B. S. T. J. BRIEFS

Approximate and Exact Results Concerning Zeros of Gaussian Noise

By A. J. RAINAL

I. INTRODUCTION

Let τ denote the interval between two successive zeros of a stationary gaussian process having zero-mean and one-sided power spectral density $W(f)$. We shall refer to such an interval as a zero-crossing interval. This brief is concerned with these probability functions:

(*i*) $P_o(\tau)$ = Probability density of a zero-crossing interval.

(*ii*) $F_o(\tau)$ = Probability that a zero-crossing interval lasts longer than τ.

Thus, $F_o(\tau)$ and $P_o(\tau)$ are related by

$$F_o(\tau) = \int_\tau^\infty P_o(x)\,dx = 1 - \int_0^\tau P_o(x)\,dx. \tag{1}$$

An exact, explicit solution for $P_o(\tau)$ or $F_o(\tau)$ in terms of arbitrary $W(f)$ is at present unknown.

In a very interesting paper, E. Wong[1] presented exact, explicit solutions for both $P_o(\tau)$ and $F_o(\tau)$ for the special case when

$$W(f) = \frac{16\sqrt{3}/3}{(\omega^2 + 3)(\omega^2 + \frac{1}{3})} \tag{2}$$

where $\omega = 2\pi f$.

The corresponding autocorrelation function $\rho(\tau)$ is given by

$$\rho(\tau) = \int_0^\infty W(f)\cos 2\pi f\tau\,df = \tfrac{3}{2}e^{-|\tau|/\sqrt{3}}\left(1 - \tfrac{1}{3}e^{-2|\tau|/\sqrt{3}}\right). \tag{3}$$

Wong's exact, explicit solutions are in terms of complete elliptic integrals, and they stemmed from a recent result in the theory of Brownian motion.

The purpose of this brief is to compare Wong's exact results with the approximate results of McFadden.[2] McFadden's approximate results stem from the numerical solution of an integral equation, and they are based on the assumption of "quasi-independence" which assumes that a given zero-crossing interval is statistically independent of the sum of the previous $(2m+2)$ zero-crossing intervals for all non-negative integral m.

We shall see that McFadden's approximate results compare well with Wong's exact results over a significant range of τ.

II. COMPARISON OF APPROXIMATE AND EXACT RESULTS

Figure 1 compares McFadden's approximate result $\hat{P}_o(\tau)$ with Wong's exact result $P_o(\tau)$. The exact first moment of $P_o(\tau)$ follows from Rice's work[3] and is indicated in Figure 1 as $E(\tau) = \pi$. Thus, the approximate and exact results for $P_o(\tau)$ compare well over a significant range of τ.

Figure 2 compares McFadden's approximate result $\hat{F}_o(\tau)$ with Wong's exact result $F_o(\tau)$. From Wong's equation 31 we have that as $\tau \to \infty$, $F_o(\tau) \sim Ce^{-\tau/(2\sqrt{3})}$ where C is a known constant. The semilog plot in Figure 2 shows this asymptotic exponential decay of $F_o(\tau)$.

III. CONCLUSION

McFadden's approximate results $\hat{P}_o(\tau)$, $\hat{F}_o(\tau)$ compare well with Wong's exact results $P_o(\tau)$, $F_o(\tau)$ over a significant range of τ.

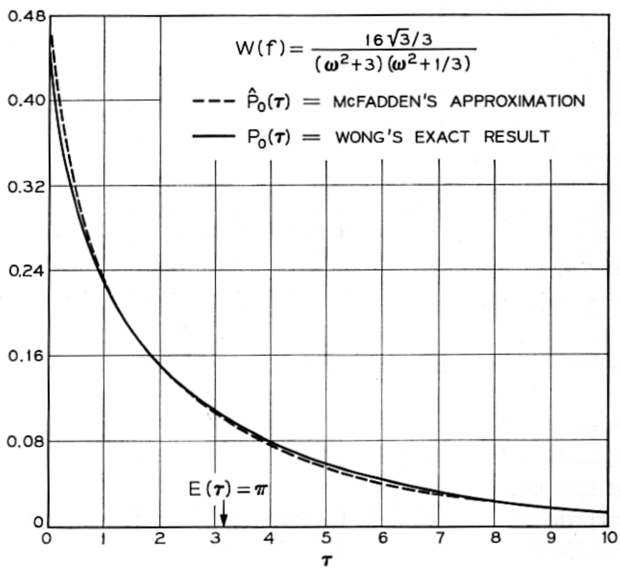

Fig. 1 — Comparison of approximate and exact results for $P_o(\tau)$, the probability density of a zero-crossing interval of gaussian noise having power spectral density $W(f)$.

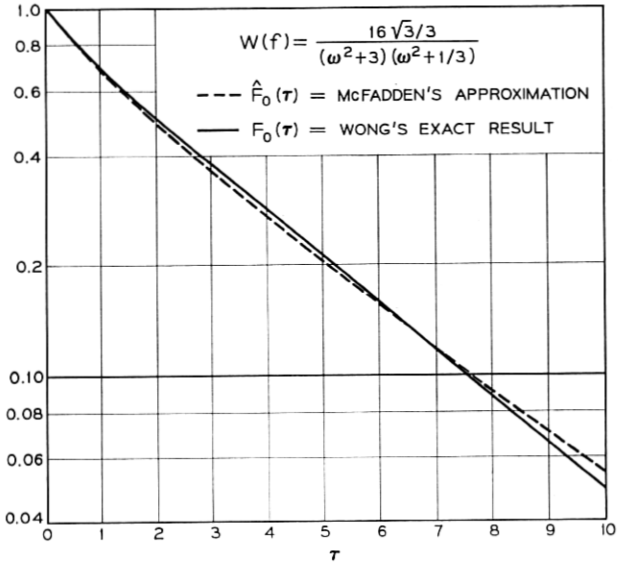

Fig. 2 — Comparison of approximate and exact results for $F_o(\tau)$, the probability that a zero-crossing interval lasts longer than τ.

IV. ACKNOWLEDGMENTS

The author wishes to thank S. O. Rice for suggesting the publication of this information. The author is indebted to Miss A. T. Seery for programming a digital computer to produce the figures.

REFERENCES

1. Wong, E., "Some Results Concerning the Zero-Crossings of Gaussian Noise," SIAM J. Appl. Math., *14*, No. 6 (November 1966), pp. 1246–1254.
2. McFadden, J. A., "The Axis-Crossing Intervals of Random Functions—II," IRE Trans. Inform. Theory, *IT-4* (March 1958), pp. 14–24.
3. Rice, S. O., "Distribution of the Duration of Fades in Radio Transmission," B.S.T.J., *37* (May 1958), pp. 581–635.

Erratum

On page 205 of the February 1968 *Bell System Technical Journal,* the drawings of Figs. 10 and 11 were inadvertently transposed. Fig. 10 is the drawing with the gate electrode marked −100V, and Fig. 11 is the drawing with the gate electrode marked +100V.