The Spectrum of a Simple Nonlinear System

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The random motion of a particle with nonlinear damping is investigated. The spectrum of the velocity of the particle is obtained by solving the associated nonstationary Fokker–Planck equation and also by using the equivalent-linearization technique. The first procedure yields an exact solution in terms of Laguerre polynomials. The second leads to simple, approximate results which are valid for cases where the small nonlinearity assumption holds. Results obtained by these two methods are compared and good agreement is observed over a large frequency range.

I. INTRODUCTION

The recent advance in space and communicational technologies has led engineers to numerous difficult but fascinating problems in regard to the structural dynamics in random environments. For example, Hempstead and Lax have investigated noise in self-sustained oscillation;[1, 2] Ariaratnam and Sanker have studied the dynamic snap-through of shallow, arch-type aircraft components under stochastic pressure.[3] In this paper the random vibration of a simple mass with nonlinear damping is studied. The nonlinearity of the system is introduced to linear viscous damping by adding to it an extra term which is inversely proportional to the first power of the current velocity. Emphasis of the analysis is placed on finding the power spectral density of the random motion.

Two different approaches are used to obtain the desired solution. First, the exact spectrum is found by solving the associated nonstationary Fokker–Planck equation in terms of the eigenfunction expansion of the degenerate ordinary differential equation. Second, approximate solutions are obtained by using the equivalent linearization technique by which the original nonlinear system is converted to an equivalent linear one. The equivalent linear system, constructed by the least mean square error criterion and based on the small nonlinearity assumption, is then solved by standard linear theory.

II. NONSTATIONARY FOKKER–PLANCK EQUATION

Consider the first-order nonlinear system described by the following differential equation:

$$\dot{x} + F(x) = f(t), \tag{1}$$

which may be thought of as the velocity equation of a unit mass with nonlinear damping $F(x)$ subject to a force $f(t)$.

Let us discuss the problem of obtaining the power spectral density of $x(t)$ when $f(t)$ is a random process. We limit the discussion to the case where $f(t)$ is a stationary white gaussian process with the first two moments defined as

$$\langle f(t) \rangle = 0 \tag{2}$$

and

$$\langle f(t_1) f(t_2) \rangle = 2\pi s_o \, \delta(r_1 - t_2) \tag{3}$$

where s_o is a constant, the symbol $\langle \ \rangle$ indicates the ensemble average and δ indicates the Dirac delta function.

Caughey and Dienes[4] have investigated a similar problem for $F(x) = k \operatorname{sgn} x$. We shall however consider a different case in which

$$F(x) = \beta x - \frac{\gamma}{x} \tag{4}$$

$$0 < x < \infty$$

where β is a constant and γ is a smaller nonlinear coefficient. In case $\gamma = 0$, equation 1 becomes the familiar linear differential equation.

The Fokker–Planck equation which governs the transition probability $p(x_o \mid x, \tau)$ with given initial velocity $x_o = x(t_o)$ for the velocity $x(t)$ at time t is

$$\dot{p} = \frac{\partial}{\partial x}\left[\left(\beta x - \frac{\gamma}{x}\right)p\right] + \pi s_o \frac{\partial^2 p}{\partial x^2} \tag{5}$$

where $\tau = t - t_o$. The initial and boundary conditions for equation 5 are

$$p_{t_o} = \delta(x - x_o) \tag{6}$$

and

$$p(0, t) = p(\infty, t) = 0, \tag{7}$$

respectively. As the time of passage t becomes sufficiently large, $p(x_o \mid x, \tau)$ in equation 5 approaches a stationary value $p_{st}(x)$ independent of t and initial condition 6. Setting $\dot{p} = 0$ in 5, such a stationary density can be found by solving the degenerate stationary equation. The result is:

$$p_{st} = C \exp\left[-\frac{1}{\pi s_o}\left(\frac{\beta x^2}{2} - \gamma \log x\right)\right] \tag{8}$$

where C is the normalization factor determined by

$$\int_0^\infty p_{st}\, dx = 1. \tag{9}$$

The power spectrum is the Fourier transform of the autocorrelation function which is determined by the joint probability density $p(x_o, x, \tau)$. Thus from the relation

$$p(x_o, x, \tau) = p_{st}(x_o) p(x_o \mid x, \tau), \tag{10}$$

we need to find the transition probability density, that is, the nonstationary solution of equation 5. Let $p(x_o \mid x, \tau) = T(t)X(x)$ in 5. It follows that

$$T(t) + \lambda T(t) = 0 \tag{11}$$

and

$$\sigma^2 \frac{\partial^2 X}{\partial x^2} + \frac{\partial}{\partial x}(xX) - \frac{\sigma^2 \gamma}{\pi s_o} \frac{\partial}{\partial x}\left(\frac{X}{x}\right) + \frac{\lambda}{\beta} X = 0 \tag{12}$$

where $\sigma^2 = \pi s_0/\beta$. If $X_m(x)$ is the eigenfunction and λ_m the corresponding eigenvalues satisfying equation 12 and the prescribed boundary condition 7, it can be shown that[5]

$$p(x_o \mid x, \tau) = \sum_{m=0}^{\infty} \frac{X_m(x) X_m(x_o)}{p_{st}(x_o)} e^{-\lambda_m (t - t_o)}. \tag{13}$$

In deriving 13, the following orthogonality condition has been used:

$$\int X_m(x) X_n(x) \frac{dx}{p_{st}(x)} = \delta_{mn}. \tag{14}$$

Following the transformations adopted by Stratonovich, let $\mu = ¼ (\gamma/\pi s_0 - 1)$, $z = x^2/2\sigma^2$, and $u = z^{-\mu} X$. Equation 12 becomes

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z} + \left[\frac{(1/2) + u + (\lambda/2\beta)}{z} + \frac{(1/4) - u^2}{z^2}\right] u = 0. \tag{15}$$

Equation 15 is a degenerate hypergeometric differential equation which has eigenfunctions $U_n(z)$ with corresponding eigenvalues $\lambda_n = 2n\beta$:

$$U_n(z) = z^{\mu+(1/2)}e^{-z}L_n^{(2\mu)}(z) \tag{16}$$

where

$$L_n^{\alpha}(z) = \frac{1}{n!}e^{z}z^{-\alpha}\frac{d^n}{dz^n}(e^{-z}z^{n+\alpha}) \tag{17}$$

is the Laguerre polynomial of degree n.

Transforming back to the original variables and applying equation 14, we obtain the following normalized eigenfunctions:

$$X_n(x) = \frac{(2)^{\frac{1}{2}}}{\sigma}\frac{z^{2\mu+(1/2)}e^{-z}L_n^{(2\mu)}(z)}{[n!\,\Gamma(n+2\mu+1)\Gamma(2\mu+1)]^{\frac{1}{2}}}. \tag{18}$$

From equations 13 and 10 the transition density and jointly density, respectively, can be found as

$$p(x_o \mid x, \tau) = \frac{(2)^{\frac{1}{2}}}{\sigma}\frac{z^{2\mu+(1/2)}e^{-z}}{L_o^{(2\mu)}(z_o)}\sum_{n=0}^{\infty}\frac{L_n^{(2\mu)}(z)L_n^{(2\mu)}(z_o)}{n!\,\Gamma(n+2\mu+1)}e^{-2n\beta|\tau|} \tag{19}$$

and

$$p(x_o, x, \tau) = \frac{2}{\sigma^2}(zz_o)^{2\mu+(1/2)}e^{-(z+z_o)} \cdot\sum_{n=0}^{\infty}\frac{L_n^{(2\mu)}(z)L_n^{(2\mu)}(z_o)}{n!\,\Gamma(n+2\mu+1)\Gamma(2\mu+1)}e^{-2n\beta|\tau|} \tag{20}$$

where

$$z = \frac{x^2}{2\sigma^2} \quad \text{and} \quad z_o = \frac{x_o^2}{2\sigma^2}.$$

From the above the autocorrelation function $R_x(\tau)$ of $x(t)$, where $\tau = t - t_o$, is

$$\begin{aligned} R_x(\tau) &= \int_0^{\infty}\int_0^{\infty} p(x_o, x, \tau)x_o x\,dx\,dx_o \\ &= 2\sigma^2\sum_{n=0}^{\infty}\frac{e^{-2n\beta|\tau|}}{\Gamma(n+2\mu+1)\Gamma(2\mu+1)n!}I(\mu, n) \end{aligned} \tag{21}$$

where

$$I(\mu, n) = \int_0^{\infty}\int_0^{\infty}(z)^{2\mu+(1/2)}e^{-z}(z_o)^{2\mu+(1/2)}e^{-z_o}L_n^{(2\mu)}(z)L_n^{(2\mu)}(z_o)\,dz\,dz_o\,. \tag{22}$$

In the appendix we show that

$$\int_0^\infty z^{2\mu+(1/2)} e^{-z} L_n^{(2\mu)}(z)\, dz = \frac{-\Gamma[2\mu + (3/2)]\Gamma[n - (1/2)]}{2(\pi)^{\frac{1}{2}} n!}, \tag{23}$$

from which it follows that

$$I(\mu, n) = \frac{1}{4\pi}\left\{\frac{\Gamma[2\mu + (3/2)]\Gamma[n - (1/2)]}{n!}\right\}^2. \tag{24}$$

Substituting 24 into 21, we obtain the following expression for the autocorrelation function $R_x(\tau)$:

$$R_x(\tau) = \frac{\sigma^2}{2\pi} \frac{\Gamma^2[2\mu + (3/2)]}{\Gamma(2\mu + 1)} \sum_{n=0}^{\infty} \frac{\Gamma^2[n - (1/2)]e^{-2n\beta|\tau|}}{(n!)^3\Gamma(n + 2\mu + 1)}, \tag{25}$$

which is a monotonically decreasing function of τ.

The power spectral density of $x(t)$ can be derived from equation 25 as the following:

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)e^{-i\omega\tau}\, d\tau = 2\sigma^2 \frac{\Gamma^2[2\mu + (3/2)]}{\Gamma(2\mu + 1)}\, \delta(\omega)$$

$$+ \left(\frac{\sigma}{2\pi}\right)^2 \frac{\Gamma^2[2\mu + (3/2)]}{\Gamma(2\mu + 1)} \sum_{n=1}^{\infty} \frac{\Gamma^2[n - (1/2)]}{(n!)^3\Gamma(n + 2\mu + 1)} \frac{4n\beta}{4n^2\beta^2 + \omega^2}. \tag{26}$$

Notice that $S_x(\omega)$ is again a monotonically decreasing function of ω and has a spike at $\omega = 0$.

The nonstationary mean value of $x(t)$ is given by

$$\langle x(\tau)\rangle = \int_0^\infty x p(x_o \mid x, \tau)\, dx. \tag{27}$$

Using equations 19 and 23 it can easily be shown that

$$\langle x(\tau)\rangle = \frac{\sigma\Gamma[2\mu + (3/2)]}{-\sqrt{2\pi}\, L_o^{(2\mu)}(z_o)} \sum_{n=0}^{\infty} \frac{\Gamma[n - (1/2)]L_n^{(2\mu)}(z_o)}{(n!)^2\Gamma(n + 2\mu + 1)} e^{-2n\beta|\tau|}. \tag{28}$$

Because $x_o = (2\sigma^2 z_o)^{1/2}$, we notice that $\langle x(\tau)\rangle$ depends on the initial velocity x_o. As $\tau \to \infty$, $\langle x(\tau)\rangle$ in equation 28 approaches its stationary value $\langle x\rangle_{st}$, which is given by

$$\langle x\rangle_{st} = \frac{(2)^{\frac{1}{2}}\Gamma[2\mu + (3/2)]}{\Gamma(2\mu + 1)}\, \sigma. \tag{29}$$

$\langle x\rangle_{st}$ is independent of the initial condition x_o. This stationary mean velocity, $\langle x\rangle_{st}$, can also be found by using the stationary density

$p_{st}(x)$ as follows:

$$\langle x \rangle_{st} = \int_0^\infty x p_{st}(x)\, dx$$
$$= \int_0^\infty x X_o(x)\, dx$$
$$= \frac{(2)^{\frac{1}{2}}\sigma}{\Gamma(2\mu + 1)} \int_0^\infty (z)^{2\mu+(1/2)} e^{-z} L_o^{2\mu}(z)\, dz$$
$$= \frac{(2)^{\frac{1}{2}}\Gamma[2\mu + (3/2)]}{\Gamma(2\mu + 1)}\, \sigma.$$

The variation of $\langle x(\tau) \rangle$ is illustrated in Fig. 1. It has a maximum value x_o at $\tau = 0$ and decreases exponentially to the stationary value $\langle x \rangle_{st}$ as given in equation 29. The nonstationary mean square value of $x(t)$, given its initial value x_0^2, is difficult to evaluate explicitly, but its stationary value $\langle x^2 \rangle_{st}$ can be found by integrating $S_x(\omega)$ in equation 25 over the entire frequency range from $-\infty$ to $+\infty$. By this procedure,

$$\langle x^2 \rangle_{st} = \int_{-\infty}^{\infty} S_x(\omega)\, d\omega = \frac{\sigma^2}{2\pi} \frac{\Gamma^2[2\mu + (3/2)]}{\Gamma(2\mu + 1)} \sum_{n=0}^{\infty} \frac{\Gamma^2[n - (1/2)]}{(n!)^3 \Gamma(n + 2\mu + 1)}. \tag{30}$$

As expected, $\langle x^2 \rangle_{st}$ is also independent of the initial condition x_o. The variations of $\langle x^2 \rangle_{st}$ with the nonlinear coefficient $k = \gamma/\pi s_o$ are shown in Fig. 2.

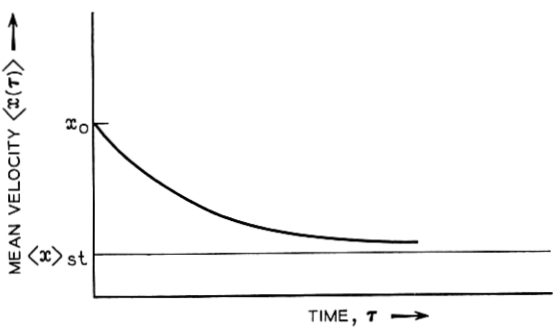

Fig. 1 — Nonstationary mean velocity of nonlinear system subjected to random force.

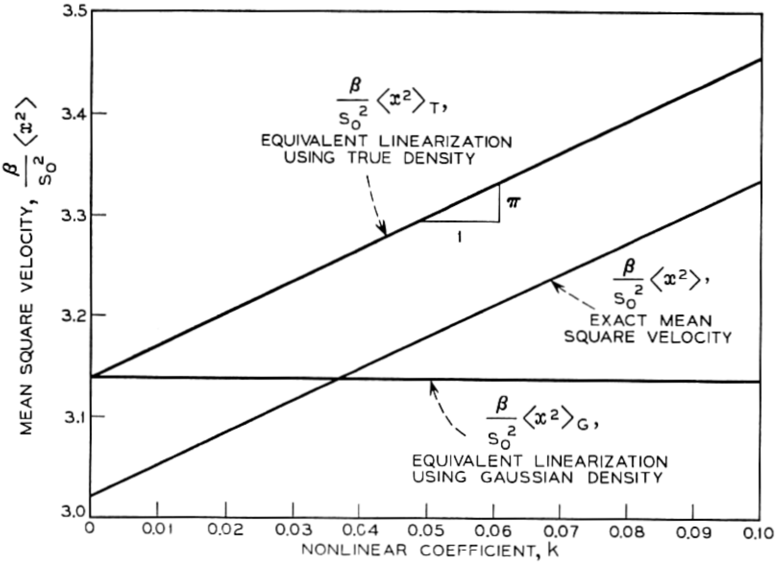

Fig. 2 — Comparison of mean square velocity nonlinear system subjected to random noise.

III. EQUIVALENT LINEARIZATION TECHNIQUE

The eigenfunction expansion of the degenerate ordinary differential equation of the governing Fokker–Planck equation is often difficult to find. In such cases it is sometimes convenient to use the perturbation[6,7] or the equivalent linearization[8,9] techniques to obtain desired quantitative results. If the nonlinearity of the system is small (that is, $\gamma/\beta \ll 1$), these techniques provide the simplest means for obtaining approximate results. In the following, an equivalent linearization procedure is used to derive the power spectral densities of the nonlinear velocity $x(t)$ in equations 1 through 4. Let

$$\dot{x} + \beta_e x + \epsilon(x) = f(t) \tag{31}$$

be the equation of motion of a system equivalent to that described by equations 1 and 4 in which β_e is the equivalent linear stiffness and

$$\epsilon(x) = \beta x + \frac{\gamma}{x} - \beta_e x \tag{32}$$

is the error function. If $\epsilon(x)$ is small and may be ignored, equation 31 becomes a linear differential equation, and its spectrum can be solved by standard technique. The equivalent linear stiffness β_e can be determined by the criterion that the mean square error is minimized. The mean square error is

$$\langle \epsilon(x)^2 \rangle = \int_0^\infty \left[(\beta - \beta_e)^2 x^2 + \frac{\gamma^2}{x^2} + 2(\beta - \beta_e)\gamma \right] p(x)\, dx.$$

Setting $\partial \langle \epsilon(x)^2 \rangle / \partial \beta_e = 0$ we obtain

$$\int_0^\infty \beta x^2 p(x)\, dx = \int_0^\infty \beta_e x^2 p(x)\, dx - \int_0^\infty \gamma p(x)\, dx;$$

therefore

$$\beta_e = \frac{\beta \int_0^\infty x^2 p(x)\, dx + \gamma}{\int_0^\infty x^2 p(x)\, dx}$$

$$= \beta + \frac{\gamma}{\langle x^2 \rangle}. \tag{33}$$

The following two cases are considered:

(*i*) Assuming $p(x)$ is gaussian, that is,

$$p(x) = \left(\frac{2\pi^2 s_o}{\beta} \right)^{-\frac{1}{2}} \exp\left(-\frac{\beta x^2}{2\pi s_o} \right),$$

then

$$\langle x^2 \rangle_G = \sigma_x^2 = \frac{\pi s_o}{\beta}. \tag{34}$$

Substitution into equation 33 yields

$$\beta_{e,G} = \beta(1 + \gamma/\pi s_o). \tag{35}$$

(*ii*) Using true (stationary) distribution, $p_{st}(x)$ given by equations 8 and 9 becomes:

$$p_{st}(x) = \frac{x^{(\gamma/\pi s_o)} \exp\left[-(\beta x^2/2\pi s_o)\right]}{A} \tag{36}$$

where

$$A = \int_0^\infty x^{(\gamma/\pi s_o)} \exp\left(\frac{-\beta x^2}{2\pi s_o} \right) dx$$

$$= \frac{1}{2} \left(\frac{2\pi s_o}{\beta} \right)^{\frac{1}{2}[1+(\gamma/\pi s_o)]} \Gamma\left[\frac{1 + (\gamma/\pi s_o)}{2} \right].$$

Therefore the mean square value of $x(t)$, using true distribution equation 36, is

$$\begin{aligned}\langle x^2\rangle_T &= \int_0^\infty x^2 p_{st}(x)\,dx \\ &= \frac{2}{A}\left[\int_0^\infty (x)^{2+(\gamma/\pi s_o)} \exp\left(-\frac{\beta}{2\pi s_o}x^2\right)dx\right] \\ &= \frac{1}{\beta}(\pi s_o + \gamma). \end{aligned} \tag{37}$$

Substitution into equation 33 yields

$$\beta_{e,T} = \beta\left(1 + \frac{\gamma}{\pi s_o + \gamma}\right) = \beta\left(1 + \frac{\gamma}{\pi s_o} - \frac{\gamma^2}{\pi^2 s_o^2} + \cdots\right). \tag{38}$$

Comparison of equation 38 with equation 35 indicates that $\beta_{e,G}$ is the first-order approximation of $\beta_{e,T}$.

Now let us consider the simple linear system

$$\dot{x} + \beta_e x = f(t) \tag{39}$$

whose transfer function is given as

$$H(i\omega) = \frac{1}{\beta_e + i\omega}. \tag{40}$$

According to the familiar linear theory of random processes, the power spectrum of $x(t)$ in equation 39 is given by

$$S_x(\omega) = \frac{s_o}{\beta_e^2 + \omega^2} \tag{41}$$

where s_o is defined in equation 3.

Substituting $\beta_{e,G}$ as given in equation 35 into equation 41, we obtain

$$S_{x,G}(\omega) = \frac{s_o}{\beta^2[1 + (\gamma/\pi s_o)]^2 + \omega^2}. \tag{42}$$

Substituting $\beta_{e,T}$ as given in equation 38 into equation 41, we obtain

$$S_{x,T}(\omega) = \frac{s_o}{[\beta(\pi s_o + 2\gamma)/(\pi s_o + \gamma)]^2 + \omega^2}. \tag{43}$$

By setting $\gamma = 0$, both equations 42 and 43 give

$$S_x(\omega) = \frac{s_o}{\beta^2 + \omega^2}$$

which is the spectrum of the corresponding linear system. The mean square value and the power spectral density of the velocity response $x(t)$, obtained by solving the Fokker–Planck equation, and the equivalent linear equations using gaussian and true distributions, respectively, are summarized in Table I. In this table $k = \gamma/\pi s_o$ is the nonlinear coefficient.

The mean square velocities $\langle x^2 \rangle$, $\langle x^2 \rangle_G$, and $\langle x^2 \rangle_T$ are compared in Fig. 2. It is seen that for $k < 0.035$, that is, in a very small nonlinearity range, both linearization cases give higher mean square velocities than the exact solutions. For $k > 0.035$, equivalent linearization methods give larger results when using true distribution and smaller results when using gaussian distribution than the exact solutions.

The power spectral density functions $S_{x,G}(\omega)$ and $S_{x,T}(\omega)$ obtained by the equivalent linearization procedure, using gaussian distribution and true distribution of $x(t)$, respectively, are compared in Fig. 3 in which $B_1 = (\beta^2/s_o)S_{x,G}(\omega)$ and $B_2 = (\beta^2/s_o)S_{x,T}(\omega)$. Notice that both $S_{x,G}(\omega)$ and $S_{x,T}(\omega)$ are monotonically decreasing functions, and the differences between them are negligible for small nonlinearity.

The exact power spectral densities are compared with the equivalent linearized solutions in Fig. 4. For the exact solution of $S_x(\omega)$ as given by equation 25, the spike at $\omega = 0$ is evaluated by normalizing $S_x(\omega)$ to an area equal to that given by Fig. 2, that is,

$$\langle x^2 \rangle = \int_0^\infty S_x(\omega)\, d\omega = 3.053 \quad \text{for} \quad k = 0.01.$$

Curves shown in Fig. 4 are seen to be monotonically decreasing. The equivalent linearized systems have higher power spectra of $x(t)$ in the low-frequency region and lower power spectra of $x(t)$ in the high-frequency region than the actual nonlinear system has.

IV. CONCLUSION

It has been shown that the exact expression for the two-dimensional nonstationary probability distribution of a class of simple nonlinear systems can be found in terms of the spatial eigenfunction expansion of the governing Fokker–Planck equation. Equivalent linearization techniques can be very useful in generating approximate response statistics for certain systems having small nonlinearities. In Figs. 2, 3, and 4 good agreement has been achieved in the comparison of the exact and approximate mean square values and of the power spectral densities of the nonlinear random response.

TABLE I—COMPARISON OF RESPONSE STATISTICS OF THE VELOCITY RESPONSE $x(t)$

	Mean square velocity, $\frac{\beta}{s_o}\langle x^2\rangle$	Power spectral density of velocity, $\frac{\beta^2}{s_o}S_x(\omega)$
Exact solution	$\frac{1}{2}\frac{\Gamma^2[2\mu+(3/2)]}{\Gamma(2\mu+1)}\sum_{n=0}^{\infty}\frac{\Gamma^2[n-(1/2)]}{(n!)^3\Gamma(n+2\mu+1)}$	$\frac{\Gamma^2[2\mu+(3/2)]}{\Gamma(2\mu+1)}\left[\frac{2\pi s_0\delta(\omega)}{\Gamma(2\mu+1)} + \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{4n\ \Gamma^2[n-(1/2)]}{(n!)^3\Gamma(n+2\mu+1)[4n^2+(\omega/\beta)^2]}\right]$
Equivalent linearization using gaussian density	π	$\frac{1}{(1+k)^2+(\omega/\beta)^2}$
Equivalent linearization using true density	$\pi(1+k)$*	$\frac{1}{[(1+2k)/(1+k)]^2+(\omega/\beta)^2}$

* $k = \gamma/\pi s_o$.

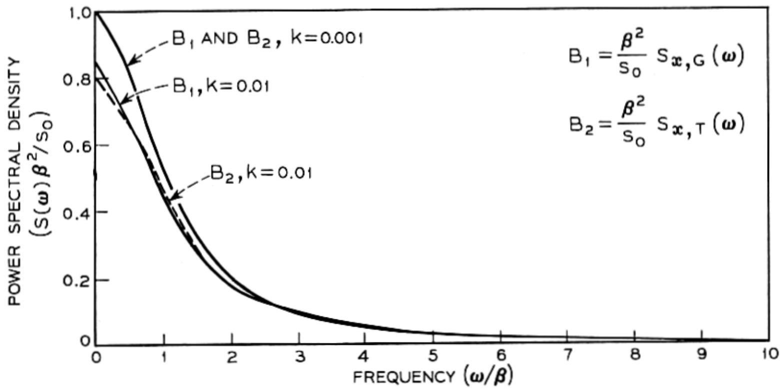

Fig. 3 — Power spectral density by equivalent linearization.

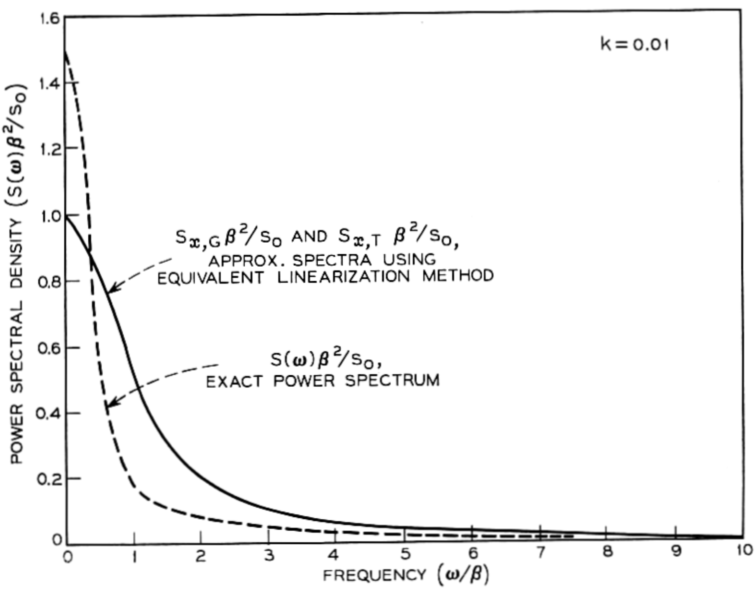

Fig. 4 — Comparison of the exact and approximate power spectra.

V. ACKNOWLEDGMENT

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APPENDIX

Derivation of Equation 23

The following formulae are used in the derivation of equation 23:

$$\int_0^\infty e^{-st} t^\beta L_n^\alpha(t)\, dt$$

$$= \frac{\Gamma(\beta + 1)\Gamma(\alpha + n + 1)}{n!\, \Gamma(\alpha + 1)} s^{-\beta-1} F\left(-n, \beta + 1; \alpha + 1; \frac{1}{s}\right)$$

$$(\mathrm{Re}\, \beta > -1,\ \mathrm{Re}\, s > 0) \qquad (44)$$

where

$$F(\alpha, \beta; \gamma; z) = {}_2F_1\,(\alpha, \beta; \gamma; z) \qquad (45)$$

is a generalized hypergeometric series which is defined as

$${}_pF_q(\alpha_1\,, \alpha_2\,, \cdots \alpha_p\,; \beta_1\,, \beta_2 \cdots \beta_q\,; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_q)_k} \frac{z^k}{k!}$$

in which

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}$$

and

$$(\beta)_k = \frac{\Gamma(\beta + k)}{\Gamma(\beta)}. \qquad (46)$$

A special case for equation 45 is when $z = 1$:

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

$$(\mathrm{Re}\, \gamma > \mathrm{Re}\, (\alpha + \beta),\ \mathrm{Re}\, \gamma > \mathrm{Re}\, \beta > 0). \qquad (47)$$

Using equations 44 through 47, the integral involved in equation 22 can be evaluated as follows:

$$\int_0^\infty z^{2\mu + (1/2)} e^{-z} L_n^{(2\mu)}(z)\, dz$$

$$= \frac{\Gamma[2\mu + (1/2) + 1]\Gamma(2\mu + n + 1)}{n!\, \Gamma(2\mu + 1)} F(-n, 2\mu + \tfrac{3}{2}; 2\mu + 1; 1)$$

$$= \frac{\Gamma[2\mu + (3/2)]\Gamma(2\mu + n + 1)}{n!\,\Gamma(2\mu + 1)} \frac{\Gamma(2\mu + 1)\Gamma[2\mu + 1 + n - 2\mu - (3/2)]}{\Gamma(2\mu + 1 + n)\Gamma[2\mu + 1 - 2\mu - (3/2)]}$$

$$= \frac{\Gamma(2\mu + (3/2)]\Gamma[n - (1/2)]}{n!\,\Gamma[-(1/2)]} = \frac{-\Gamma[2\mu + (3/2)]}{2(\pi)^{\frac{1}{2}}} \frac{\Gamma[n - (1/2)]}{n!}$$

which is equation 23.

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