First and Second Passage Times of Sine Wave Plus Noise

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This paper is concerned with the first and second passage times of a stationary random process, $I(t, a)$, consisting of a sinusoidal signal of amplitude $(2a)^{\frac{1}{2}}$ plus stationary Gaussian noise with a finite expected number of zeros per unit time. This type of random process is present at the output of the IF amplifier of a radio or radar receiver during the reception of a sinusoidal signal immersed in Gaussian noise. Approximate integral equations are developed whose solutions yield approximate probability densities concerning the first and second passage times of $I(t, a)$. The resulting probability functions are presented in graphs for the case when the frequency of the sine wave is located in the center of a band of noise. Related results concerning the approximate distribution function of the absolute minimum or absolute maximum of $I(t, a)$ in the closed interval $[0, \tau]$ are also presented.

I. INTRODUCTION

Exact, explicit, results concerning the first passage times of a Markov or "Markov-like" random process have been given by many authors.[1-7] But very little is known about the first passage times of a random process consisting of a sinusoidal signal plus stationary gaussian noise. This random process is of interest because it serves as a realistic model for the output of the IF amplifier of a typical radio or radar receiver during the reception of a sinusoidal signal immersed in Gaussian noise.

Let $I(t, a)$ denote the stationary random process consisting of a sinusoidal signal of amplitude $(2a)^{\frac{1}{2}}$ and angular frequency q plus stationary gaussian noise, $I_N(t)$, of zero mean and unit variance. Thus,

$$I(t, a) = (2a)^{\frac{1}{2}} \cos (qt + \theta_0) + I_N(t). \tag{1}$$

θ_0 denotes a random phase angle which is distributed uniformly in the interval $(-\pi, \pi)$. a denotes the signal-to-noise power ratio.

The first and second passage times of $I(t, a)$ are indicated in Fig. 1 and are defined as:

(*i*) τ^+ represents the time $I(t, a)$ takes in going from an upcrossing of the level I_1 to the first crossing of the level $I_2 < I_1$.
(*ii*) τ^- represents the time $I(t, a)$ takes in going from a downcrossing of the level I_1 to the first crossing of the level $I_2 < I_1$.
(*iii*) τ_1^+ represents the time $I(t, a)$ takes in going from an upcrossing of the level I_1 to the second crossing of the level $I_2 < I_1$.
(*iv*) τ_1^- represents the time $I(t, a)$ takes in going from a downcrossing of the level I_1 to the second crossing of the level $I_2 < I_1$.

For fixed I_1, $I_2 < I_1$, and a we denote the probability densities of τ^+, τ^-, τ_1^+, and τ_1^- by $W^+(\tau, I_1, I_2, a)$, $W^-(\tau, I_1, I_2, a)$, $W_1^+(\tau, I_1, I_2, a)$, and $W_1^-(\tau, I_1, I_2, a)$, respectively. These four probability densities arise in many branches of science and technology.

Because the random process $I(t, a)$ is symmetrical about its mean value of zero, we need only discuss the case when $I_1 > I_2$ as is indicated in Fig. 1. The case when $I_1 < I_2$ can always be converted to the case under discussion by considering the random process $-I(t, a)$.

For the general process $I(t, a)$, exact, explicit expressions for the four probability densities are unknown. However, as already mentioned, exact, explicit results concerning the first passage times of $I(t, a)$ are known for the special cases when $a = 0$ and $I_N(t)$ is a Markov or Markov-like random process.

The purpose of this paper is to present theoretical approximations for the four probability densities of the first and second passage times of

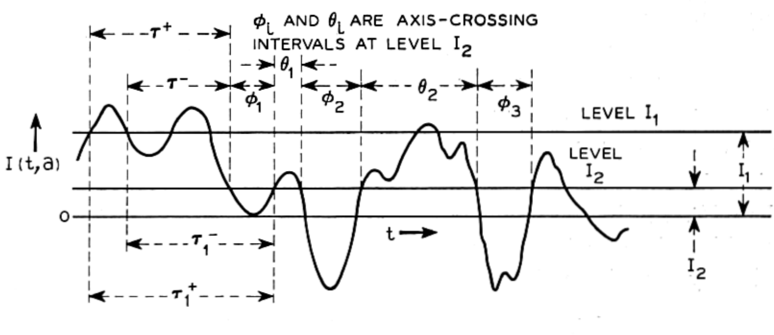

Fig. 1 — τ^+ and τ^- are the first passage times of $I(t, a) = (2a)^{1/2} \cos (qt + \theta_0) + I_N(t)$ defined by the levels I_1 and I_2. Similarly, τ_1^+ and τ_1^- are the second passage times.

$I(t, a)$ for the cases when $a \geq 0$ and $I_N(t)$ is a stationary, gaussian process having a finite expected number of zeros per unit time.

II. AUXILIARY PROBABILITY FUNCTIONS

Using a notation consistent with Ref. 8, we define the following auxiliary probability functions concerning the stationary random process $I(t, a)$:

(*i*) $P_2^{+-}(\tau, I_1, I_2, a)d\tau$, the conditional probability that a downward crossing of the level I_2 occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward crossing of the level I_1 at t.

(*ii*) $P_2^{-+}(\tau, I_1, I_2, a)d\tau$, the conditional probability that an upward crossing of the level I_2 occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward crossing of the level I_1 at t.

(*iii*) $P_2^{--}(\tau, I_1, I_2, a)d\tau$, the conditional probability that a downward crossing of the level I_2 occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward crossing of the level I_1 at t.

(*iv*) $P_2^{++}(\tau, I_1, I_2, a)d\tau$, the conditional probability that an upward crossing of the level I_2 occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward crossing of the level I_1 at t.

These auxiliary probability functions were given in Ref. 8 for the case when $I_1 = I_2$. Here, we need to merely generalize to the case when $I_1 \neq I_2$. The reader should see Rice's work for the definition of all notation which is not defined in this paper.[4] When $a \geq 0$ and $I_1 \neq I_2$, Rice's equation 38 generalizes to:

$$P_2^{+-}(\tau, I_1, I_2, a)$$
$$= -[2\pi N_{I_1}]^{-1} \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dI_1' \int_{-\infty}^{0} dI_2' I_1' I_2' \, p(I_1, I_1', I_2', I_2) \tag{2}$$

where

N_{I_1} = Rice's equation 2.7 of Ref. 9 for the expected number of up-crossings (or downcrossings) of the level I_1 per second

$$= \frac{(\beta)^{\frac{1}{2}} \exp(-I_1^2/2)}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^n (n!)^3} \left(\frac{a}{2}\right)^n$$
$$\cdot {}_1F_1\left(-n; \frac{1}{2}; \frac{I_1^2}{2}\right) {}_1F_1\left(-\frac{1}{2}; n+1; -\frac{aq^2}{\beta}\right)$$

${}_1F_1$ = the confluent hypergeometric function

$$p(I_1, I_1', I_2', I_2) = (2\pi)^{-2}M^{-\frac{1}{2}}$$

$$\cdot \exp\left\{-\frac{1}{2M}[M_{22}(I_1'^2 + I_2'^2) + 2M_{22}r_1I_1'I_2' + 2D_2I_1' + 2E_2I_2' + F_2]\right\}$$

$$r_1 = \frac{M_{23}}{M_{22}} \qquad Q = (2a)^{\frac{1}{2}}$$

$$D_2 = M_{12}(I_1 - Q\cos\theta) + M_{13}[Q\cos(q\tau + \theta) - I_2] + M_{22}Qq\sin\theta + M_{23}Qq\sin(q\tau + \theta)$$

$$E_2 = M_{12}[Q\cos(q\tau + \theta) - I_2] + M_{13}(I_1 - Q\cos\theta) + M_{22}Qq\sin(q\tau + \theta) + M_{23}Qq\sin\theta$$

$$\begin{aligned} F_2 = {} & M_{11}\{I_1^2 + I_2^2 - 2Q[I_1\cos\theta + I_2\cos(q\tau + \theta)] \\ & + Q^2[\cos^2\theta + \cos^2(q\tau + \theta)]\} \\ & + 2M_{12}Qq\{[I_1 - Q\cos\theta]\sin\theta + [Q\cos(q\tau + \theta) - I_2]\sin(q\tau + \theta)\} \\ & + 2M_{13}Qq\{[I_1 - Q\cos\theta]\sin(q\tau + \theta) + [Q\cos(q\tau + \theta) - I_2]\sin\theta\} \\ & + 2M_{14}[I_1 - Q\cos\theta][I_2 - Q\cos(q\tau + \theta)] \\ & + M_{22}(Qq)^2[\sin^2\theta + \sin^2(q\tau + \theta)] + 2M_{23}(Qq)^2\sin\theta\sin(q\tau + \theta). \end{aligned}$$

The M's are given in Ref. 4, Appendix I with

$$m(\tau) = \int_0^\infty W(f)\cos 2\pi f\tau\, df, \tag{3}$$

where $W(f)$ = one-sided power spectral density of $I_N(t)$. Also, $\beta = -m''(0)$. The primes denote differentiations.

Equation 2 can be put in the form:

$$P_2^{+-}(\tau, I_1, I_2, a) = [4\pi^2 N_{I_1}]^{-1}M_{22}(1 - m^2)^{-\frac{1}{2}} \cdot \int_{-\pi}^{\pi} \exp(-G_2/2M)J(r_1, h_3, k_3)\, d\theta \tag{4}$$

where

$$J(r_1, h_3, k_3) \equiv \frac{1}{2\pi(1 - r_1^2)^{\frac{1}{2}}}\int_{h_3}^\infty dx \int_{k_3}^\infty dy(x - h_3)(y - k_3)e^z$$

$$z = -\frac{x^2 + y^2 - 2r_1xy}{2(1 - r_1^2)}$$

$$h_3 = M_{22}^{-1}[1 - r_1^2]^{-1}[D_2 - r_1E_2]\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}}$$

$$k_3 = -M_{22}^{-\frac{1}{2}}[1 - r_1^2]^{-1}[E_2 - r_1 D_2]\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}}$$

$$G_2 = M_{22}^{-1}[1 - r_1^2]^{-1}[2r_1 D_2 E_2 - D_2^2 - E_2^2] + F_2 .$$

$P_2^{-+}(\tau, I_1, I_2, a)$ is obtained from equation 2 by changing the signs of the ∞'s in the limits of integration. We find that $P_2^{-+}(\tau, I_1, I_2, a)$ is equal to the right side of equation 4 with h_3, k_3 replaced by $-h_3$, $-k_3$.

$P_2^{--}(\tau, I_1, I_2, a)$ is obtained from equation 2 by changing the upper limit of integration of I_1' to $-\infty$. We find that $P_2^{--}(\tau, I_1, I_2, a)$ is equal to the right side of equation 4 with the function $J(r_1, h_3, k_3)$ replaced by the function $J_1(r_1, h_3, k_3)$, where

$$J_1(r_1, h_3, k_3) \equiv \frac{1}{2\pi(1 - r_1^2)^{\frac{1}{2}}} \int_{h_3}^{-\infty} dx \int_{k_3}^{\infty} dy\, (x - h_3)(y - k_3)e^{z}. \tag{5}$$

$P_2^{++}(\tau, I_1, I_2, a)$ is obtained from equation 2 by changing the lower limit of integration of I_2' to $+\infty$. We find that $P_2^{++}(\tau, I_1, I_2, a)$ is equal to the right side of equation 4 with the function $J(r_1, h_3, k_3)$ replaced by the function $J_1(r_1, -h_3, -k_3)$.

The functions $J(r_1, h_3, k_3)$ and $J_1(r_1, h_3, k_3)$ are expressed in terms of Karl Pearson's well-known tabulated function (d/N) in Ref. 10.

A considerable simplification occurs when $a = 0$. For this case we find

$$P_2^{+-}(\tau, I_1, I_2, 0) = M_{22}\beta^{-\frac{1}{2}}(1 - m^2)^{-\frac{3}{2}} \cdot \exp\left[\frac{I_1^2}{2} - \frac{(I_1^2 + I_2^2 - 2mI_1I_2)}{2(1 - m^2)}\right] J(r_1, h_3, k_3) \tag{6}$$

where

$$h_3 = \frac{-m'(mI_1 - I_2)}{[M_{22}(1 - m^2)]^{\frac{1}{2}}}$$

$$k_3 = \frac{m'(I_1 - mI_2)}{[M_{22}(1 - m^2)]^{\frac{1}{2}}}.$$

Equation 6 reduces to Rice's[4] equation 47 when $I_1 = I_2 = I$.

III. APPROXIMATE RESULTS VIA INTEGRAL EQUATIONS

3.1 *Probability Densities*

Let us assume that each of the random variables τ^+, τ^-, τ_1^+, and τ_1^- (see Fig. 1), is statistically independent of the sum of the following $(2N + 2)$ axis-crossing intervals at level I_2 when $N = 0, 1, 2, \cdots$. Under this "quasi-independence" assumption, approximate theoretical results for the probability densities of τ^+, τ^-, τ_1^+, and τ_1^-, namely $W^+(\tau, I_1, I_2, a)$, $W^-(\tau, I_1, I_2, a)$, $W_1^+(\tau, I_1, I_2, a)$, and $W_1^-(\tau, I_1, I_2, a)$ are given by the following integral equations:

$$P_2^{+-}(\tau, I_1, I_2, a) = W^+(\tau, I_1, I_2, a) + W^+(\tau, I_1, I_2, a) * P_2^{--}(\tau, I_2, I_2, a) \tag{7}$$

$$P_2^{--}(\tau, I_1, I_2, a) = W^-(\tau, I_1, I_2, a) + W^-(\tau, I_1, I_2, a) * P_2^{--}(\tau, I_2, I_2, a) \tag{8}$$

$$P_2^{++}(\tau, I_1, I_2, a) = W_1^+(\tau, I_1, I_2, a) + W_1^+(\tau, I_1, I_2, a) * P_2^{++}(\tau, I_2, I_2, a) \tag{9}$$

$$P_2^{-+}(\tau, I_1, I_2, a) = W_1^-(\tau, I_1, I_2, a) + W_1^-(\tau, I_1, I_2, a) * P_2^{++}(\tau, I_2, I_2, a). \tag{10}$$

The P_2's are the auxiliary probability functions presented in Section II, and the symbol * denotes the convolution operator, that is,

$$f * g \equiv \int_{-\infty}^{\infty} f(t)g(\tau - t)\, dt. \tag{11}$$

From symmetry we have

$$P_2^{++}(\tau, I_2, I_2, a) = P_2^{--}(\tau, I_2, I_2, a). \tag{12}$$

Integral equations 7 through 10 are analogous to McFadden's equation 47 and Rice's equation 84 in Refs. 11 and 4, respectively.

Let us define two additional probability densities defined by the first and second passage times of $I(t, a)$ between the levels I_1 and I_2:

(*i*) $W(\tau, I_1, I_2, a)\,d\tau$, the conditional probability that the first crossing of the level I_2 occurs between $t + \tau$ and $t + \tau + d\tau$ given a crossing of the level $I_1 > I_2$ at t.

(*ii*) $W_1(\tau, I_1, I_2, a)\,d\tau$, the conditional probability that the second crossing of the level I_2 occurs between $t + \tau$ and $t + \tau + d\tau$ given a crossing of the level $I_1 > I_2$ at t.

Clearly, we have

$$W(\tau, I_1, I_2, a) = \tfrac{1}{2}W^+(\tau, I_1, I_2, a) + \tfrac{1}{2}W^-(\tau, I_1, I_2, a) \tag{13}$$

$$W_1(\tau, I_1, I_2, a) = \tfrac{1}{2}W_1^+(\tau, I_1, I_2, a) + \tfrac{1}{2}W_1^-(\tau, I_1, I_2, a). \tag{14}$$

3.2 *Absolute Minimum In a Closed Interval*

Let us define the following probability functions concerning the absolute minimum of $I(t, a)$ in a closed interval $[0, \tau]$:

$$F^+(\tau, I_1, I_2, a) \equiv \Pr\{\min_{0 \leq t \leq \tau} I(t, a) > I_2 \mid I(0, a) = I_1 > I_2, I'(0, a) > 0\} \quad (15)$$

$$F^-(\tau, I_1, I_2, a) \equiv \Pr\{\min_{0 \leq t \leq \tau} I(t, a) > I_2 \mid I(0, a) = I_1 > I_2, I'(0, a) < 0\} \quad (16)$$

$$F(\tau, I_1, I_2, a) \equiv \Pr\{\min_{0 \leq t \leq \tau} I(t, a) > I_2 \mid I(0, a) = I_1 > I_2\} \quad (17)$$

where $\Pr\{\cdot\}$ denotes the probability of the event inside the brace. Clearly, we have

$$\begin{aligned} F^+(\tau, I_1, I_2, a) &= \int_\tau^\infty W^+(\tau, I_1, I_2, a)\, d\tau \\ &= 1 - \int_0^\tau W^+(\tau, I_1, I_2, a)\, d\tau \end{aligned} \quad (18)$$

$$\begin{aligned} F^-(\tau, I_1, I_2, a) &= \int_\tau^\infty W^-(\tau, I_1, I_2, a)\, d\tau \\ &= 1 - \int_0^\tau W^-(\tau, I_1, I_2, a)\, d\tau \end{aligned} \quad (19)$$

$$\begin{aligned} F(\tau, I_1, I_2, a) &= \int_\tau^\infty W(\tau, I_1, I_2, a)\, d\tau \\ &= 1 - \int_0^\tau W(\tau, I_1, I_2, a)\, d\tau. \end{aligned} \quad (20)$$

Because we are discussing only the case when $I_1 > I_2$ as is indicated in Fig. 1, we discuss only the probability functions concerning the absolute minimum of $I(t, a)$ in a closed interval $[0, \tau]$. The corresponding probability functions concerning the absolute maximum of $I(t, a)$ in a closed interval $[0, \tau]$ are associated with the case when $I_1 < I_2$, and they can be obtained from symmetry by considering the random process $-I(t, a)$.

IV. RESULTS FOR SINUSOIDAL SIGNAL CENTERED IN LOW-PASS NOISE

For purposes of computation we set the angular frequency, q, of the sinusoidal signal in the center of a band of gaussian noise with an ideal low-pass power spectral density of cutoff frequency f_0. Thus,

$$q = \pi f_0 \quad (21)$$

and

$$W(f) = \begin{cases} f_0^{-1} & 0 \leqq f \leqq f_0 \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Accordingly, from equation 3,

$$m(\tau) = \frac{\sin 2\pi f_0 \tau}{2\pi f_0 \tau}. \tag{23}$$

From equation 23 we see that it is convenient to define normalized time as $u_0 = 2\pi f_0 \tau$. All our results are plotted with respect to normalized time u_0.

4.1 *Experimental Verification of Auxiliary Probability Functions*

The auxiliary probability functions for the case when $I_1 = I_2 = 0$ are useful for studying the zero-crossing intervals, the axis-crossing intervals defined by the level $I_1 = I_2 = 0$, of $I(t, a)$. Figures 2, 3, and 4 present P_2^{+-}, P_2^{-+}, P_2^{++}, and P_2^{--} for the case when $I_1 = I_2 = 0$ and $a = 0, 1$, and 4. These results were computed by using Simpson's rule. The results compare satisfactorily with the initial behavior of the experimental probability densities presented in Figs. 34, 35, 42, 43, 46, and 47 of Ref. 12. Notice that the experimental probability densities pertain to the case when the power spectral density of the noise is

$$W_0(f) = \frac{1}{1 + \left(\frac{f}{f_0}\right)^{14}} \tag{24}$$

rather than the power spectral density defined by equation 22.

4.2 *Results When* a $=$ *0 and* a $=$ *1*

Figures 5 through 13 present the results when $a = 0$, signal absent, and $I_1 = 1, I_2 = 0$; $I_1 = 1, I_2 = -1$; and $I_1 = 0, I_2 = -1$. The P_2's were computed by using Simpson's rule, the integral equations defining the W's were solved numerically by using the trapezoidal rule, and the F's were computed by using Simpson's rule.

Similarly, Figs. 14 through 22 present the results when $a = 1$, signal present, and $I_1 = 1, I_2 = 0$; $I_1 = 1, I_2 = -1$; and $I_1 = 0, I_2 = -1$.

As an example of the interpretation of these results, we see from Fig. 16 that the median time, τ_m, for the random process $I(t, 1)$ to go from the level $I_1 = 1$ to the level $I_2 = 0$ for the first time is given by

$$u_0 = 2\pi f_0 \tau_m \doteq \pi \tag{25}$$

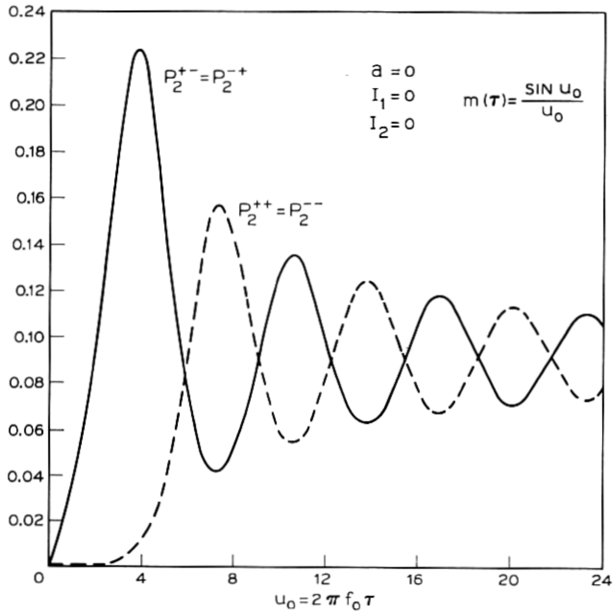

Fig. 2 (also Figs. 3 and 4) — Plots of the probability functions $P_2^{+-}(u_0, I_1, I_2, a)$, $P_2^{-+}(u_0, I_1, I_2, a)$, $P_2^{++}(u_0, I_1, I_2, a)$, and $P_2^{--}(u_0, I_1, I_2, a)$ associated with the crossings of the levels I_1 and I_2 by a stationary random process consisting of a sinusoidal signal of frequency $f_0/2$ plus stationary gaussian noise with autocorrelation function $m(\tau)$. a denotes the signal-to-noise power ratio.

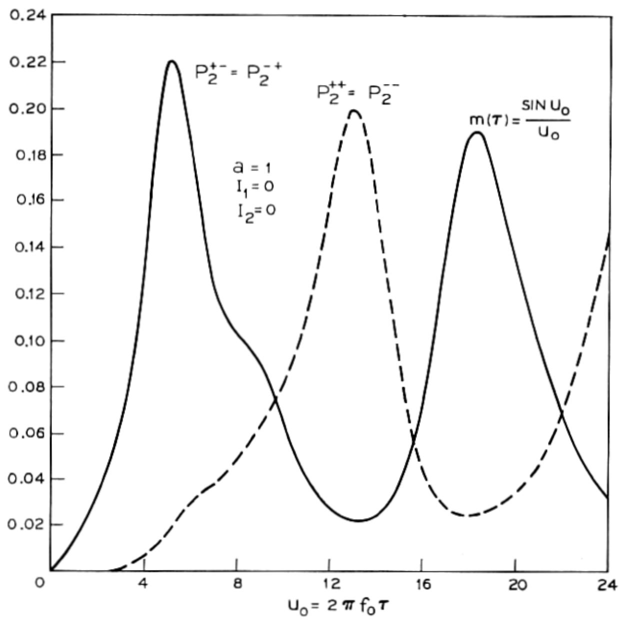

Fig. 3 — (See Fig. 2.)

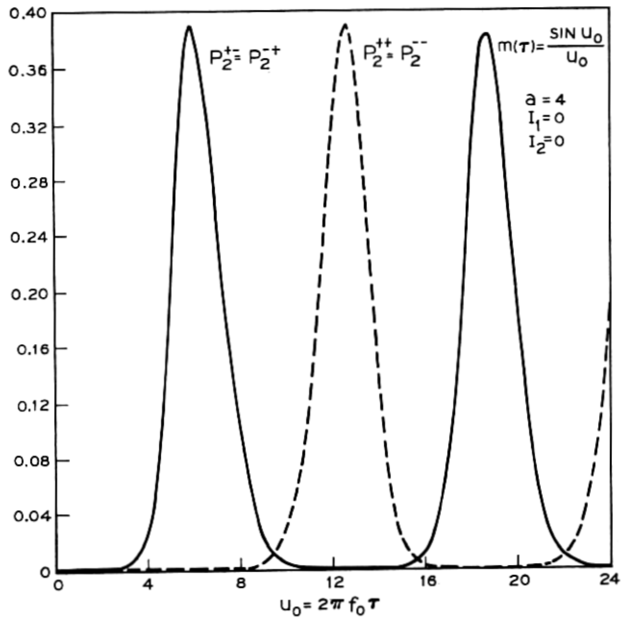

Fig. 4 — (See Fig. 2.)

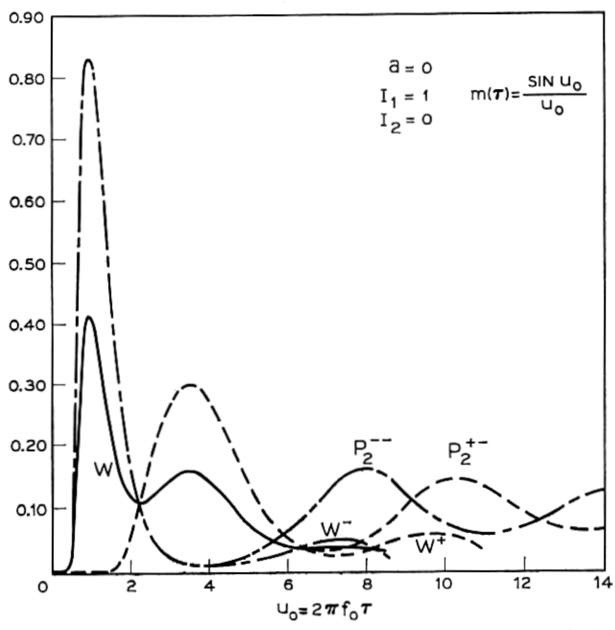

Fig. 5 (also Figs. 8, 11, 14, 17, and 20) — Plots of the probability functions $P_2^{--}(u_0, I_1, I_2, a)$, $W^-(u_0, I_1, I_2, a)$, $P_2^{+-}(u_0, I_1, I_2, a)$, $W^+(u_0, I_1, I_2, a)$, and $W(u_0, I_1, I_2, a)$ associated with the crossings of the levels I_1 and I_2 by a stationary random process consisting of a sinusoidal signal of frequency $f_0/2$ plus stationary gaussian noise with autocorrelation function $m(\tau)$. a denotes the signal-to-noise power ratio.

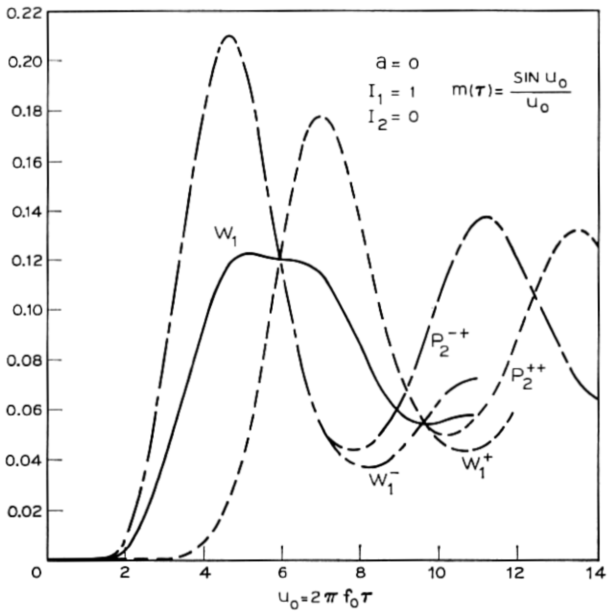

Fig. 6 (also Figs. 9, 12, 15, 18, and 21) — Plots of the probability functions $P_2^{-+}(u_0, I_1, I_2, a)$, $W_1^-(u_0, I_1, I_2, a)$, $P_2^{++}(u_0, I_1, I_2, a)$, $W_1^+(u_0, I_1, I_2, a)$ and $W_1(u_0, I_1, I_2, a)$ associated with the crossings of the levels I_1 and I_2 by a stationary random process consisting of a sinusoidal signal of frequency $f_0/2$ plus stationary gaussian noise with autocorrelation function $m(\tau)$. a denotes the signal-to-noise power ratio.

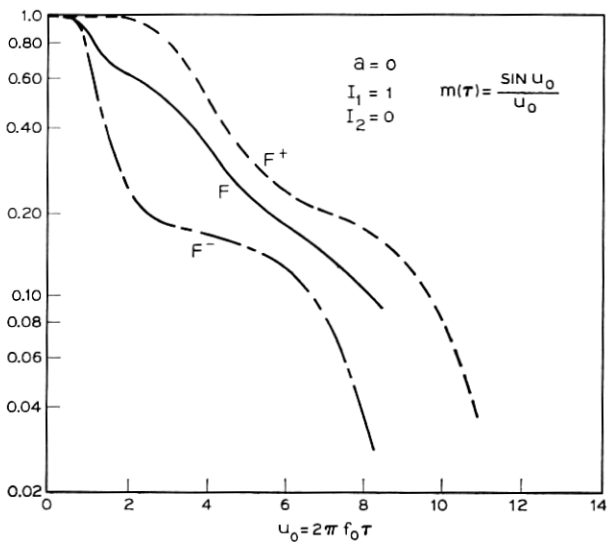

Fig. 7 (also Figs. 10, 13, 16, 19, and 22) — Plots of the probability functions $F^+(u_0, I_1, I_2, a)$, $F^-(u_0, I_1, I_2, a)$, and $F(u_0, I_1, I_2, a)$ associated with the crossings of the levels I_1 and I_2 by a stationary random process consisting of a sinusoidal signal of frequency $f_0/2$ plus stationary gaussian noise with autocorrelation function $m(\tau)$. a denotes the signal-to-noise power ratio.

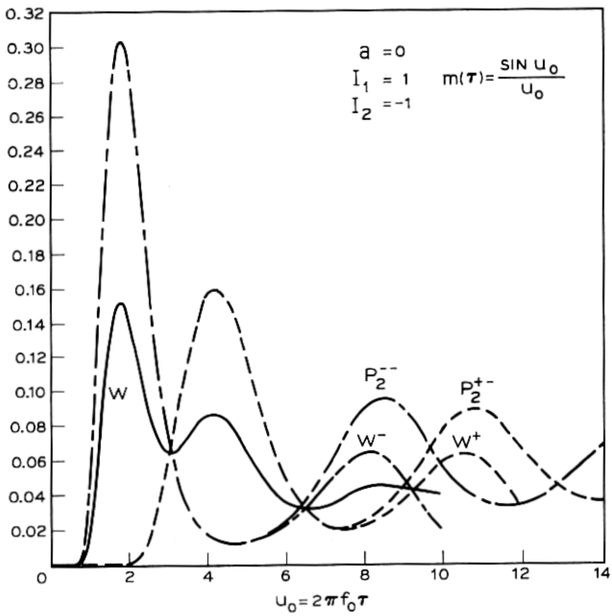

Fig. 8 — (See Fig. 5.)

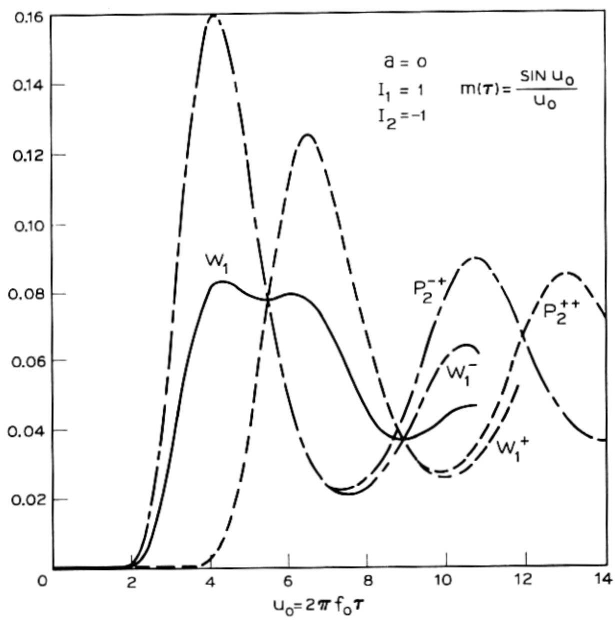

Fig. 9 — (See Fig. 6.)

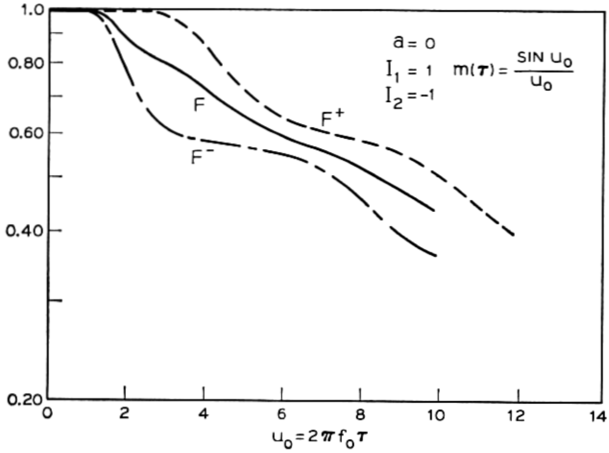

Fig. 10 — (See Fig. 7.)

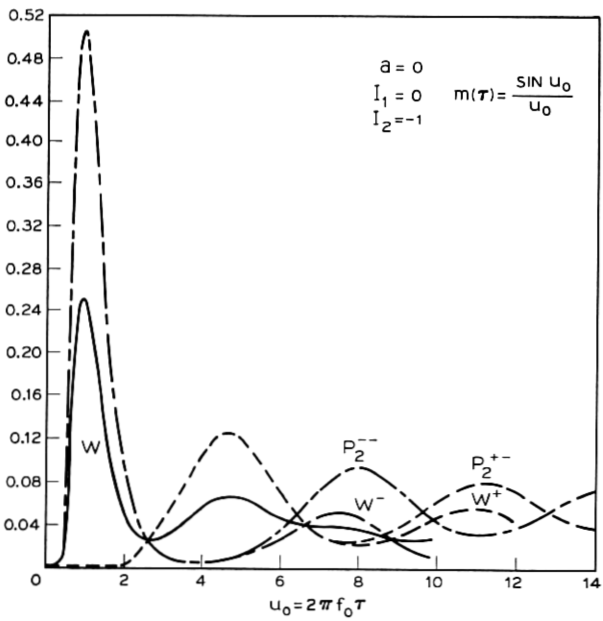

Fig. 11 — (See Fig. 5.)

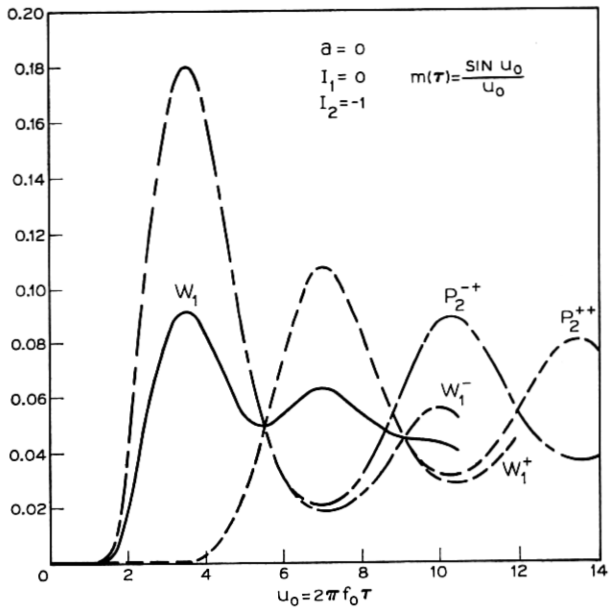

Fig. 12 — (See Fig. 6.)

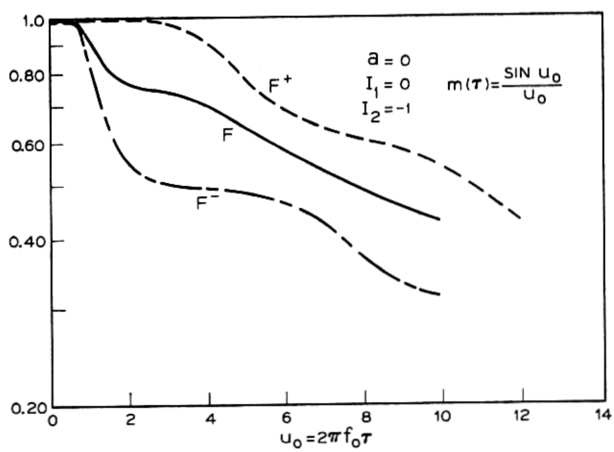

Fig. 13 — (See Fig. 7.)

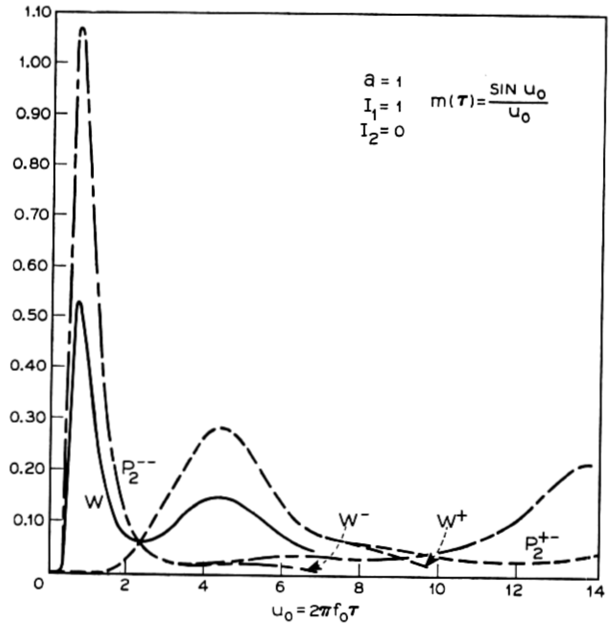

Fig. 14 — (See Fig. 5.)

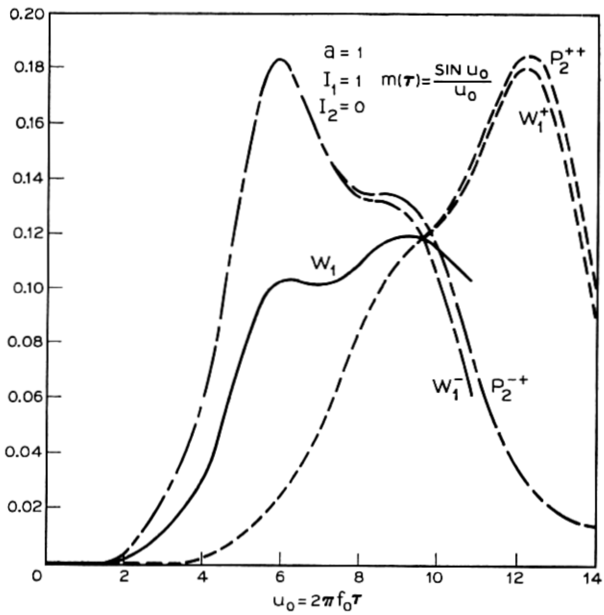

Fig. 15 — (See Fig. 6.)

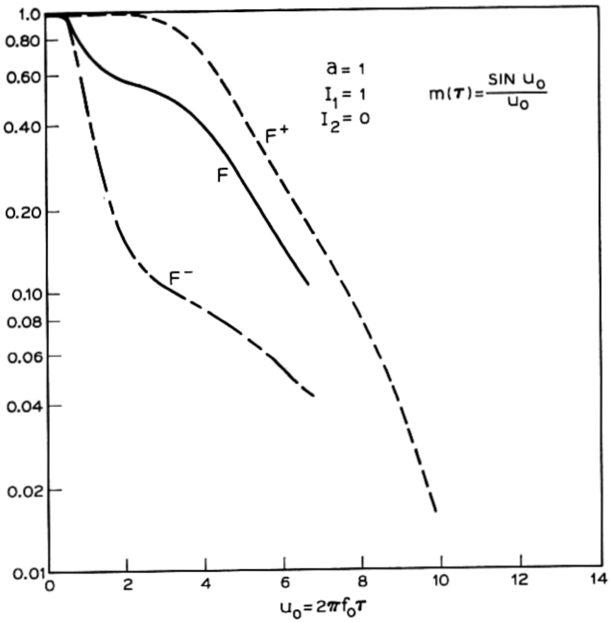

Fig. 16 — (See Fig. 7.)

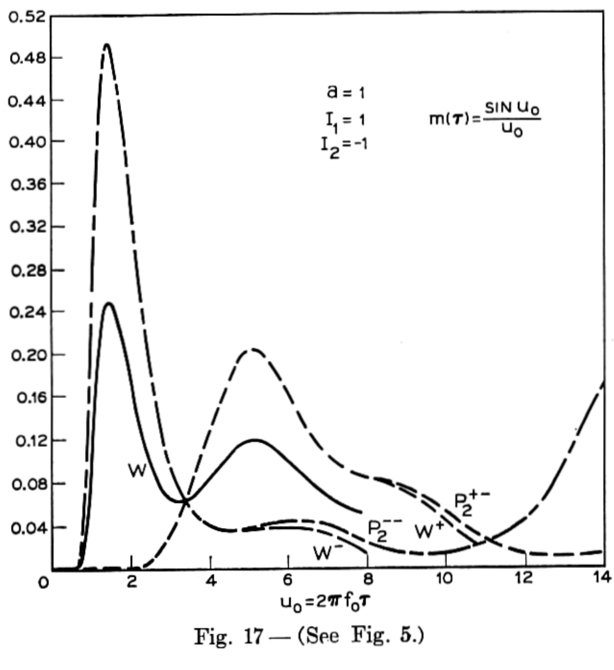

Fig. 17 — (See Fig. 5.)

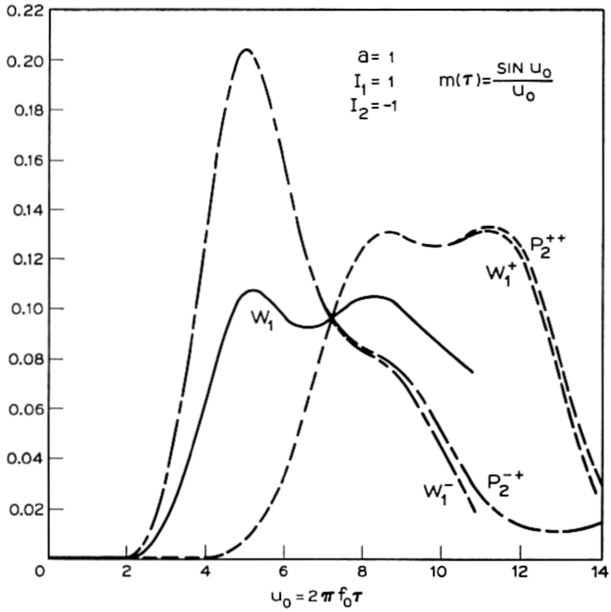

Fig. 18 — (See Fig. 6.)

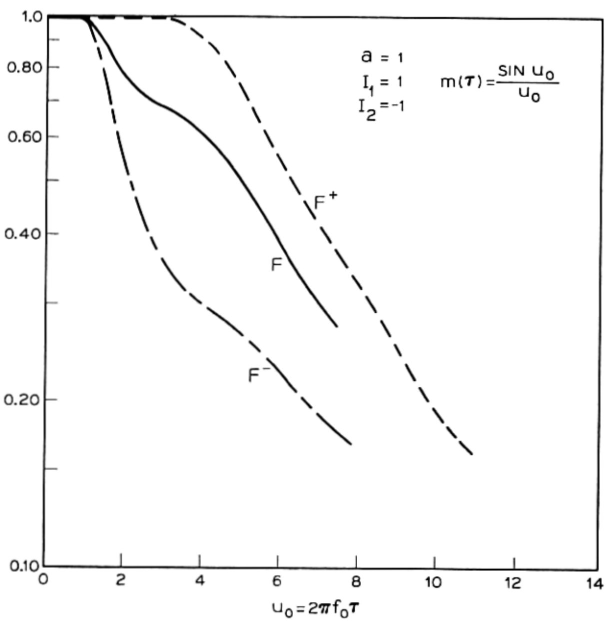

Fig. 19 — (See Fig. 7.)

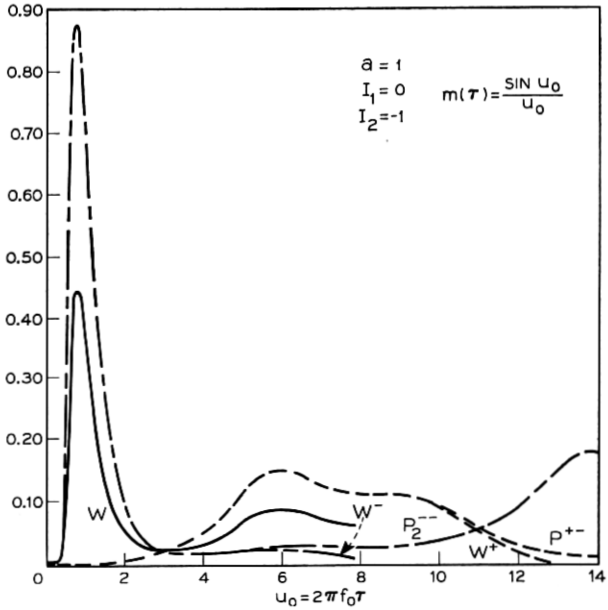

Fig. 20 — (See Fig. 5.)

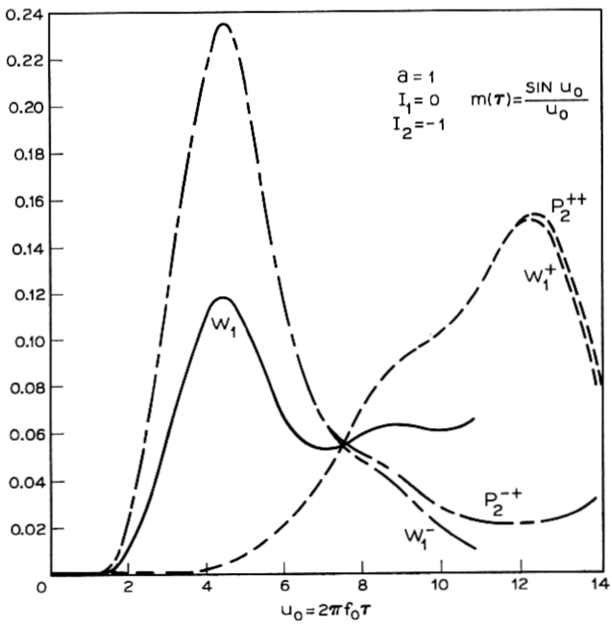

Fig. 21 — (See Fig. 6.)

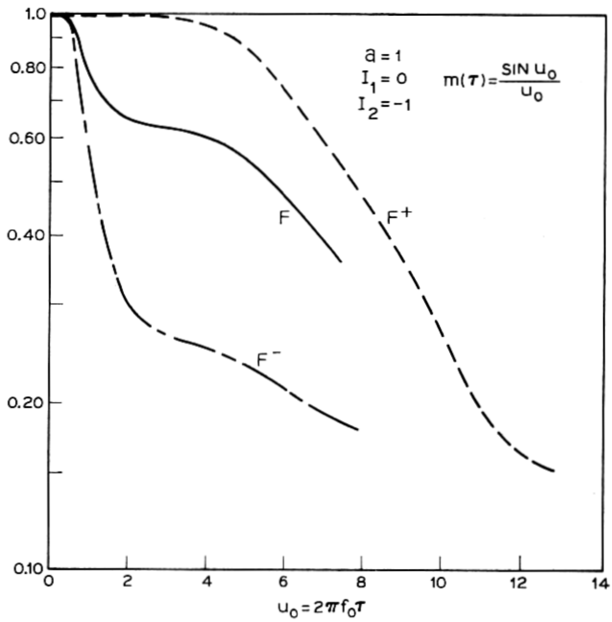

Fig. 22 — (See Fig. 7.)

or

$$\tau_m \doteq (2f_0)^{-1}. \tag{26}$$

τ_m also represents the median time during which the random process $I(t, 1)$ remains continually above the level $I_2 = 0$ when it starts at the level $I_1 = 1$. From symmetry, τ_m also represents the median time during which the random process $I(t, 1)$ remains continually below the level $I_2 = 0$ when it starts at the level $I_1 = -1$.

V. CONCLUSIONS

The exact auxiliary probability functions can be used in approximate integral equations in order to deduce approximate probability densities of the first and second passage times of sine wave plus stationary, gaussian noise with a finite expected number of zeros per unit time. These approximate probability densities can be used to deduce the approximate median times associated with the first and second passage times. Also, the approximate probability densities of the first passage times can be used to deduce approximate distribution functions for the absolute minimum or absolute maximum of sine wave plus noise in a closed interval.

The corresponding exact results are not yet known.

VI. ACKNOWLEDGMENT

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REFERENCES

1. Wang, Ming Chen, and Uhlenbeck, G. E., "On the Theory of the Brownian Motion II," Rev. Modern Phys., *17* (April–July 1945), pp. 323–342.
2. Siegert, A. J. F., "On the First Passage Time Probability Problem," Phys. Rev., *81* (February 15, 1951), pp. 617–623.
3. Darling, D. A. and Siegert, A. J. F., "The First Passage Problem for a Continuous Markov Process," Ann. Math. Stat., *24* (December 1953), pp. 624–639.
4. Rice, S. O. "Distribution of the Duration of Fades in Radio Transmission," B.S.T.J. *37*, No. 3(May 1958), pp. 581–635.
5. Slepian, D., "First Passage Time for a Particular Gaussian Process, Ann. Math. Stat., *32* (June 1961), pp. 610–612.
6. Kac, M., "Probability Theory as a Mathematical Discipline and as a Tool in Engineering and Science," *Proc. First Symp. on Eng. Applications of Random Function Theory and Prob.*, ed. J. L. Bogdanoff and F. Kozin, New York: Wiley 1963, pp. 31–67.
7. Mehr, C. B. and McFadden, J. A., "Certain Properties of Gaussian Processes and Their First-Passage Times," J. Royal Stat. Soc., Series B., *27*, No. 3 (1965), pp. 505–522.
8. Rainal, A. J. "Axis-Crossing Intervals of Sine Wave Plus Noise," B.S.T.J., *46*, No. 7 (September 1967), pp. 1655–1658.
9. Rice, S. O., "Statistical Properties of a Sine Wave Plus Random Noise," B.S.T.J., *27*, No. 1 (January 1948), pp. 109–157.
10. Rainal, A. J., "Axis-Crossing Intervals of Rayleigh Processes," B.S.T.J., *44*, No. 6 (July–August 1965), pp. 1219–1224
11. McFadden, J. A., "The Axis-Crossing Intervals of Random Functions-II," IRE Trans. Inform. Theory, *IT-4* (March 1958), pp. 14–23.
12. Rainal, A. J., "Zero-Crossing Intervals of Random Processes," Technical Report AF-102, DDC No. AD-401-148, Baltimore, Maryland: Johns Hopkins University, Carlyle Barton Laboratory, April 1963. Abstracted in IEEE Trans. Inform. Theory., *IT-9* (October 1963), p. 295.