36) and (37) are substituted in (43), after some algebra, the following equation results:

$$\left(\frac{Z_0^+ Z_0^-}{R^2} - \frac{R^2}{Z_0^+ Z_0^-}\right) \sinh \gamma^+ l \sinh \gamma^- l + \left(\frac{Z_0^-}{R} - \frac{R}{Z_0^-}\right) \sinh \gamma^- l \cosh \gamma^+ l$$
$$+ \left(\frac{Z_0^+}{R} - \frac{R}{Z_0^+}\right) \sinh \gamma^+ l \cosh \gamma^- l = 0. \tag{44}$$

If a matched directional coupler at all frequencies is desired, (44) must be satisfied at all frequencies. Some possible mathematical solutions are the following:

(*i*) Make the three terms in parentheses in (44) vanish. This happens if

$$Z_0^+ = Z_0^- = R. \tag{45}$$

(*ii*) A second possibility is to make

$$\gamma^+ = \gamma^- = \gamma. \tag{46}$$

Then (44) reduces to

$$\left(\frac{Z_0^+ Z_0^-}{R^2} - \frac{R^2}{Z_0^+ Z_0^-}\right) \sinh^2 \gamma l + \left(\frac{Z_0^+}{R} + \frac{Z_0^-}{R} - \frac{R}{Z_0^+} - \frac{R}{Z_0^-}\right) \sinh \gamma l \cosh \gamma l = 0. \tag{47}$$

To make the quantities in the parentheses in (47) vanish, R is given by

$$R = \sqrt{Z_0^+ Z_0^-}\,. \tag{48}$$

Equation (45) or (46) and (48) give possible conditions for directional coupler effect at all frequencies. If a narrow band directional coupler is desired one may match the coupler at the discrete frequencies which satisfy (47). Since (47) is a transcendental equation it is not unreasonable to expect an infinity of roots. For instance, suppose (46) holds, and l is made so that

$$\sinh \gamma l = 0. \tag{49}$$

Then the device will be a directional coupler at the frequencies that are roots of (49).

Some explicit relationships for two identical lossless lines of inductance per unit length L_{11}, mutual inductance per unit length L_{12}, capacitance per unit length of one line alone C, and capacitance between the two lines per unit length C_M, are:

$$\gamma^+ = j\sqrt{(L_{11} + L_{12})C}\;\omega, \tag{50}$$

$$\gamma^- = j\sqrt{(L_{11} - L_{12})(C + 2C_M)}\;\omega, \tag{51}$$

$$Z_0^+ = \frac{(L_{11} + L_{12})}{\sqrt{(L_{11} + L_{12})C}}, \tag{52}$$

$$Z_0^- = \frac{(L_{11} - L_{12})}{\sqrt{(L_{11} - L_{12})(C + 2C_M)}}. \tag{53}$$

The condition expressed by (45) implies, equating (52) and (53)

$$-\frac{L_{11}}{L_{12}} = \frac{C + C_M}{C_M}, \tag{54}$$

$$R = \sqrt{\frac{L_{11}}{C + C_M}}. \tag{55}$$

While the condition expressed by (46) implies

$$\frac{L_{11}}{L_{12}} = \frac{C + C_M}{C_M}, \tag{56}$$

and (48) reads

$$R = \sqrt{\frac{L_{11}}{C + C_M}},$$

which is the same as (55). Equation (54) implies that the lines have negative mutual inductance, which is not achievable with parallel lines. This condition can, however, be satisfield with counter-wound lumped elements.

If (56) holds, which implies (46) then (49) implies

$$\omega_k = \frac{k\pi}{l\sqrt{L_{12}C_M - L_{11}(C + C_M)}}, \qquad k = 0, 1, 2, \cdots . \tag{57}$$

That is, for the frequencies given by (57), independent of the value of the loads (as long as they are all equal) the lines will be matched and will exhibit directional coupler effect.

3.2 *Frequency Dependency of the Coupling Between the Ports*

Equations (54) and (55) or alternatively (56) and (55) are not frequency-dependent. This means that the resulting circuit will be matched for all frequencies. Therefore, the directional coupler effect will exist for all frequencies, meaning that the coupling between uncoupled ports is zero at all frequencies. However, the coupling between coupled ports is frequency-dependent. This dependency is derived as follows.

Considering the coupled transmission lines as the load to four uncoupled lines, each of characteristic impedance R as shown in Fig. 7, the scattering matrix of the load is calculated according to (24) and (22) with

$$\mathbf{Y}_0 = \begin{bmatrix} \frac{1}{R} & 0 & 0 & 0 \\ 0 & \frac{1}{R} & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ 0 & 0 & 0 & \frac{1}{R} \end{bmatrix} \tag{58}$$

and $\mathbf{Z}_L$ given by $\boldsymbol{\zeta}$ of (28). Applying the methods of the appendix the matrix $\boldsymbol{\zeta}$ may be expressed as follows:

$$\boldsymbol{\zeta} = \mathbf{R}_a \lambda_{\zeta a} + \mathbf{R}_b \lambda_{\zeta b} + \mathbf{R}_c \lambda_{\zeta c} + \mathbf{R}_d \lambda_{\zeta d} , \tag{59}$$

where $\mathbf{R}_a$, $\mathbf{R}_b$, $\mathbf{R}_c$, $\mathbf{R}_d$ are the spectral set of $\boldsymbol{\zeta}$ and they are given by (165)–(168) of the appendix and $\lambda_{\zeta a}$, $\lambda_{\zeta b}$, $\lambda_{\zeta c}$, $\lambda_{\zeta d}$ are the eigenvalues of $\boldsymbol{\zeta}$ and are given by

$$\lambda_{\zeta a} = \coth (\gamma^+ l) Z_0^+ + \operatorname{cosch} (\gamma^+ l) Z_0^+ , \tag{60}$$

$$\lambda_{\zeta b} = \coth (\gamma^+ l) Z_0^+ - \operatorname{csch} (\gamma^+ l) Z_0^+ , \tag{61}$$

$$\lambda_{\zeta c} = \coth (\gamma^- l) Z_0^- + \operatorname{csch} (\gamma^- l) Z_0^- , \tag{62}$$

$$\lambda_{\zeta c} = \coth (\gamma^- l) Z_0^- - \operatorname{csch} (\gamma^- l) Z_0^- . \tag{63}$$

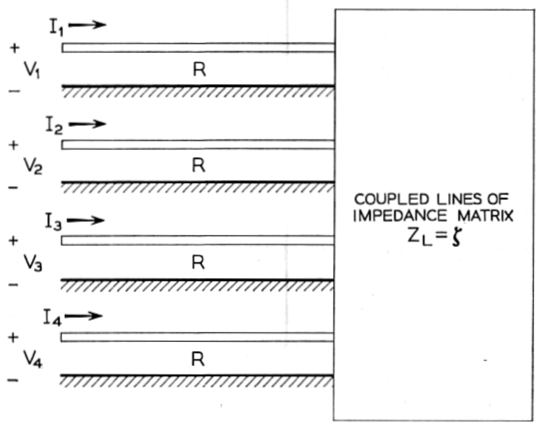

Fig. 7 — Two coupled lines above a ground plane of impedance matrix given by equation (28) serve as load to four uncoupled lines of characteristic impedance R for calculation of scattering matrix of coupled lines.

The matrix $\mathbf{S}$ may be written

$$\mathbf{S} = \mathbf{R}_a\lambda_{Sa} + \mathbf{R}_b\lambda_{Sb} + \mathbf{R}_c\lambda_{Sc} + \mathbf{R}_d\lambda_{Sd}\,, \tag{64}$$

where the eigenvalues of $\mathbf{S}$ are

$$\lambda_{Sa} = \frac{\lambda_{\zeta a} - R}{\lambda_{\zeta a} + R}\,, \tag{65}$$

$$\lambda_{Sb} = \frac{\lambda_{\zeta b} - R}{\lambda_{\zeta b} + R}\,, \tag{66}$$

$$\lambda_{Sc} = \frac{\lambda_{\zeta c} - R}{\lambda_{\zeta c} + R}\,, \tag{67}$$

$$\lambda_{Sd} = \frac{\lambda_{\zeta d} - R}{\lambda_{\zeta d} + R}. \tag{68}$$

If the values of $\lambda_{\zeta a}$, $\lambda_{\zeta b}$, $\lambda_{\zeta c}$, $\lambda_{\zeta d}$ are substituted in (65)–(68) and those are in turn substituted into (64) while the conditions indicated by (46) and (48) are imposed, the following result is obtained after some algebra

$$\mathbf{S} = \begin{bmatrix} 0 & S_{12} & S_{13} & 0 \\ S_{12} & 0 & 0 & S_{13} \\ S_{13} & 0 & 0 & S_{12} \\ 0 & S_{13} & S_{12} & 0 \end{bmatrix}; \tag{69}$$

where

$$S_{12} = \frac{\left(\sqrt{\dfrac{Z_0^+}{Z_0^-}} - \sqrt{\dfrac{Z_0^-}{Z_0^+}}\right)\sinh\gamma l}{2\cosh\gamma l + \left(\sqrt{\dfrac{Z_0^+}{Z_0^-}} + \sqrt{\dfrac{Z_0^-}{Z_0^+}}\right)\sinh\gamma l}\,, \tag{70}$$

$$S_{13} = \frac{2}{2\cosh\gamma l + \left(\sqrt{\dfrac{Z_0^+}{Z_0^-}} + \sqrt{\dfrac{Z_0^-}{Z_0^+}}\right)\sinh\gamma l}\,; \tag{71}$$

where

$$\gamma = j\omega\sqrt{(L_{11} + L_{12})C}, \tag{72}$$

$$Z_0^+ = \sqrt{\frac{L_{11} + L_{12}}{C}}\,, \tag{73}$$

$$Z_0^- = \frac{L_{11} - L_{12}}{\sqrt{C(L_{11} + L_{12})}}. \tag{74}$$

If, instead of the condition of (46), the one of (45) is imposed plus (48) then the following result is obtained

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & S_{13} & S_{14} \\ 0 & 0 & S_{14} & S_{13} \\ S_{13} & S_{14} & 0 & 0 \\ S_{14} & S_{13} & 0 & 0 \end{bmatrix}; \tag{75}$$

where

$$S_{13} = \frac{\frac{1}{2}}{\cosh \gamma^+ l + \sinh \gamma^+ l} + \frac{\frac{1}{2}}{\cosh \gamma^- l + \sinh \gamma^- l}, \tag{76}$$

$$S_{14} = \frac{\frac{1}{2}}{\cosh \gamma^+ l + \sinh \gamma^+ l} - \frac{\frac{1}{2}}{\cosh \gamma^- l + \sinh \gamma^- l}; \tag{77}$$

where

$$Z_0 = \sqrt{\frac{L_{11}}{C + C_M}}, \tag{78}$$

$$\gamma^+ = \frac{j\omega(L_{11} + L_{12})}{Z_0}, \tag{79}$$

$$\gamma^- = \frac{j(L_{11} - L_{12})\omega}{Z_0}. \tag{80}$$

The matrices of (69) and (75) with the aid of (70)–(74) and (76)–(80) give the frequency dependency of the coupling between the ports for the two types of directional couplers derived (matched at all frequencies)

IV. EXPLANATION OF THE EFFECT

Two sets of conditions have been derived for obtaining the directional coupler effect at all frequencies for transmission lines. Fig. 8 represents a pair of lossless lines c, d which are coupled from $x = -l$ to $x = 0$. The coupled lines are connected to four (uncoupled) lossless transmission lines a, b, e, f, each of characteristic impedance R and each terminated in R. The matrix $\mathbf{Z}_0$ is the characteristic impedance matrix of the set of coupled lines (c and d in Fig. 8) and $\mathbf{\Gamma}$ its propagation matrix.

Consider a case in which the eigenvalues of the matrix $\mathbf{Z}_0$ of the lines c, d satisfy

$$R = Z_0^+ = Z_0^-,$$

which implies

$$\mathbf{Z}_0 = R\mathbf{I}^*,$$

where $\mathbf{I}$ is the unit matrix. Assume that a pulse travelling down line a occupies at time t_1 the position shown in Fig. 8. Lines a and b together may be considered as a particular case of a set of multiple lines, in which both $(\mathbf{Z}_0)_{ab}$ and $(\mathbf{\Gamma})_{ab}$—the characteristic impedance and propagation matrices of lines a, b—are diagonal because lines a and b are uncoupled. Thus it is convenient to think of the pulse traveling down line a as a vector of pulses traveling down the two lines a, b; the second component of the vector which corresponds to line b being zero. When the vector of pulses arrives at the position $x = -l$, indicated by M in Fig. 8, the vector of pulses continue "seeing" the same characteristic

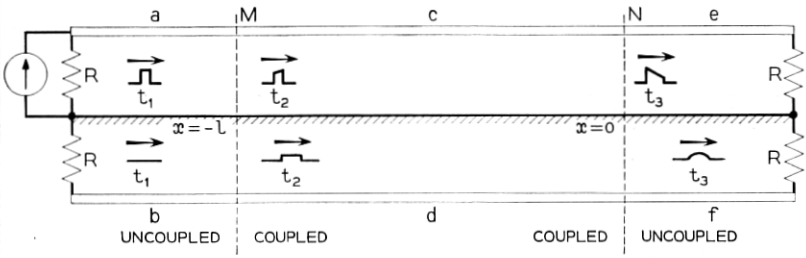

Fig. 8 — Pair of coupled lossless lines c and d above a ground plane which has uncoupled terminated lines a, b, e, f of characteristic impedances R. This shows the progress of a pulse to explain the directional coupler effect in physical terms for a coupler with diagonal characteristic impedance matrix.

impedance matrix $R\mathbf{I}$ as before and hence no reflection of the vector is created at M.

In a vector formulation one speaks of reflections in a multidimensional sense. The voltage on line a may give rise to a reflected voltage on line a [self reflection] or to a reflected voltage on line b [mutual reflection]. If all the lines are self matched and mutually matched there will be no reflections whatever. Often a line might be self matched but not mutually matched; then no self reflection will occur, but a mutual reflection will.

At time t_2 the second component of the vector pulse is no longer zero because the pulse on line c which is coupled to line d induces a pulse on line d as shown in Fig. 8. These component pulses will be

* Although for simplicity in the explanations, parallel lines are assumed, this kind of coupler requires negative mutual inductances which are in general achieved with counter-wound helices.

distorted because continually varying portions of them travel at different speeds on the lines c and d.* When they reach $x = 0$ marked by N in Fig. 8, the vector again continues to see the same characteristic impedance matrix and hence no reflections are caused at N. Finally the pulses travel out on lines e and f and are dissipated at the resistances R.

From this it is clear that the two ports at M are uncoupled and also the two ports at N are uncoupled. However a port at M is coupled to both ports at N and vice versa. All the ports are self matched. This explanation suggests a simple way of determining the conditions for a nonsymmetric coupler realized with transmission lines. Assume lines c and d are no longer identical but that the characteristic impedance matrix of the set is still diagonal

$$\mathbf{Z}_0 = \begin{bmatrix} (Z_0)_{11} & 0 \\ 0 & (Z_0)_{22} \end{bmatrix}$$

while the propagation matrix $\mathbf{\Gamma}$ is not. If the lines a and e have characteristic impedance $(Z_0)_{11}$ and lines b and f have characteristic impedance $(Z_0)_{22}$ and a, b, e, f are properly terminated, the coupled lines c, d will constitute a matched nonsymmetric directional coupler since the discussion above holds for this case without modification.

A physical account of the directional coupler satisfying

$$\gamma^+ = \gamma^- = \gamma,$$

$$R = \sqrt{\frac{L_{11}}{C + 2C_M}},$$

is as follows.

Fig. 9 shows the same arrangement as Fig. 8. A pulse traveling to the right on line a is shown at $t = t_1$. The second component of the incident pulse corresponding to line b is zero since lines a and b are uncoupled. As the vector of pulses reaches the position M, the vector of voltage is reflected according to a reflection matrix because for this case, the vector no longer sees the same characteristic impedance matrix in the transition from lines a, b to lines c, d. However, because line a matches line c there is no self reflection; only a mutual reflection appears on line b. Besides the reflected pulse, an identical transmitted pulse appears on line d at time t_2 as indicated in Fig. 9. The appearance

* The propagation matrix should not have equal eigenvalues, otherwise it will be diagonal which, together with a diagonal characteristic impedance, implies uncoupled lines.

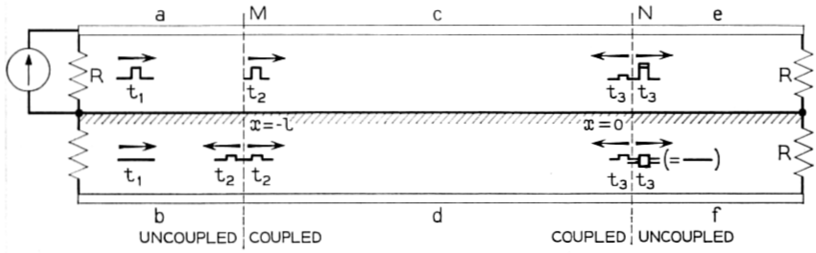

Fig. 9—Same structure as in Fig. 8 showing the progress of a pulse for a coupler with scalar propagation matrix.

of the mutual reflection indicates that the two ports at M are coupled.

After t_2, as the two forward pulses travel along lines c and d they do so at the same speed, without distortion and do not interact with each other since the propagation matrix is diagonal. As the two pulses arrive at point N they encounter a reflection matrix of opposite sign to the one they encountered at M. This means that the pulse on line c passes through N undisturbed but creates on lines d and f reflected and transmitted pulses identical to the ones created at M but of opposite sign. Likewise, the incident pulse on line d goes right through N (since each individual line is matched) creating transmitted and reflected pulses on lines e and c, but being cancelled on line f by the transmitted pulse created by the incident pulse on line c; thus, nothing comes out of line f.

At time t_3 right after the reflection at N the situation is depicted in Fig. 9. After t_3 the reflected pulses are traveling to the left at the same speed undisturbed and undistorted on lines c and d. As they arrive from the right at point M the pulse on line d goes out line b undisturbed but creating transmitted and reflected pulses on lines a and c. Likewise the pulse on line c goes out line a undisturbed creating transmitted and reflected pulses on lines b and d, but being cancelled on line a by the transmitted pulse created by the incident pulse on line d. This eliminates any delayed reflections on line a to the original incident pulse. The process continues in the same manner, the outgoing pulses on lines a and f always being such that they cancel. This means that the ports associated with lines a and f are uncoupled, but the ports of lines a, b and e are coupled.

It is clear that what is necessary for directional coupler effect on this type of coupler is: all ports self matched, equal propagation velocities without attenuation or distortion. Hence it should be possible to realize a nonsymmetrical directional coupler of this type whose propagation matrix is a scalar matrix* by self matching its ports. This may be useful for interconnecting lines a and b of different characteristic impedances.

Because the reasoning above was made in the time domain with pulses of arbitrary shape, the results hold for all frequencies. This is an example of the gain in insight owing to the vector-matrix formulation.

V. EXTENSION OF THE THEORY

5.1 *Matrices for a 2N-Port*

Let us generalize several concepts introduced in Section 2.1. Consider a $(2N + 1)$-terminal network in which terminal $2N + 1$ will be grounded and ports from terminals 1 through $2N$ to ground will be considered. Ports 1 to N will be considered input ports and Ports $N + 1$ to $2N$ output ports. Suppose the $2N$-port is characterized by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ $N \times N$ matrices. (Extensions to $2N$-ports of the A, B, C, D parameters of a two-port). Assume the circuit is such that

$$\mathbf{A} = \mathbf{D}, \tag{81}$$

$$\mathbf{A}^2 - \mathbf{BC} = \mathbf{I}, \tag{82}$$

where $\mathbf{I}$ is the $N \times N$ unit matrix and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are $N \times N$ symmetric matrices which commute.

By analogy with a multiple transmission line the characteristic impedance matrix $\mathbf{Z}_0$ and the propagation matrix $\mathbf{\Gamma}$ are defined so that they satisfy the following equations:

$$\mathbf{A} = \cosh \mathbf{\Gamma}, \tag{83}$$

$$\mathbf{B} = \sinh \mathbf{\Gamma}, \tag{84}$$

$$\mathbf{C} = \mathbf{Z}_0^{-1} \sinh \mathbf{\Gamma}. \tag{85}$$

Solving for $\mathbf{\Gamma}$ and $\mathbf{Z}_0$

$$\mathbf{\Gamma} = \cosh^{-1} \mathbf{A} = \sinh^{-1} \mathbf{B}, \tag{86}$$

$$\mathbf{Z}_0 = \sqrt{\mathbf{BC}^{-1}}. \tag{87}$$

The $N \times N$ matrices $\mathbf{\Gamma}$ and $\mathbf{Z}_0$ will also be symmetric and commute. The matrix $\mathbf{Z}_0$ is the open circuit impedance matrix of that network which, when connected to the output ports $N + 1$ through $2N$ of

* A scalar matrix is the unit matrix multiplied by a scalar.

the circuit whose **A**, **B**, **C** matrices are those of (86) and (87), will result in an open circuit impedance matrix of $\mathbf{Z}_0$ when the circuit is viewed at its ports 1 through N. This property is analogous to the one of the characteristic impedance matrices of a set of multiple coupled transmission lines. The matrix $\mathbf{\Gamma}$, has virtually the same properties of the matrix $\mathbf{\Gamma}l$ of a set of coupled lines (although the single quantity l looses its significance as a length in case the $2N$-port is a lumped circuit). For instance, if n identical $2N$-ports are cascaded the resulting $2N$-port has a propagation matrix equal to $n\mathbf{\Gamma}$.

The matrices $\mathbf{\Gamma}$ and $\mathbf{Z}_0$ may be expressed in terms of the $2N \times 2N$ impedance matrix **Z** of the $2N$-port with the aid of (38) through (41)

$$\mathbf{\Gamma} = \cosh^{-1} (\mathbf{Z}_{11}\mathbf{Z}_{21}^{-1}), \tag{88}$$

$$\mathbf{Z}_0 = \sqrt{\mathbf{Z}_{11}^2 - \mathbf{Z}_{12}^2} ; \tag{89}$$

where the **Z** matrix is partitioned as follows:

$$\mathbf{Z} = \left[\begin{array}{c|c} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \hline \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{array}\right], \tag{90}$$

the submatrices $\mathbf{Z}_{11}$, $\mathbf{Z}_{12}$, $\mathbf{Z}_{21}$, $\mathbf{Z}_{22}$ are $N \times N$ symmetric matrices and commute with each other. The characteristic impedance matrix may also be expressed in terms of the so-called open and short impedance matrices. If the N-vector $\mathbf{V}_1$ and $\mathbf{I}_1$ denote the voltages and currents at the N input ports and $\mathbf{V}_2$ and $\mathbf{I}_2$ denote the voltages and currents at the output ports, then if $\mathbf{I}_2 = \mathbf{0}$, that is, the terminals on the output ports are open then

$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 , \tag{91}$$

$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 . \tag{92}$$

Solving for $\mathbf{V}_2$ in (92) and substituting in (91)

$$\mathbf{V}_1 = \mathbf{A}\mathbf{C}^{-1}\mathbf{I}_1 . \tag{93}$$

which shows that the $N \times N$ impedance matrix $\mathbf{Z}_{0c}$ seen at the input ports is

$$\mathbf{Z}_{0c} = \mathbf{A}\mathbf{C}^{-1}. \tag{94}$$

Now, if the output ports are shorted, that is $\mathbf{V}_2 = \mathbf{0}$ then

$$\mathbf{V}_1 = \mathbf{B}\mathbf{I}_2 , \tag{95}$$

$$\mathbf{I}_1 = \mathbf{A}\mathbf{I}_2 , \tag{96}$$

from which

$$\mathbf{V}_1 = \mathbf{B}\mathbf{A}^{-1}\mathbf{I}_1 \,. \tag{97}$$

That is the $N \times N$ impedance matrix with the output terminals shorted $\mathbf{Z}_{Sc}$ is

$$\mathbf{Z}_{Sc} = \mathbf{B}\mathbf{A}^{-1}. \tag{98}$$

From Eqs. (94) and (98) it is seen that

$$\mathbf{Z}_0 = \sqrt{\mathbf{Z}_{Sc}\mathbf{Z}_{0c}} \,. \tag{99}$$

Equation (99) gives an experimental method of determining $\mathbf{Z}_0$ if it is known that the network satisfies equations (81) and (82), and the symmetry and commutativity conditions.

5.2 *Multiport Circuits and Multiple Transmission*

It is often convenient to analyze some lumped or distributed (or combinations of lumped and distributed) systems as though they were multiple transmission lines using such concepts as reflection matrix and incident voltage.

Consider the connection shown in Fig. 10. Each network is an $(2N+1)$-terminal network in which ports from each terminal to ground are made. Ports 1 through N and $1'$ through N' are considered input ports. Ports $N + 1$ through $2N$ and $(N + 1)'$ through $(2N)'$ are considered output ports. The voltage vector at the junction B whose components are the voltages of nodes $N + 1$, $N + 2$, $\cdots$, $2N$ to ground is denoted by $\mathbf{V}_B$.

The vectors $\mathbf{v}_+$, $\mathbf{v}_-$, $\mathbf{i}_+$, $\mathbf{i}_-$, called incident voltage, reflected voltage, incident current, and reflected current at the junction B (assuming the direction of propagation from left to right), are defined to satisfy

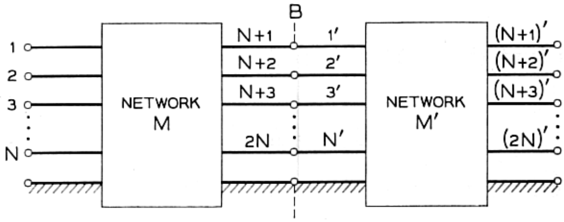

Fig. 10 — Connection of two $2N$ ports in cascade used to define the reflection matrix.

$$\mathbf{V}_B = \mathbf{v}_+ + \mathbf{v}_- , \tag{100}$$

$$\mathbf{I}_B = \mathbf{i}_+ - \mathbf{i}_- , \tag{101}$$

$$\mathbf{i}_+ = \mathbf{Z}^{-1}\mathbf{v}_+ , \tag{102}$$

$$\mathbf{i}_- = \mathbf{Z}^{-1}\mathbf{v}_- ; \tag{103}$$

where $\mathbf{I}_B$ is the N-vector whose components are the currents flowing out of terminals $N+1, N+2, \cdots, 2N$ and into terminals $1', 2', \cdots, N'$; $\mathbf{Z}$ is the $N \times N$ open-circuit impedance matrix of network M seen from the ports $N+1, N+2, \cdots, 2N$ with network M' disconnected.

The reflection matrix $\mathbf{\Gamma}_{MM'}$ is defined according to

$$\mathbf{v}_+ = \mathbf{\Gamma}_{MM'}\mathbf{v}_- . \tag{104}$$

The matrix $\mathbf{\Gamma}_{MM'}$ satisfies the following relationship:

$$\mathbf{\Gamma}_{MM'} = (\mathbf{Z}'\mathbf{Z}^{-1} + \mathbf{I})^{-1}(\mathbf{Z}'\mathbf{Z}^{-1} - \mathbf{I}), \tag{105}$$

where $\mathbf{Z}'$ is the $N \times N$ open-circuit impedance matrix of network M' as seen from ports $1', 2', \cdots, N'$ with network M disconnected. $\mathbf{I}$ is the $N \times N$ unit matrix. The indices MM' on $\mathbf{\Gamma}_{MM'}$ indicate the direction of propagation from M to M'. If the indices are reversed the roles of $\mathbf{Z}'$ and $\mathbf{Z}$ are reversed, that is

$$\mathbf{\Gamma}_{M'M} = (\mathbf{Z}(\mathbf{Z}')^{-1} + \mathbf{I})(\mathbf{Z}(\mathbf{Z}')^{-1} - \mathbf{I}). \tag{106}$$

The transmission matrix $\mathbf{T}_{MM'}$ is defined by

$$\mathbf{V}_B = \mathbf{T}_{MM'}\mathbf{v}_+ . \tag{107}$$

Hence $\mathbf{T}_{MM'}$ satisfies

$$\mathbf{T}_{MM'} = \mathbf{I} + \mathbf{\Gamma}_{MM'} . \tag{108}$$

5.3 *Directional Coupler Equations*

Taking advantage of the derivations done for transmission line directional couplers and the analogies introduced in Sections 5.1 and 5.2, it is possible to write without further work, the equations of a directional coupler having the same mathematical symmetry of a multiple transmission line directional coupler but which may have lumped components or combinations of lumped and distributed components. Suppose a four-port is characterized by its $\mathbf{E}$ matrix whose 2×2 submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ have the form of the matrix $\mathbf{K}$ of equation (1) of the Appendix and satisfy equations (81) and (82). Without any further work it can be stated that if the load impedance z and the four port satisfy for some frequency the equation

$$z = Z_0^+ = Z_0^- , \tag{109}$$

the device will be a directional coupler for that frequency.

This result is deduced from equation (45). Likewise from equation (46) it may be deduced that if the four port is such that

$$\gamma^+ = \gamma^-, \tag{110}$$

and

$$z = \sqrt{Z_0^+ Z_0^-} , \tag{111}$$

then the device will also be a directional coupler for those frequncies for which (110) and (111) are satisfied.

It is convenient at this point to exemplify with a simple lumped circuit.

Consider the four-port lumped circuit shown in Fig. 11. The **E** matrix of the circuit of Fig. 11 may be calculated by cascading 3 sections, the first and third containing only capacitors and the second containing the inductors and mutuals. By proceeding carefully much labor can be saved using the spectral sets given in the Appendix. The results are

$$\mathbf{E} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{Y} & \mathbf{I} \end{array}\right]\left[\begin{array}{c|c} \mathbf{I} & \mathbf{Z} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right]\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{Y} & \mathbf{I} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} + \mathbf{ZY} & \mathbf{Z} \\ \hline (\mathbf{YZ} + 2\mathbf{I})\mathbf{Y} & \mathbf{YZ} + \mathbf{I} \end{array}\right]; \tag{112}$$

where

$$\mathbf{Y} = S\begin{bmatrix} C + C_M & -C_M \\ -C_M & C + C_M \end{bmatrix},$$

$$\mathbf{Z} = S\begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{11} \end{bmatrix}.$$

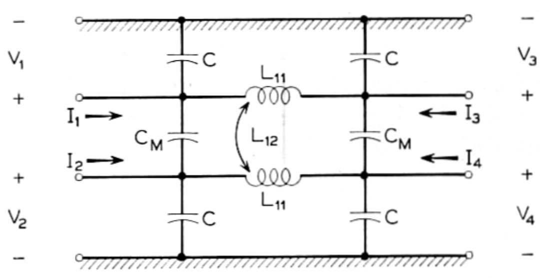

Fig. 11 — Four-port lumped circuit that can be used as a directional coupler.

From (86) the hyperbolic sine and cosine of the propagation matrix is

$$\cosh \mathbf{\Gamma} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}[1 + S^2C(L_{11} + L_{12})] + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}[1 + S^2(C + 2C_M)(L_{11} - L_{12})], \tag{113}$$

$$\sinh \mathbf{\Gamma} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} S(L_{11} + L_{12}) + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} S(L_{11} - L_{12}). \tag{114}$$

From (87) the characteristic impedance matrix is found to be

$$\mathbf{Z}_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(\frac{(L_{11} + L_{12})}{[S^2C(L_{11} + L_{12}) + 2]C}\right)^{\frac{1}{2}} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \left(\frac{(L_{11} - L_{12})}{[S^2(C + 2C_M)(L_{11} - L_{12}) + 2](C + 2C_M)}\right)^{\frac{1}{2}} \tag{115}$$

The condition expressed by (109) is, for this case,

$$z = \left(\frac{L_{11} + L_{12}}{[S^2C(L_{11} + L_{12}) + 2]C}\right)^{\frac{1}{2}} = \left(\frac{L_{11} - L_{12}}{[S^2(C + 2C_M)(L_{11} - L_{12}) + 2](C + 2C_M)}\right)^{\frac{1}{2}}. \tag{116}$$

The second and third members of (116) imply

$$[S^2(L_{11}^2 - L_{12}^2)(C + 2C_M) + 2(L_{11} + L_{12})](C + 2C_M) = [S^2C(L_{11}^2 - L_{12}^2) + 2(L_{11} - L_{12})]C. \tag{117}$$

If this condition is to be satisfied at all frequencies then

$$(L_{11}^2 - L_{12}^2)(C + 2C_M)^2 = (L_{11}^2 - L_{12}^2)C^2, \tag{118}$$

$$2(L_{11} + L_{12})(C + 2C_M) = 2(L_{11} - L_{12})C. \tag{119}$$

Both (118) and (119) are satisfied for the following choice: $L_{11} = -L_{12}$, $C = 0$. Thus the circuit of Fig. 11 with $C = 0$ and $L_{11} = -L_{12}$ (perfectly coupled counterwound inductors) is a directional coupler at all frequencies, provided it is loaded at all ports with the impedance

$$z = \left(\frac{L_{11}}{4L_{11}C_MS^2 + 2C_M}\right)^{\frac{1}{2}}. \tag{120}$$

The impedance z is frequency dependent. The voltage ratios will be given by (76) and (77),* that is

$$S_{13} = \frac{1}{2}\left(1 + \frac{1}{4C_M L_{11} S^2 + 1}\right), \tag{121}$$

$$S_{14} = \frac{1}{2}\left(1 - \frac{1}{4C_M L_{11} S^2 + 1}\right). \tag{122}$$

5.4 *Equations in Terms of* **A, B, C, D** *Matrices*

It is often convenient to express the equations of a directional coupler in terms of the **A, B, C, D** matrices directly instead of the $\mathbf{Z}_0$ and $\mathbf{\Gamma}$ matrices. For this purpose assume a lossless reciprocal four-port is characterized in terms of its **A, B, C, D** matrices which are of the form of the matrix **K** of equation (150) of the Appendix. Assume $\mathbf{A} = \mathbf{D}$. The condition $\mathbf{A}^2 - \mathbf{BC} = \mathbf{I}$ is automatically satisfied if the circuit is reciprocal. **A, B, C, D** commute, since they have the same eigenvectors. Because all matrices commute they may be treated without ambiguity as scalars. The open circuit impedance matrix is

$$\boldsymbol{\zeta} = \left[\begin{array}{c|c} \mathbf{AC}^{-1} & \mathbf{C}^{-1} \\ \hline \mathbf{C}^{-1} & \mathbf{AC}^{-1} \end{array}\right]. \tag{123}$$

Suppose the ports are loaded with equal impedance z. The impedance matrix $\boldsymbol{\zeta}$ normalized with respect to the matrix $z\mathbf{I}$ is

$$\boldsymbol{\zeta}_n = \left[\begin{array}{c|c} \mathbf{AC}^{-1}z^{-1} & \mathbf{C}^{-1}z^{-1} \\ \hline \mathbf{C}^{-1}z^{-1} & \mathbf{AC}^{-1}z^{-1} \end{array}\right]. \tag{124}$$

The eigenvalues of $\boldsymbol{\zeta}_n$ are

$$\lambda_a = \frac{A^+ + 1}{zC^+}, \tag{125}$$

$$\lambda_b = \frac{A^+ - 1}{zC^+}, \tag{126}$$

$$\lambda_c = \frac{A^- + 1}{zC^-}, \tag{127}$$

$$\lambda_d = \frac{A^- - 1}{zC^-}; \tag{128}$$

where A^+, A^-, C^+, C^- are the eigenvalues of **A** and **C** associated with the sum and difference modes. (See Appendix.) The reflection matrix

* Because for lumped elements $\mathbf{\Gamma}$ corresponds to $\mathbf{\Gamma}l$ in using the formulas derived for distributed elements for circuits with lumped elements one should take $l = 1$.

(or scattering matrix) is

$$\mathbf{\Gamma}_R = (\boldsymbol{\zeta}_n - \mathbf{I})(\boldsymbol{\zeta}_n + \mathbf{I})^{-1}, \tag{129}$$

which may be written

$$\mathbf{\Gamma}_R = \mathbf{R}_a \frac{\lambda_a - 1}{\lambda_a + 1} + \mathbf{R}_b \frac{\lambda_b - 1}{\lambda_b + 1} + \mathbf{R}_c \frac{\lambda_c - 1}{\lambda_c + 1} + \mathbf{R}_d \frac{\lambda_d - 1}{\lambda_d + 1}; \tag{130}$$

where $\mathbf{R}_a$, $\mathbf{R}_b$, $\mathbf{R}_c$, $\mathbf{R}_d$ are the members of the spectral set of $\boldsymbol{\zeta}_n$ and which are given by equations (165) to (168) of the appendix.

To find the condition for self match at port 1 (which because of the symmetry gives the condition of self match at any port) the eigenvalues given by equations (125) to (128) are substituted in (130) and the upper left corner of $\mathbf{\Gamma}_R$ is equated to zero. After some algebra this yields

$$[(A^- + zC^-)^2 - 1][(A^+)^2 - (zC^+)^2 - 1] + [(A^+ + zC^+)^2 - 1][(A^-)^2 - (zC^-)^2 - 1] = 0. \tag{131}$$

Equation (131) is a quartic in z which may be rewritten

$$z^4(C^+C^-) + z^3(C^+A^- + C^-A^+) - z(A^+B^- + B^+A^-) - B^+B^- = 0 \tag{132}$$

The solutions of Equation (118) give the values of the impedances which will match the four ports in terms of the eigenvalues of the matrices **A**, **B**, **C**. Although a quartic algebraic equation can be solved in terms of the coefficients, the solution is extremely cumbersome algebraically and it would be very difficult to see the effect of varying the quantities A^+, A^-, B^+, B^-, C^+, C^-. A sounder approach is probably to look at particular simple cases. For instance if

$$C^+ = 0 \quad \text{and} \quad B^- = 0, \tag{133}$$

Equation (132) reduces to

$$z^3(C^-A^+) - z(B^+A^-) = 0,$$

whose solutions are

$$z = 0 \quad \text{and} \quad z = \sqrt{\frac{B^+A^-}{C^-A^+}}. \tag{134}$$

A second possibility is

$$C^+ = 0, \qquad A^+ = 0. \tag{135}$$

Eq. (132) then reduces to

$$-z(B^+A^-) - B^+B^- = 0,$$

whose solution is

$$z = -\frac{B^-}{A^-}. \tag{136}$$

Obviously there are many possibilities. Some of the solutions are not immediately apparent. For instance from equations (46) and (48) if

$$A^+ = A^-, \tag{137}$$

then

$$z = \sqrt[4]{\frac{B^+B^-}{C^+C^-}} \tag{138}$$

is a root of equation (132). This fact can only be seen after a good deal of algebra, for this reason it is convenient to express (132) in different ways so that different possibilities may be "seen." With this in mind equation (129) may be written

$$\mathbf{\Gamma}_R = \mathbf{I} - 2(\boldsymbol{\zeta}_n + \mathbf{I})^{-1}. \tag{139}$$

Using the spectral set of $\boldsymbol{\zeta}_n$ the following alternative expression for the condition for the self-match of all ports is obtained

$$\frac{1}{4}\frac{2}{\dfrac{A^+ + 1}{zC^+} + 1} + \frac{1}{4}\frac{2}{\dfrac{A^+ - 1}{zC^+} + 1} + \frac{1}{4}\frac{2}{\dfrac{A^- + 1}{zC^-} + 1} + \frac{1}{4}\frac{2}{\dfrac{A^- - 1}{zC^-} + 1} = 1,$$

which after some algebra may be written

$$\frac{(A^+ + zC^+)zC^+}{(A^+ + zC^+)^2 - 1} + \frac{(A^- + zC^-)zC^-}{(A^- + zC^-)^2 - 1} = 1. \tag{140}$$

It is simpler (although not trivial) to verify that the conclusions associated with equations (137) and (138) are true from (140) than from (132).

The reflected voltages caused by an incident voltage at port 1 may be obtained from equation (130).

$$\frac{V_{2-}}{V_{1+}} = \frac{(A^+)^2 - (zC^+)^2 - 1}{2[(A^+ + zC^+)^2 - 1]} - \frac{(A^-)^2 - (zC^-)^2 - 1}{2[(A^- + zC^-)^2 - 1]}, \tag{141}$$

$$\frac{V_{3-}}{V_{1+}} = \frac{zC^+}{(A^+ + zC^+)^2 - 1} + \frac{zC^-}{(A^- + zC^-)^2 - 1}, \tag{142}$$

$$\frac{V_{4-}}{V_{1+}} = \frac{zC^+}{(A^+ + zC^+)^2 - 1} - \frac{zC^-}{(A^- + zC^-)^2 - 1}. \tag{143}$$

Equations (123) through (143) all are good for any four-port, whether lumped, distributed, or made with combinations of lumped and distributed elements.

5.5 *A Degenerate Situation:*

Consider the circuit of Fig. 12. Notice the structure is not physically symmetrical. The **A, B, C, D** matrices of the circuit are

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{Y} & \mathbf{I} \end{array}\right]\left[\begin{array}{c|c} \mathbf{I} & \mathbf{Z} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{Z} \\ \hline \mathbf{Y} & \mathbf{YZ} + \mathbf{I} \end{array}\right] \tag{144}$$

where

$$\mathbf{Z} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} 2SL_{11}, \qquad \mathbf{Y} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} 2SC_M$$

Although in equation (144) **A** and **D** are apparently not equal, it turns out that **Y** and **Z** are orthogonal and therefore $\mathbf{YZ} = 0$. Thus (144) reads

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{Z} \\ \hline \mathbf{Y} & \mathbf{I} \end{array}\right]. \tag{145}$$

The matrices **A, B, C, D** commute and satisfy equations (81) and (82). Thus, although the structure is not physically symmetrical, it is electrically symmetrical. When one attempts to use equation (87) to determine $\mathbf{Z}_0$ one finds that the matrix $\mathbf{C}^{-1}$ does not exist because $\mathbf{C} = \mathbf{Y}$ is singular. Since $\mathbf{Z}_0$ does not exist, equations (48), (70), and

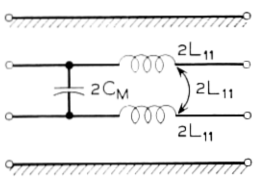

Fig. 12 — Lumped directional coupler which, when connected to a resistive load, exhibits directional coupler effect of all frequencies.

(71) cannot be used. Equation (132) can be used instead. The eigenvalues of **A**, **B**, **C** are:

$$A^+ = 1, \qquad A^- = 1$$
$$B^+ = 4SL_{11}, \qquad B^- = 0$$
$$C^+ = 0, \qquad C^- = 4SC_M$$

Thus (132) reads

$$4SC_M z^3 - 4SL_{11} z = 0,$$

the solutions of which are

$$z_1 = 0, \qquad z_2 = \sqrt{\frac{L_{11}}{C_M}}, \qquad z_3 = -\sqrt{\frac{L_{11}}{C_M}}. \tag{146}$$

The load impedances are frequency invariant, which indicates that the circuit may be matched with a constant at all frequencies and should exhibit directional coupler effect at all frequencies when loaded with a positive resistance of value $\sqrt{L_{11}/C_M}$. Using equations (141)–(143) the voltage ratios are found to be

$$\frac{V_{2-}}{V_{1+}} = \frac{2S\sqrt{L_{11}C_M}}{1 + 2S\sqrt{L_{11}C_M}}, \tag{147}$$

$$\frac{V_{3-}}{V_{1+}} = \frac{1}{1 + 2S\sqrt{L_{11}C_M}}, \tag{148}$$

$$\frac{V_{4-}}{V_{1+}} = 0. \tag{149}$$

Equation (149) corroborates that the coupler exhibits directional coupler effect at all frequencies. This example illustrates the use of the directional coupler equations in terms of the **A**, **B**, **C**, **D** matrices.

5.6 *Lumped and Distributed Elements*

The formulation that has been developed allows the handling of circuits with both lumped and distributed elements without any changes because the formulas are good for "black boxes." For example, for the circuit shown in Fig. 13, the total **E** matrix is found by multiplying the individual **E** matrices of the sections. The **E** matrix of section P or T is given by equation (145) while that of Q is given by equation (29). Once the total **E** matrix is known, it is partitioned into **A**, **B**, **C**, **D** matrices and equation (132) applied to determine the proper z for terminating the coupler. When the coupler is thus terminated, equations (141)–(143) yield the voltage ratios.

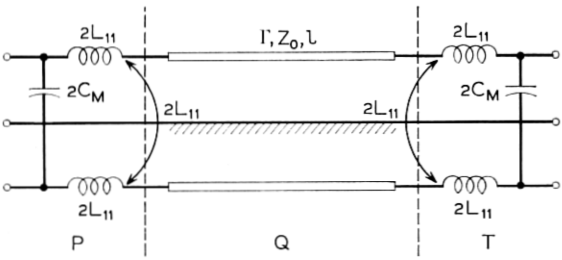

Fig. 13 — Directional coupler configuration containing both lumped and distributed elements.

In general, the algebra will get quite unmanageable and one will have to resort to numerical calculations on a digital computer at discrete frequencies.[13] The z may be found by numerically solving the quartic equation (132) at a set of discrete frequencies and then realizing it as a driving point impedance through successive approximations, or some similar procedure. The processes of normalizing the impedance z and of making frequency transformations can be used very effectively in the realization of directional couplers of this sort.

VI. CONCLUSIONS

Although strictly speaking all physical devices are distributed in space and thus, in general, have transcendental transfer functions for certain frequency regions, it might be possible to model the devices accurately enough with conventional ideal lumped elements, or more generally with elements having given frequency curves, which may be given analytically or numerically. In this paper we give a matrix theory for lumped and distributed circuits, keeping this fact in mind.

By using matrix formulation and treating the circuits as black boxes, it is possible to extend the classic theory of stripline directional couplers to more general circuits while still keeping many of the concepts (such as even and odd characteristic impedance) that have been found useful.

The paper makes evident the fact that the concepts of odd and even mode arise because of the special symmetry of the matrices and that they correspond to their eigenvectors and eigenvalues. We indicate in the appendix that when such special symmetry is lost, the odd and even modes are also lost and it might be necessary to introduce one set of modes for the currents and another for the voltages. This fact is not simple to see without the matrix formulation.

By thinking in vector-matrix terms, we explain the directional coupler effect and gain considerable insight which is useful for realizing directional couplers lacking double symmetry.

We have given general equations for analyzing and designing black-box directional couplers in terms of the characteristic impedance and propagation matrices, and in terms of the transmission **A**, **B**, **C**, **D** matrices. The latter may be necessary to analyze circuits whose characteristic impedance matrix (hence even or odd characteristic impedances) does not exist but which have a scattering matrix.

The topic of the actual design of directional couplers with lumped or with lumped and distributed elements, and more specifically the design of multisection directional couplers using computer aids, is not treated because it is the subject of a forthcoming paper.

APPENDIX

Here are some spectral properties of the principal matrices in the paper for ease of reference.[14]

The symmetric matrix

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{11} \end{bmatrix} \tag{150}$$

has eigenvalues

$$\lambda_1 = k_{11} + k_{12} \tag{151}$$

$$\lambda_2 = k_{11} - k_{12} \tag{152}$$

and normalized eigenvectors

$$\mathbf{U}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{U}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{153}$$

This can be seen by verifying the following identity

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} k_{11}+k_{12} & 0 \\ 0 & k_{11}-k_{12} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}} \equiv \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{11} \end{bmatrix} \tag{154}$$

The spectral set of **K** is:

$$\mathbf{R}_1 = \mathbf{U}_1\mathbf{U}_1' \;; \qquad \mathbf{R}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{155}$$

$$\mathbf{R}_2 = \mathbf{U}_2\mathbf{U}_2' \;; \qquad \mathbf{R}_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{156}$$

Hence an analytic function f of the matrix $\mathbf{K}$ may be written

$$f(\mathbf{K}) = \mathbf{R}_1 f(\lambda_1) + \mathbf{R}_2 f(\lambda_2) \tag{157}$$

For example

$$\cosh\left(\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{11} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{1}{2}[\cosh(k_{11}+k_{12})+\cosh(k_{11}-k_{12})] & \frac{1}{2}[\cosh(k_{11}+k_{12})-\cosh(k_{11}-k_{12})] \\ \frac{1}{2}[\cosh(k_{11}+k_{12})-\cosh(k_{11}-k_{12})] & \frac{1}{2}[\cosh(k_{11}+k_{12})+\cosh(k_{11}-k_{12})] \end{bmatrix}.$$

The quantities associated with the vector $\mathbf{U}_1$ are called the "even mode" or "sum mode." The quantities associated with the vector $\mathbf{U}_2$ are called the "odd mode" or "difference mode."

In our main paper, the eigenvalues of the matrices $\mathbf{Z}_0$ and $\mathbf{\Gamma}$ which are in the form of equation (1), are denoted by Z_0^+ and γ^+ for the sum mode and Z_0^- and γ^- for the difference mode.

The partitioned matrix

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \hline \mathbf{M}_{12} & \mathbf{M}_{11} \end{array}\right], \tag{158}$$

where $\mathbf{M}_{11}$ and $\mathbf{M}_{12}$ are the following 2×2 symmetric matrices

$$\mathbf{M}_{11} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}, \qquad \mathbf{M}_{12} = \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix}, \tag{159}$$

has the following eigenvalues

$$\lambda_a = (\alpha + \beta) + (\gamma + \delta), \tag{160}$$

$$\lambda_b = (\alpha + \beta) - (\gamma + \delta), \tag{161}$$

$$\lambda_c = (\alpha - \beta) + (\gamma - \delta), \tag{162}$$

$$\lambda_d = (\alpha - \beta) - (\gamma - \delta), \tag{163}$$

and the corresponding eigenvectors

$$\mathbf{U}_a = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{U}_b = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{U}_c = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{U}_d = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}. \tag{164}$$

This can be verified by a matrix multiplication similar to that of equation (154).

The spectral set of **M** is

$$\mathbf{R}_a = \mathbf{U}_a\mathbf{U}_a' = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tag{165}$$

$$\mathbf{R}_b = \mathbf{U}_b\mathbf{U}_b' = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \tag{166}$$

$$\mathbf{R}_c = \mathbf{U}_c\mathbf{U}_c' = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \tag{167}$$

$$\mathbf{R}_d = \mathbf{U}_d\mathbf{U}_d' = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{168}$$

Any analytic function f of **M** may be written

$$f(\mathbf{M}) = \mathbf{R}_a f(\lambda_a) + \mathbf{R}_b f(\lambda_b) + \mathbf{R}_c f(\lambda_c) + \mathbf{R}_d f(\lambda_d). \tag{169}$$

Concerning the so-called "modes of propagation" of a set of two coupled lines, when the $\mathbf{\Gamma}$ matrix of the two lines has the double symmetry exhibited by the matrix **K** of equation (1), the eigenvectors of the matrix are those given by equation (4). Since the matrix is symmetrical, the eigenvectors of the transposed matrix $\mathbf{\Gamma}'$ are the same; therefore, one may speak of the "sum mode voltages and currents" and "difference mode voltages and currents." However, it might happen that $\mathbf{\Gamma}$ does not have the form of **K** in equation (1). Then the concept of sum and difference modes disappear because the eigenvectors that will result will not be quite so simple. If he wishes, one may then speak of "first mode" and "second mode," associating each mode with each eigenvector.

If the matrix $\mathbf{\Gamma}$ is symmetrical the voltage and current modes will coincide. However, if $\mathbf{\Gamma}$ is not symmetrical then the eigenvectors of $\mathbf{\Gamma}'$ will not be the same as those of $\mathbf{\Gamma}$. It will be necessary to speak of "first voltage mode," "second voltage mode," "first current mode," and "second current mode" because the voltage modes will differ from the current modes if $\mathbf{\Gamma}$ is not symmetrical. The eigenvalues of a matrix and its transpose are always the same, hence no distinction is necessary for the propagation constants of the voltage and current modes.

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