Phase Principle for Measuring Location or Spectral Shape of a Discrete Radio Source

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This paper describes a phase principle for measuring the location or the spectral shape of a discrete radio source. The phase principle is relatively simple to implement and leads to a measurement of location or spectral shape which is insensitive to receiver gain fluctuations. For measuring the location of a weak, discrete radio source, the theoretical accuracy is slightly better than the theoretical accuracy resulting from the Ryle interferometer. For measuring the spectral shape of a weak, discrete radio source, the theoretical accuracy is slightly better than the theoretical accuracy resulting from either the Ryle interferometer or the Dicke radiometer. Furthermore, the implementation of the phase principle doesn't require input switching. Also, the calibration curve associated with the phase principle is independent of changes in the average receiver gains of the two receivers.

I. INTRODUCTION

In many branches of science and technology observations of a discrete radio source provide fundamental knowledge. In the field of radar the illuminated target serves as the discrete radio source. In the field of space exploration the radio transmitter on-board the space vehicle serves as the discrete radio source. In the field of radio astronomy the "radio star" serves as the discrete radio source.

The "radio star" is a remarkable example of a discrete radio source. In the past twenty years radio astronomers have discovered that nature provided many discrete radio sources or radio stars at certain locations in the sky. What are the locations of these radio stars? What is the power spectrum of the observed radiation from a particular radio star? Answers to such questions are of fundamental importance in the field of radio astronomy.[1]

In order to measure relatively small values of radiated power from a discrete radio source, one must compete with the inevitable background noise and the inevitable radio receiver noise. It is well known that one requires a method of measurement which is relatively insensitive to receiver gain fluctuations. The papers by Dicke[2] and Ryle[3] discuss this important point in more detail. In fact, the present day method for measuring relatively small values of radiated power from a discrete radio source makes use of some form of the Dicke[2] radiometer or the Ryle[3] interferometer.

The purpose of this paper is to describe a phase principle for measuring the location or the spectral shape of a discrete radio source. We shall see that the phase principle is relatively simple to implement and leads to a measurement of location or spectral shape which is insensitive to receiver gain fluctuations. For measuring the location or spectral shape of a weak, discrete radio source, we shall see that the theoretical accuracy associated with the phase principle is slightly better than the theoretical accuracy associated with the Dicke radiometer or the Ryle interferometer. We shall also see that for measuring the location or spectral shape of a weak, discrete radio source using only phase information, the accuracy associated with the phase principle is essentially equal to the accuracy associated with the maximum likelihood principle.

II. MEASUREMENTS BASED ON THE PHASE PRINCIPLE

2.1 *Implementation*

Fig. 1 illustrates a simplified implementation of the phase principle for measuring the location or spectral shape of a discrete radio source. $S(t)$, $N_1(t)$, and $N_2(t)$ represent zero mean, independent, narrow-band, stationary Gaussian processes. $N_1(t)$ and $N_2(t)$ each represent the sum of background noise plus receiver noise. $N_1(t)$ and $N_2(t)$ are assumed to have equal variances. The spacing, d, between the two antennas is many wavelengths in order that $N_1(t)$ and $N_2(t)$ can be considered as independent processes. $S(t - \Delta t)$ and $S(t + \Delta t)$ are due to the presence of a discrete radio source located at a small angle θ with respect to boresight. We assume that the receivers preserve the phase difference between the antenna excitations.

$S(t - \Delta t) + N_1(t)$ and $S(t + \Delta t) + N_2(t)$ represent the outputs of the two receivers. η_i represents the ith independent sample of the phase difference between $S(t - \Delta t) + N_1(t)$ and $S(t + \Delta t) + N_2(t)$.

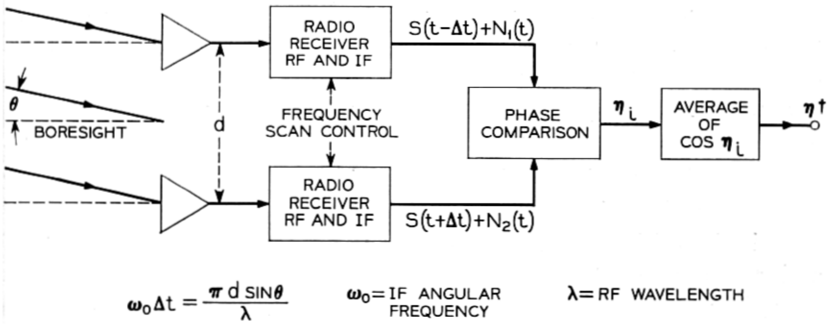

Fig. 1 — Simplified implementation of a phase principle for measuring location or spectral shape of a discrete radio source. When source is located at $d \sin \theta = \lambda/4$, $\eta^\dagger \doteq 0$. When source is located at $\theta = 0$, receivers scan in frequency and $\eta^\dagger$ traces out a measure of spectral shape.

η_i is taken to be in the primary interval $(-\pi, \pi)$. After n such samples the output $\eta^\dagger$ is given by

$$\eta^\dagger = \frac{1}{n} \sum_{i=1}^{n} \cos \eta_i \,, \tag{1}$$

where

$$n = B\tau$$
$$B = \text{IF bandwidth}$$
$$\tau = \text{observation time.}$$

We shall assume that n is relatively large like $n \geqq 10^4$, since we are primarily interested in observing relatively weak, discrete radio sources.

Fig. 2 illustrates a relatively simple method for generating $\eta^\dagger$ from the inputs $S(t - \Delta t) + N_1(t)$ and $S(t + \Delta t) + N_2(t)$. R_1, ω_0, and θ_1 represent the envelope, IF angular frequency, and phase angle, respec-

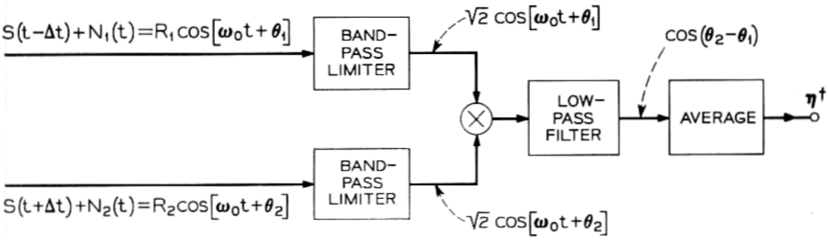

Fig. 2 — A method for generating $\eta^\dagger$ from the inputs $S(t-\Delta t)+N_1(t)$ and $S(t+\Delta t)+N_2(t)$. The band-pass limiters remove all amplitude information.

tively, of the narrow-band Gaussian process $S(t - \Delta t) + N_1(t)$. Similarly, R_2, ω_0, and θ_2 represent the envelope, IF angular frequency, and phase angle respectively of the narrow-band Gaussian process $S(t + \Delta t) + N_2(t)$. The band-pass limiters shown in Fig. 2 are well-known devices for removing all amplitude information and preserving the phase information as is indicated in Fig. 2. See Davenport and Root[4] for a discussion of the band-pass limiter.

As indicated in Fig. 2, if one takes the product of $\sqrt{2} \cos (\omega_0 t + \theta_1)$ and $\sqrt{2} \cos (\omega_0 t + \theta_2)$ and passes the result through a suitable low-pass filter the result is $\cos (\theta_2 - \theta_1) \equiv \cos \eta(t)$. By taking the average of this result, one can generate $\eta^\dagger$.

This method of generating $\eta^\dagger$ can also be used to help implement the phase principle described in Ref. 5 in order to detect the presence of a discrete radio source located at $\theta = 0$.

Fig. 2 indicates clearly that $\eta^\dagger$ is independent of receiver gain fluctuations or changes in the average receiver gain of each receiver. Also, the two receivers need not have the same average gain. Thus, an unusually long observation time τ is advantageous when using the phase principle.

Notice that if the band-pass limiters in Fig. 2 are shorted out, we have the well-known correlator configuration.[6,7,8,9]

We shall now show that a measurement of $\eta^\dagger$ leads to a measurement of location or spectral shape of the discrete radio source.

2.2 *Statistical Properties of* η_i

In order to simplify the analysis, we shall always assume that the discrete radio source is at a small angle θ with respect to boresight. To begin, we shall state some known statistical properties of the angle η_i.

Equation (34) of Ref. 10 gives the probability density $p_2(\eta)$ of each independent sample η_i as

$$p_2(\eta) = \frac{1 - l^2}{2\pi} (1 - \beta_2^2)^{-\frac{3}{2}} \left[\beta_2 \sin^{-1} \beta_2 + \frac{\pi \beta_2}{2} + \sqrt{1 - \beta_2^2} \right], \tag{2}$$

where

$$\beta_2 = l \cos (\eta - \eta_\theta)$$

$$l = \frac{a}{1 + a}$$

$$a = \frac{\text{Var } S(t)}{\text{Var } N_1(t)} = \frac{\text{Var } S(t)}{\text{Var } N_2(t)}$$

$$\text{Var} = \text{Variance}$$

$$\eta_\theta/2 = \frac{\pi}{\lambda} d \sin \theta = \omega_0 \, \Delta t$$

$$-\frac{\pi}{2} \leqq \sin^{-1} \beta_2 \leqq \frac{\pi}{2}.$$

The Fourier series development of (2) follows from Middleton's[11] equation (9.33)

$$p_2(\eta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{l^n \Gamma^2(n/2 + 1)}{n!} \, {}_2F_1\left(\frac{n}{2}, \frac{n}{2}; n + 1; l^2\right) \cos n(\eta - \eta_\theta), \tag{3}$$

where ${}_2F_1$ is the Gaussian hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; x) \equiv 1 + \frac{\alpha\beta}{\gamma} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)} \frac{x^2}{2!} + \cdots$$

and Γ is the gamma function.

The expectations $E \cos \eta_i$ and $E \cos^2 \eta_i$ follow from (3)

$$E \cos \eta_i = \frac{\pi l}{4} \, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 2; l^2) \cos \eta_\theta \tag{4}$$

$$E \cos^2 \eta_i = \frac{1}{2} + \frac{l^2}{4} \, {}_2F_1(1, 1; 3; l^2) \cos 2\eta_\theta . \tag{5}$$

We shall see that the phase principle for measuring location or spectral shape of a discrete radio source is based on (4). Equation (4) should be compared with (4) of Ryle,[3] the equation which characterizes the output of a Ryle interferometer. Both equations are proportional to $\cos \eta_\theta$.

2.3 *Measurement of Location*

Let us first consider the problem of measuring the location of a discrete radio source whose true location is some small positive angle θ. From (4) we see that $E \cos \eta_i = 0$ when $\eta_\theta = \pi/2$ or $d \sin \theta = \lambda/4$. This suggests that we observe $\eta^\dagger$ and conclude that $d \sin \theta = \lambda/4$ when $\eta^\dagger \doteq 0$. How accurately can we form an estimate $\hat{\theta}$ of θ in this manner?

For η_θ near $\pi/2$ let the estimate $\hat{\eta}_\theta$ of η_θ be determined from the linear equation

$$\eta^\dagger = \frac{\pi l}{4} \, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 2; l^2)\left(\frac{\pi}{2} - \hat{\eta}_\theta\right). \tag{6}$$

Thus,

$$\text{Var } \hat{\eta}_\theta = (2\pi)^2 \left(\frac{d}{\lambda}\right)^2 \text{Var } \hat{\theta} = \left[\text{Var } \eta^{\dagger}\right]\left[\left(\frac{\pi l}{4}\right) {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 2; l^2)\right]^{-2} \tag{7}$$

or, in a more suitable form,

$$n\left(\frac{ad}{\lambda}\right)^2 \text{Var } \hat{\theta} = \frac{4}{\pi^4}(1 + a)^2 (E \cos^2 \eta_i) {}_2F_1^{-2}(\tfrac{1}{2}, \tfrac{1}{2}; 2; l^2), \tag{8}$$

where $E \cos^2 \eta_i$ is given by (5) with $\eta_\theta = \pi/2$. Equation (8) characterizes the theoretical accuracy associated with the phase principle for measuring the location of a discrete radio source and is plotted in Fig. 3.

2.4 *Measurement of Spectral Shape*

Now let us consider the problem of measuring the spectral shape of a discrete radio source located at $\theta = 0$. We shall assume that the variances of the background noises and receiver noises are invariant over the frequency region of interest. Under these conditions the estimate $\hat{a}$ of the signal-to-noise power ratio "a" can serve as an estimate of spectral shape by using the well-known frequency scan technique

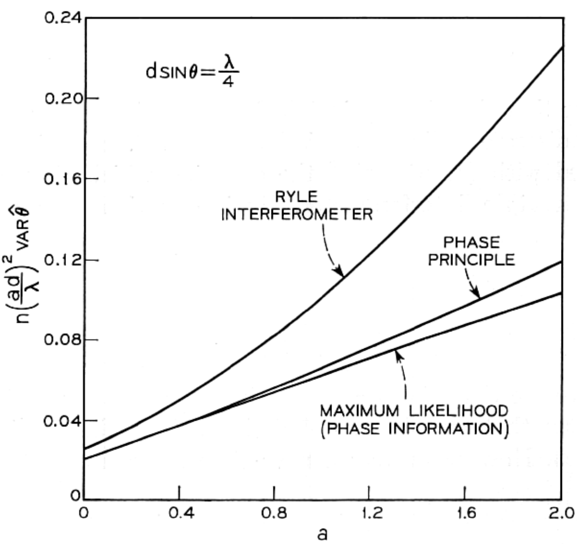

Fig. 3 — Theoretical accuracies of Ryle interferometer, phase principle, and maximum likelihood principle (using only phase information) for measuring the angular location of a discrete radio source.

indicated in Fig. 1. How accurately can we form an estimate $\hat{a}$ of "a" in this manner?

Equation (4) with $\theta = 0$ or $\eta_\theta = 0$ defines "a" as an implicit function of $E \cos \eta_i$ which we shall indicate by

$$a = H(E \cos \eta_i). \tag{9}$$

Equation (9) suggests that we form an estimate $\hat{a}$ of "a" from the equation

$$\hat{a} = H(\eta^\dagger). \tag{10}$$

A plot of (10) is presented in Fig. 4. This figure can be considered as a theoretical calibration curve. Notice that the theoretical calibration curve is independent of changes in the average receiver gain of each receiver. This is indeed unusual. One measures $\eta^\dagger$ and reports the corresponding value of $\hat{a}$. Assuming that Var N_1 and Var N_2 are invariant with frequency over the frequency range of interest, $\hat{a}$ will then trace out the spectral shape of the discrete radio source as the receivers scan in frequency. We shall now characterize the accuracy of the estimate $\hat{a}$.

For large n, the only case of interest in this paper, Cramér's[12] work shows that the estimate $\hat{a}$ is characterized, approximately, by a Gaus-

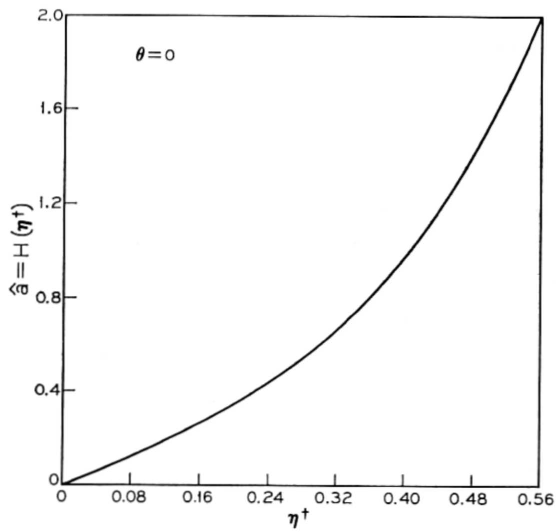

Fig. 4 — The theoretical calibration curve associated with the phase principle for measuring spectral shape of a discrete radio source. The receivers scan in frequency and â traces out the spectral shape.

sian probability density having the following expectation and variance:

$$E\hat{a} = H(E \cos \eta_i) + O(n^{-1}) \doteq a \tag{11}$$

$$\operatorname{Var} \hat{a} = H_1^2 \operatorname{Var} \eta^\dagger + O(n^{-\frac{3}{2}}) \doteq H_1^2 \operatorname{Var} \eta^\dagger, \tag{12}$$

where

$$H_1 = \frac{d\hat{a}}{d\eta^\dagger}\bigg|_{E \cos \eta_i} = \left[\frac{dE \cos \eta_i}{da}\bigg|_0\right]^{-1}$$

$$= \frac{4(1+a)^2}{\pi}\left[\frac{l^2}{4}\, {}_2F_1(\tfrac{3}{2}, \tfrac{3}{2}; 3; l^2) + {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 2; l^2)\right]^{-1}.$$

This last result follows by differentiating (4) with respect to "a", setting $\eta_\theta = 0$, and then taking its reciprocal. Equation (11) implies that the estimate $\hat{a}$ is essentially unbiased for large n.

From (12) and (1) we have

$$n \operatorname{Var} \hat{a} \doteq H_1^2(E \cos^2 \eta_i - E^2 \cos \eta_i), \tag{13}$$

where $E \cos \eta_i$ and $E \cos^2 \eta_i$ are given by (4) and (5) with $\eta_\theta = 0$.

Equation (13) characterizes the theoretical accuracy associated with the phase principle for measuring the signal-to-noise power ratio "a" or the spectral shape of the discrete radio source and is plotted in Fig. 5.

III. MEASUREMENTS BASED ON THE RYLE INTERFEROMETER OR DICKE RADIOMETER

3.1 *Measurement of Location*

When θ is small and $\eta_\theta = \pi/2$ or $d \sin \theta = \lambda/4$, the theoretical accuracy associated with the Ryle interferometer for measuring the location of the discrete radio source was derived by Manasse.[13] In our notation Manasse's[13] (60) becomes

$$n\left(\frac{ad}{\lambda}\right)^2 \operatorname{Var} \hat{\theta} = (2\pi)^{-2}(1+a)^2. \tag{14}$$

Equation (14) characterizes the theoretical accuracy associated with the Ryle interferometer for measuring the location of the discrete radio source and is plotted in Fig. 3.

3.2 *Measurement of Spectral Shape*

One can measure the spectral shape of the discrete radio source located at $\theta = 0$ by using the Ryle[3] interferometer or the Dicke[2] radi-

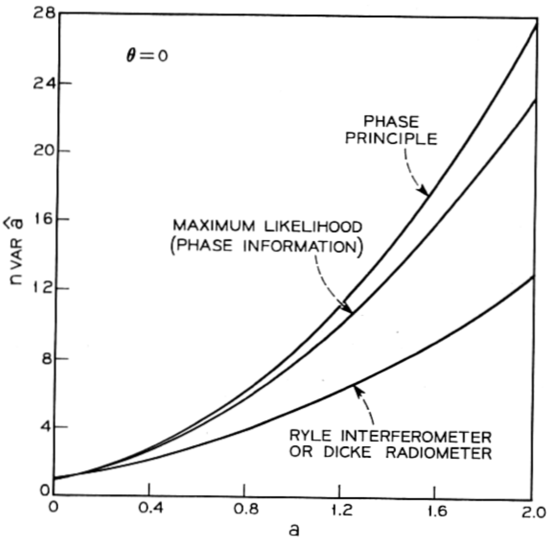

Fig. 5 — Theoretical accuracies of Ryle interferometer, Dicke radiometer, phase principle, and maximum likelihood principle (using only phase information) for measuring the spectral shape of a discrete radio source. Both the Ryle interferometer and the Dicke radiometer require some amplitude information.

ometer. In fact, these methods are at present the accepted methods. We shall go on to characterize the accuracy associated with these methods of measuring spectral shape.

In order to simplify the notation of this section, let $P_s = \text{Var } S(t)$ and $P_N = \text{Var } N_1(t) = \text{Var } N_2(t)$. Then,

$$a = \frac{P_s}{P_N} \quad \text{and} \quad \hat{a} = \frac{\hat{P}_s}{P_N}. \tag{15}$$

$\hat{P}_s$ denotes the unbiased estimate of P_s and P_N is regarded as a parameter.

The Ryle[3] interferometer utilizes a phase reversing switch to produce, periodically, the following inputs to a square-law detector

$$\frac{2S(t) + N_1(t) + N_2(t)}{2} \tag{16}$$

or

$$\frac{N_1(t) + N_2(t)}{2}. \tag{17}$$

Thus, using some of Rice's[14] results, the mean values at the output of the square-law detectors are, periodically,

$$\frac{2P_s + P_N}{2} \tag{18}$$

or

$$\frac{P_N}{2}. \tag{19}$$

The difference in the outputs of the square-law detector is taken as the unbiased estimate $\hat{P}_s$. Thus, $E\hat{P}_s = P_s$, and $E\hat{a} = a$.

Also, using some of Rice's[14] results, the variances at the output of the square-law detector are, periodically,

$$\frac{[2P_s + P_N]^2}{4(n/2)} \tag{20}$$

or

$$\frac{[P_N]^2}{4(n/2)}. \tag{21}$$

The factor $n/2$ appears because the output is in either position only one half of the time. Since the difference in the outputs of the square-law detector is taken as the unbiased estimate $\hat{P}_s$, the variance of $\hat{P}_s$ is given by the sum of expressions (20) and (21):

$$\text{Var } \hat{P}_s = \frac{[2P_s + P_N]^2 + [P_N]^2}{2n} \tag{22}$$

or

$$n \text{ Var } \hat{a} = nP_N^{-2} \text{ Var } \hat{P}_s = 2a^2 + 2a + 1. \tag{23}$$

Notice that the Ryle interferometer can be considered as a Dicke[2] radiometer switching between the two inputs given by expressions (16) and (17). Thus, (23) characterizes the accuracy of both the Ryle interferometer and the Dicke radiometer for measuring the spectral shape of the discrete radio source. Equation (23) is plotted in Fig. 5.

IV. MEASUREMENTS BASED ON THE MAXIMUM LIKELIHOOD PRINCIPLE USING ONLY PHASE INFORMATION

4.1 *Measurement of Location*

If one uses the maximum likelihood principle[12] to process a large number n of independent samples of the phase difference η_i in order to estimate the location of the discrete radio source when $\eta_\theta = \pi/2$ or $d \sin \theta = \lambda/4$, one finds

$$n\left(\frac{ad}{\lambda}\right)^2 \text{Var } \hat{\theta} = \frac{n}{(2\pi)^2} a^2 \text{ Var } \hat{\eta}_\theta$$
$$= \frac{1}{(2\pi)^2}\left\{\int_{-\pi}^{\pi}\left[\frac{1}{p_2}\left(\frac{1}{a}\frac{\partial p_2}{\partial \eta_\theta}\right)^2\right]_{\pi/2} d\eta\right\}^{-1}, \tag{24}$$

where p_2 is given by (2) and the integrand of the integral in equation (24) is to be evaluated at $\eta_\theta = \pi/2$. For various values of "a", the definite integral appearing in (24) was evaluated numerically by using a digital computer and Simpson's rule. The resulting curve is plotted in Fig. 3. Incidentally, this curve applies for all values of η_θ.

As $a \to 0$ we find that (8) and (24) both yield

$$\lim_{a\to 0}\left[n\left(\frac{ad}{\lambda}\right)^2 \text{Var } \hat{\theta}\right] = \frac{2}{\pi^4} \doteq 0.02053. \tag{25}$$

Thus, as $a \to 0$, the phase principle and the maximum likelihood principle using only phase information are essentially equivalent.

4.2 *Measurement of Spectral Shape*

If one uses the maximum likelihood principle[12] to process a large number n of independent samples of the phase difference η_i in order to estimate the signal-to-noise power ratio "a" or the spectral shape of a discrete radio source when $\theta = 0$, one finds

$$n \text{ Var } \hat{a} = \left\{2\int_0^{\pi}\frac{1}{p_2}\left[\frac{\partial p_2}{\partial a}\right]^2 d\eta\right\}^{-1}, \tag{26}$$

where p_2 is given by equation (2) with $\eta_\theta = 0$. For various values of "a", the definite integral appearing in (26) was evaluated numerically by using a digital computer and Simpson's rule. The resulting curve is plotted in Fig. 5. This curve also applies for all values of η_θ.

As $a \to 0$ we find that (13) and (26) both yield

$$\lim_{a\to 0}[n \text{ Var } \hat{a}] = \frac{8}{\pi^2} \doteq 0.81057. \tag{27}$$

Thus, as $a \to 0$ the phase principle and the maximum likelihood principle using only phase information are essentially equivalent.

V. COMPARISONS OF THEORETICAL ACCURACIES

5.1 *Measurements of Location*

When using the Ryle interferometer to measure the location of a weak, discrete radio source located at $d \sin \theta = \lambda/4$, we have, from (14),

$$\lim_{a \to 0} \left[n \left(\frac{ad}{\lambda} \right)^2 \text{Var } \hat{\theta} \right] = \frac{1}{(2\pi)^2} \doteq 0.02533. \tag{28}$$

Whereas, when using the phase principle or the maximum likelihood principle to measure the location, we have, from (25),

$$\lim_{a \to 0} \left[n \left(\frac{ad}{\lambda} \right)^2 \text{Var } \hat{\theta} \right] = \frac{2}{\pi^4} \doteq 0.02053. \tag{29}$$

Thus, the phase principle and the maximum likelihood principle are essentially equivalent, and they are both slightly more accurate than the Ryle interferometer.

See Fig. 3 for a comparison of the theoretical accuracies at other values of "a".

5.2 *Measurements of Spectral Shape*

When using the Ryle interferometer or the Dicke radiometer to measure the signal-to-noise power ratio "a" or the spectral shape of a weak, discrete radio source located at $\theta = 0$, we have, from (23),

$$\lim_{a \to 0} [n \text{ Var } \hat{a}] = 1. \tag{30}$$

Whereas, when using the phase principle or the maximum likelihood principle to measure the signal-to-noise power ratio "a" or the spectral shape, we have, from (27),

$$\lim_{a \to 0} [n \text{ Var } \hat{a}] = \frac{8}{\pi^2} \doteq 0.81057. \tag{31}$$

Again, the phase principle and the maximum likelihood principle are essentially equivalent, and they are both slightly more accurate than either the Ryle interferometer or the Dicke radiometer.

See Fig. 5 for a comparison of the theoretical accuracies at other values of "a".

For values of "a" away from zero, Fig. 5 shows that the Ryle interferometer or Dicke radiometer are more accurate than the maximum likelihood principle using only phase information. Thus, one must conclude that the Ryle interferometer or the Dicke radiometer require some amplitude information. Consequently, their accuracy is subject to deterioration by gain variations.

Notice that (29) divided by (28) equals $8/\pi^2$, and (31) divided by (30) also equals $8/\pi^2$. Thus, for measuring the location or the spectral shape of a weak, discrete radio source, Var $\hat{\theta}$ and Var $\hat{a}$ associated with the phase principle are lower, by the same factor $8/\pi^2$, than the corresponding variances associated with the Ryle interferometer.

VI. CONCLUSIONS

For measuring the location or the spectral shape of a discrete radio source, the phase principle leads to a measurement which is insensitive to receiver gain fluctuations.

For measuring the location or the spectral shape of a weak, discrete radio source, the accuracy associated with the phase principle is slightly better than the accuracy associated with the Ryle interferometer or the Dicke radiometer. Also, the accuracy associated with the phase principle is essentially equal to the accuracy associated with the maximum likelihood principle using only phase information.

The phase principle is relatively simple to implement, and the implementation doesn't require input switching.

The calibration curve associated with the phase principle is independent of changes in the average receiver gain of each receiver. The two receivers need not have the same average gain.

An unusually long observation time is advantageous when using the phase principle.

For measuring spectral shape, both the Ryle interferometer and the Dicke radiometer require some amplitude information. Consequently, their accuracy is subject to deterioration by gain variations.

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