An Upper Bound on the Zero-Crossing Distribution*

By NICHOLAS A. STRAKHOV and LUDWIK KURZ†

Let $Q(T)$ equal the probability that a random process, $x(t)$, does not cross the zero axis in a given interval of length T. A family of upper bounds for $Q(T)$ is derived with only weak restrictions imposed on $x(t)$ and it is shown that for gaussian random processes only one member of the family provides useful formulae. Specific results are obtained for $x(t)$ representing a number of interesting random processes.

I. INTRODUCTION

Let $Q(T)$ equal the probability that a random process, $x(t)$, does not cross the zero axis in a given interval of length T. The problem of determining $Q(T)$ (and related functions) has important applications in communications theory and has been investigated by many authors.[1–6] Reference 5 gives an extensive bibliography of most of the related work on this subject prior to 1962. Despite all this effort, $Q(T)$ is known only when $x(t)$ is a simple nongaussian process (such as a process whose zero-crossings obey the Poisson distribution) or a stationary gaussian zero-mean process with one of four explicit correlation functions.[5, 6] Most of the rest of the results obtained are either approximate or form upper or lower bounds.[5]

In this paper, we develop a whole family of upper bounds on $Q(T)$. For computational purposes, however, only one member of the family has been found to provide useful results for most cases of interest.

II. DERIVATION OF AN UPPER BOUND ON Q(T)

Consider the transformation

$$z_T = \frac{1}{T}\int_0^T \operatorname{sgn}\,[x(t)]\,dt \tag{1}$$

* An abbreviated version of this paper was presented at the Fifth Allerton Conference on System and Circuit Theory, Monticello, Ill., October 4, 1967.

† New York University.

where $x(t)$ is a sample function of a stochastic process,* T is a fixed observation interval, and z_T is a random variable defined by the stochastic integral (1). The function $sgn[x(t)]$ is defined as

$$\mathrm{sgn}\,[x] = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Since z_T is a random variable, it has a cumulative distribution function, $P(z_T)$, associated with it. From (1), two properties of $P(z_T)$ are immediately apparent, regardless of the statistics governing $x(t)$:

(*i*) $$P(z_T) = 0 \quad \text{for} \quad z_T < -1$$

and (2)

$$P(z_T) = 1 \quad \text{for} \quad z_T > 1$$

(*ii*) $$\lim_{\epsilon \to 0} [P(1 + \epsilon) - P(1 - \epsilon)] = Q_U(T) \tag{3}$$

and

$$\lim_{\epsilon \to 0} [P(-1 + \epsilon) - P(-1 - \epsilon)] = Q_L(T) \tag{4}$$

where

$$Q_U(T) = \mathrm{Prob}\,\{x(t) \geqq 0 \quad \text{for} \quad 0 \leqq t \leqq T\} \tag{5}$$

and

$$Q_L(T) = \mathrm{Prob}\,\{x(t) \leqq 0 \quad \text{for} \quad 0 \leqq t \leqq T\}. \tag{6}$$

Obviously, $Q(T)$ as defined previously is related to the last two quantities by

$$Q(T) = Q_U(T) + Q_L(T). \tag{7}$$

If $x(t)$ is a symmetric† process, then

$$Q_U(T) = Q_L(T) = \tfrac{1}{2}Q(T). \tag{8}$$

As a consequence of properties (*i*) and (*ii*), $P(z_T)$ can be represented by

$$P(z_T) = G(z_T) + Q_L(T)u(z_T + 1) + Q_U(T)u(z_T - 1) \tag{9}$$

---

* Throughout this paper, we assume that almost all sample functions of the stochastic pocess are continuous. Thus, (1), (5), and (6) are well defined.

† The stochastic process $x(t)$ will be called symmetric if the probability measures that govern it also govern the process $-x(t)$.

where $G(z_T)$ is continuous at $z_T = \pm 1$ and

$$u(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geqq 0. \end{cases}$$

We assume throughout that the function $G(z_T)$ is not identically equal to zero. If it were, then it would be easy to show that $Q(T)$ is known exactly; that is, $Q(T) = 1$.

Next, consider the even-order moments of $P(z_T)$, denoted by the Stieltjes integral

$$q_{2k} = \int_{-1}^{1} z_T^{2k}\, dP(z_T) \qquad k = 0, 1, 2, \cdots . \tag{10}$$

By substituting (9) into (10) one obtains

$$q_{2k} = \int_{-1}^{1} z_T^{2k}\, dG(z_T) + Q_U(T) + Q_L(T) \qquad k = 0, 1, 2, \cdots . \tag{11}$$

Neglecting the first term in the right side of (11) (which is always positive) and taking (7) into account leads to a family of upper bounds for $Q(T)$ expressed by

$$Q(T) \leqq q_{2k} \qquad k = 0, 1, 2, \cdots . \tag{12}$$

For $k = 0$, (12) reduces to the obvious result

$$Q(T) \leqq 1.$$

Before discussing the usefuless of the inequality (12), an expression for the moments will be derived.

From its definition, (10), q_{2k} can be expressed as

$$q_{2k} = E\{z_T^{2k}\}$$

where $E\{\cdot\}$ denotes the expected value of the quantity enclosed in braces. Substitution for z_T from (1) results in

$$q_{2k} = \frac{1}{T^{2k}} E\left\{\int_0^T \int_0^T \cdots \int_0^T y(t_1)y(t_2) \cdots y(t_{2k})\, dt_1\, dt_2 \cdots dt_{2k}\right\} \tag{13}$$

where

$$y(t_i) = \operatorname{sgn}[x(t_i)] \qquad i = 1, 2, \cdots, 2k.$$

Interchanging the order of integration and expectation yields

$$q_{2k} = \frac{1}{T^{2k}} \int_0^T \int_0^T \cdots \int_0^T R(t_1, t_2, \cdots, t_{2k})\, dt_1\, dt_2 \cdots dt_{2k} \tag{14}$$

where

$$R(t_1, t_2, \cdots, t_{2k}) = E\{y(t_1)y(t_2) \cdots y(t_{2k})\}. \tag{15}$$

We now make some remarks concerning the ordering of the family of inequalities expressed by (12).

Denote the first term in the right side of (11) by s_{2k}, or

$$s_{2k} = \int_{-1}^{1} z_T^{2k} \, dG(z_T). \tag{16}$$

We next establish that $s_{2k} > s_{2k+2}$ $(k = 0, 1, 2, \ldots)$ which in turn establishes

$$1 > q_2 > q_4 \cdots > Q(T). \tag{17}$$

The former inequality follows directly from

$$\begin{aligned} s_{2k+2} &= \int_{-1}^{1} z^2 z^{2k} \, dG(z_T) \\ &\leqq \int_{-1}^{1} z^{2k} \, dG(z_T) \\ &= s_{2k} \end{aligned}$$

with equality if, and only if, $G(z_T)$ is of the form

$$G(z_T) = Au(z_T + 1) + Bu(z_T - 1). \tag{18}$$

Since $G(z_T)$ is continuous at $z_T = \pm 1$, equality is not possible and therefore

$$s_{2k+2} < s_{2k}$$

which, together with (11), establishes (17). We next establish the readily proven fact that

$$\lim_{k \to \infty} s_{2k} < \epsilon, \qquad \epsilon > 0 \tag{19}$$

and therefore,

$$\lim_{k \to \infty} q_{2k} = Q(T).$$

We begin by choosing an $\alpha(k_0) > 0$ such that

$$\int_{-1}^{-1+\alpha(k_0)} z_T^{2k_0} \, dG(z_T) < \frac{\epsilon}{2}, \qquad \epsilon > 0.$$

This can always be done because $G(z_T)$ is continuous at $z_T = -1$. Using the definition of s_{2k}, (16), obvious symmetry properties, and the fact that

$$\int_{-1}^{-1+\alpha(k_0)} z_T^{2k}\, dG(z_T) < \int_{-1}^{-1+\alpha(k_0)} z_T^{2k_0}\, dG(z_T)$$

for each $k > k_0$, it follows immediately that

$$\lim_{k\to\infty} \int_{-1}^{1} z_T^{2k}\, dG(z_T) < \epsilon + \lim_{k\to\infty} \int_{-1+\alpha(k_0)}^{1-\alpha(k_0)} z_T^{2k}\, dG(z_T).$$

Since the sequence of functions $\{z_T^{2k}\}$, $k = 0, 1, 2, \cdots$ is uniformly convergent to zero on the interval $[-1 + \alpha(k_0), 1 - \alpha(k_0)]$, the limit and the integral may be interchanged yielding (19).

In light of (17) and (19), it appears that (12) should be evaluated for as large a value of k as possible. For the special case when $x(t)$ is a stationary gaussian random process (assumed to be zero-mean without loss of generality), it does not seem to be possible to evaluate q_{2k} for $k > 1$ as evidenced by the following discussion.

As shown by McFadden,[7] the quantity $R(t_1, t_2, \cdots, t_n)$, defined in (15), is equal to the sum of some simple terms plus a quantity $P_n(\mathbf{r})$, which is defined as

$$P_n(\mathbf{r}) = (2\pi)^{-n/2} |\mathbf{r}|^{-\frac{1}{2}} \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \exp\left[-\frac{1}{2} \sum_{i,j}^{n} r_{ij}^{-1} x_i x_j\right]$$

where $\mathbf{r}$ is a covariance matrix with elements

$r_{ij} = r(t_i - t_j) = E\{x(t_i)x(t_j)\}, \qquad i, j = 1, \cdots, n$

$|\mathbf{r}|$ is the determinant of $\mathbf{r}$

$\sum_{i,j}^{n} r_{ij}^{-1} x_i x_j$ is the quadratic form associated with the inverse of $\mathbf{r}$

and

$$x_i = x(t_i), \qquad i = 1, 2, \cdots, n$$

In other words, $P_n(\mathbf{r})$ is the probability that the n jointly distributed gaussian random variables, $x(t_i)$ $(i = 1, \cdots, n)$ are all positive.

As discussed by McFadden,[7] and even more thoroughly by Slepian,[5] expressions for $P_n(\mathbf{r})$ have not been obtained in terms of elementary functions for $n > 3$. Because of this fact, it seems unlikely that an expression for (14) with $k > 1$ can be obtained for a general gaussian process, $x(t)$. It should be pointed out, however, that an expression for q_4 has been derived[8] for $\rho(\tau) = \exp(-|\tau|)$, but without first evaluating $P_4(\mathbf{r})$. This result is not included because, for this correlation function, $Q(T)$ is known exactly.[5]

III. APPLICATION TO GAUSSIAN RANDOM PROCESS

Assume that $x(t)$ in (1) is a stationary, zero-mean, gaussian random process, normalized so that $\rho(0) = 1$ where $\rho(\tau) = E\{x(t)x(t+\tau)\}$. The relationship (14) will be evaluated for the case $k = 1$, that is, for

$$q_2 = \frac{1}{T^2} \int_0^T \int_0^T R(t_1, t_2)\, dt_1\, dt_2 \tag{20}$$

where

$$R(t_1, t_2) = E\{\operatorname{sgn} x(t_1) \operatorname{sgn} x(t_2)\}. \tag{21}$$

The latter expression has been evaluated by many authors (see page 58 of Lawson and Uhlenbeck's book,[9] for example) and the result is

$$R(t_1, t_2) = \frac{2}{\pi} \sin^{-1} [\rho(t_1 - t_2)]. \tag{22}$$

Substituting (22) into (20) and making the obvious simplifications in integration results in

$$q_2 = \frac{4}{\pi} \int_0^1 (1 - u) \sin^{-1} [\rho(Tu)]\, du,$$

or, in light of (12),

$$Q(T) \leqq \frac{4}{\pi} \int_0^1 (1 - u) \sin^{-1} [\rho(Tu)]\, du. \tag{23}$$

This result has been obtained by Slepian,[5] who states it as Theorem 5.* Slepian's proof, however, is long and complicated, as opposed to the simplicity of the proof given here. Furthermore, extensions to other cases can be obtained using the new method.

IV. APPLICATION TO SINE WAVE PLUS GAUSSIAN RANDOM PROCESS

We now turn to applying (12) with $k = 1$ to the case where

$$x(t) = w(t) + A \cos (2\pi ft + \varphi) \tag{24}$$

where

- $w(t)$ is a stationary, zero-mean, gaussian random process with normalized correlation function, $\rho(\tau)$,
- φ is a random phase constant uniformly distributed on $[0, 2\pi]$,
- f is the sine-wave frequency, and
- A is the sine-wave amplitude.

* Notice that Slepian's $P[T, r(\tau)]$ equals one half of our $Q(T)$.

As derived in the next subsection, the result obtained is

$$Q(T) \leqq q_2^{(1)} + q_2^{(2)} \tag{25}$$

where

$$q_2^{(2)} = 2 \int_0^1 (1 - u)H(uT)\, du \tag{26}$$

with

$$H(uT) = \frac{2}{\pi} \int_0^{\sin^{-1} \rho(uT)} \exp\left\{-\frac{A^2}{2} \frac{1 - \sin\theta \cos 2\pi bu}{\cos^2\theta}\right\} \cdot I_o\left\{\frac{A^2}{2} \frac{\sin\theta - \cos 2\pi bu}{\cos^2\theta}\right\} d\theta$$

$$b = fT \tag{27}$$

and $I_o(x)$ = modified Bessel function of the first kind, zero order.

The expression given below for $q_2^{(1)}$ is approximate, except for $T = k/f$, $k = 0, 1, 2, \cdots$ where it is exact and consequently the function is most accurate in a neighborhood of these points. In addition, the accuracy of the approximation improves as A increases. For small A, where the approximation is least accurate, (26) dominates $q_2^{(1)}$ and so very little error results in the upper bound, (25) for all values of A. The expression for $q_2^{(1)}$ is given by

$$q_2^{(1)} \cong \frac{2}{b^2} \times \begin{cases} \int_0^{b-n} (b - n - v)S(v)\, dv, & n \leqq b < n + \frac{1}{4} \\ -\int_{n+\frac{1}{2}-b}^{\frac{1}{4}} (b - n - \frac{1}{2} + v)S(v)\, dv + \int_0^{\frac{1}{4}} (b - n - v)S(v)\, dv, & n + \frac{1}{4} \leqq b < n + \frac{1}{2} \\ -\int_0^{b-n-\frac{1}{2}} (b - n - \frac{1}{2} - v)S(v)\, dv + 2\int_0^{\frac{1}{4}} (\frac{1}{4} - v)S(v)\, dv, & n + \frac{1}{2} \leqq b < n + \frac{3}{4} \\ \int_{n+1-b}^{\frac{1}{4}} (b - n - 1 + v)S(v)\, dv - \int_0^{\frac{1}{4}} (b - n - 1 + v)S(v)\, dv, & n + \frac{3}{4} \leqq b < n + 1 \end{cases} \tag{28}$$

where

$$n = 0, 1, 2, \cdots$$

$$S(v) = (1 - 4v) - \frac{e^{-A^2/2}}{\pi} G(v) \tag{29}$$

$$G(v) = \int_0^{\pi - 2\pi v} e^{-K(v) \cos x} \, dx - \int_0^{2\pi v} e^{K(v) \cos x} \, dx \tag{30}$$

and

$$K(v) = \frac{A^2}{2} \cos 2\pi v. \tag{31}$$

While (28) appears to be a formidable equation, it turns out to be easily computed, partly because $S(v)$ does not depend on b and hence needs only to be computed once for each value of A. The expression (26), on the other hand, turns out to be time-consuming to compute, particularly for large values of b.

4.1 *Derivation of Upper Bound, Given by (25)*

The expression for q_2, (14), with $x(t)$ specified by (24) is

$$q_2 = \frac{1}{T^2} \int_0^T \int_0^T R(t_1, t_2) \, dt_1 \, dt_2 \tag{32}$$

with $R(t_1, t_2)$ given by (15). Notice that the expectation in this case ranges over the three random variables, $w(t_1)$, $w(t_2)$ and φ. For convenience, define

$$R(t_1, t_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} r(t_1, t_2) \, d\varphi \tag{33}$$

where

$$r(t_1, t_2) = E \{\text{sgn} [w_1 + a_1] \text{ sgn} [w_2 + a_2]\} \tag{34}$$

and

$$w_i = w(t_i)$$
$$a_i = A \cos (2\pi f t_i + \varphi) \qquad i = 1, 2.$$

The latter expectation is with respect to w_1 and w_2 only. Writing out (34) in terms of the definition of $E\{\cdot\}$ results in

$$r(t_1, t_2) = \frac{1}{2\pi[1 - \rho^2(\tau)]^{\frac{1}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sgn} [w_1 + a_1] \text{ sgn} [w_2 + a_2] \cdot \exp \left\{ - \frac{w_1^2 + w_2^2 - 2\rho(\tau) w_1 w_2}{2[1 - \rho^2(\tau)]} \right\} dw_1 \, dw_2 \tag{35}$$

where $\tau = t_2 - t_1$.

Applying Price's theorem,[10] to this equation results in

$$\frac{\partial r(t_1, t_2)}{\partial \rho(\tau)} = \frac{2}{\pi[1 - \rho^2(\tau)]^{\frac{1}{2}}} \exp\left\{-\frac{a_1^2 + a_2^2 - 2\rho(\tau)a_1a_2}{2[1 - \rho^2(\tau)]}\right\}. \tag{36}$$

Integrating (36) and applying the appropriate boundary condition yields

$$r(t_1, t_2) = r(t_1, t_2)|_{\rho(\tau)=0} + \int_0^{\rho(\tau)} \frac{\partial r(t_1, t_2)}{\partial \rho(\tau)}\, d\rho$$

or,

$$r(t_1, t_2) = 4 \operatorname{erf}(a_1) \operatorname{erf}(a_2) + \frac{2}{\pi}\int_0^{\rho(\tau)} \frac{1}{(1-\alpha^2)^{\frac{1}{2}}} \exp\left\{-\frac{a_1^2 + a_2^2 - 2\alpha a_1 a_2}{2[1-\alpha^2]}\right\} d\alpha \tag{37}$$

where

$$\operatorname{erf}(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^x e^{-y^2/2}\, dy.$$

As a result of the natural separation of (37) into the sum of two quantities, define

$$q_2 = q_2^{(1)} + q_2^{(2)} \tag{38}$$

where, by substituting (33) and (37) into (32), the terms in (38) may be defined as

$$q_2^{(1)} = \frac{2}{\pi T^2}\int_0^T \int_0^T \int_{-\pi}^{\pi} \operatorname{erf}(a_1) \operatorname{erf}(a_2)\, d\varphi\, dt_1\, dt_2 \tag{39}$$

$$q_2^{(2)} = \frac{1}{\pi^2 T^2}\int_0^T \int_0^T \int_{-\pi}^{\pi} \int_0^{\rho(\tau)} \frac{1}{(1-\alpha^2)^{\frac{1}{2}}} \cdot \exp\left\{-\frac{a_1^2 + a_2^2 - 2\alpha a_1 a_2}{1-\alpha^2}\right\} d\alpha\, d\varphi\, dt_1\, dt_2 . \tag{40}$$

The detailed steps of simplifying (39) and (40) are relegated to Appendices A and B, respectively. In Appendix A we discuss the nature of the approximation made in arriving at (28).

Before applying the results just obtained to specific situations, a power series representation for (32) will be given. The series may be derived from (39) and (40) by expanding the integrands of these functions in their respective Taylor series, evaluating the resulting terms and adding the expansions for (39) and (40) together. This procedure results in

$$q_2 = \frac{4}{\pi}\left\{\int_0^1 (1 - u)\sin^{-1}\rho(Tu)\,du + \frac{A^2}{2}\int_0^1 \frac{(1 - u)[\cos 2\pi f Tu - \rho(Tu)]}{\sqrt{1 - \rho^2(u)}}\,du + 0(A^4)\right\}.$$

4.2 *Numerical Results and Comparisons*

We evaluated the inequality (25) with the aid of a digital computer. For the first case considered, $\rho(\tau) = e^{-|\tau|}$, $f = 2$ Hz, T ranged between 0 and 3.5 seconds and A, the sine-wave amplitude, was either 1 or 10. The results of these computations are plotted in Fig. 1.

Let us first discuss the $A = 10$ case. The quantity $q_2^{(1)}(T)$ exhibits a damped oscillatory behavior much like a plot of $[\sin(\pi fT)/(\pi fT)]^2$ while the quantity $q_2^{(2)}(T)$ decays toward zero quite smoothly. For

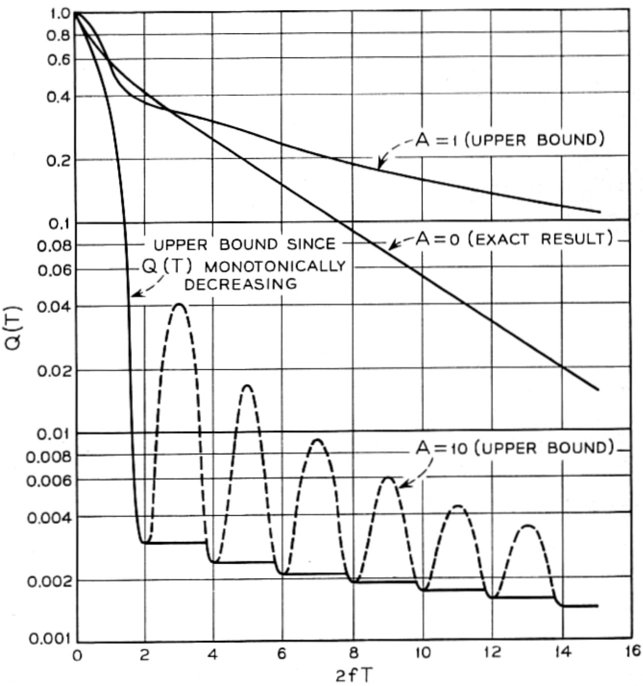

Fig. 1 — Upper bound of $Q(T)$ for zero crossings of sine wave plus gaussian noise.

large A, the former term dominates except at its zeros located at $T = k/f$, $k = 1, 2, \cdots$. The values of $q_2(T)$ are shown as dotted lines on Fig. 1. Since $Q(T)$ is a monotonically decreasing function, its upper bound can be constructed by drawing horizontal lines between the local minima of $q_2(T)$ and the first intersection of this line with $q_2(T)$ to the right of the minimum. This accounts for the step-like curve drawn in Fig. 1 representing the upper bound for $Q(T)$ with $A = 10$.

The curve representing $A = 1$, while not exhibiting as fast a decay as the curve for $A = 10$, shows some interesting features. As contrasted to the last case, the $q_2^{(2)}(T)$ term dominates the $q_2^{(1)}(T)$ term and consequently much of the oscillatory behavior noted earlier has disappeared.

Another interesting observation can be made when the $A = 1$ curve is compared with the $A = 0$ curve (gaussian noise alone) for which $Q(T)$ is known to equal $(2/\pi) \sin^{-1} (e^{-T})$ when $\rho(\tau) = e^{-|\tau|}$. (See Reference 5.) Notice that for $0 < T < 0.25$ the $A = 1$ curve lies above the $A = 0$ curve while for $0.25 < T < 0.75$ the reverse is true.

This result can be explained by recalling that $T = 0.25$ represents one-half the period of $\cos (4\pi t)$. For intervals shorter than this, the sine wave is not likely to cross zero and the effect is to cause fewer zero crossings than would be obtained if the sine wave were absent. Conversely, for time intervals longer than one-half the period ($T = 0.25$ in this case), the sine wave is sure to cross zero and therefore tend to increase the number of zero crossings over the noise-alone case.

As a result of this observation, it seems reasonable to conjecture that for T greater than one half the sine-wave period $Q(T)$, for noise alone, also forms an upper bound to $Q(T)$ for the sum of a sine wave plus noise.

Additional calculations were made for comparison with Cobb's previously reported approximate results.[1] The quantity that Cobb derived is an approximate expression for the probability distribution function of zero-crossing intervals, denoted by $P_0(T)$. Rice gives the relationship between $Q(T)$ and $P_0(T)$ in Reference 4 as

$$Q(T) = 1 - 2\nu T + 2\nu \int_0^T \int_0^x P_0(t)\, dt\, dx \tag{41}$$

where ν = expected number of zero crossings of (24).

As observed in Fig. 1 and 2 of Reference 1,

$$\nu \cong f$$

for the large sine-wave amplitudes where Cobb's approximation is valid. Thus,

$$Q(T) \cong 1 - 2fT + \int_0^{2fT} \int_0^{y} P_0(s)\, ds\, dy. \tag{42}$$

Cobb shows in equation 52 of Reference 1 that

$$P_0(s) \cong \frac{1}{(2\pi\sigma)^{\frac{1}{2}}} \exp\left[-\frac{(s-1)^2}{\sigma^2}\right] \tag{43}$$

where

$$\sigma = \frac{[2(1+\rho_1)]^{\frac{1}{2}}}{\pi A}$$

$$\rho_1 = \rho(2fT).$$

The approximation (43) is only valid for $\sigma \ll 1$.

Substituting (43) into (42), we obtain

$$Q(T) \cong (1 - 2fT)\left[0.5 + \operatorname{erf}\left(\frac{1-2fT}{\sigma}\right)\right] + \frac{\sigma}{(2\pi)^{\frac{1}{2}}}\left\{\exp\left[-\frac{1}{2}\left(\frac{1-2fT}{\sigma}\right)^2\right] - \exp\left(-\frac{1}{2\sigma^2}\right)\right\}. \tag{44}$$

As in Reference 1, set

$$\rho(\tau) = \frac{\sin \tau}{\tau} \tag{45}$$

$$A = 3 \tag{46}$$

$$2\pi f = 0.875 \text{ rad/sec}. \tag{47}$$

Figure 2 compares the approximate solution based on Cobb's results (44), and our upper bound (25). For $2fT < 1$, the approximate solution is somewhat smaller than the upper bound. For $2fT > 1$, the approximation becomes negative and therefore of little interest while the upper bound gradually approaches zero as T increases.

V. EXTENSIONS TO OTHER CASES

The specific applications discussed should not be considered exhaustive. For example, the case where $x(t)$ is the sum of a sine wave plus gaussian noise could easily be extended to $x(t)$ being the sum of a square wave plus gaussian noise. Although the specific formulae may be more complex, the general result (equations 12 and 14) is still applicable for $x(t)$ nonstationary or nongaussian.

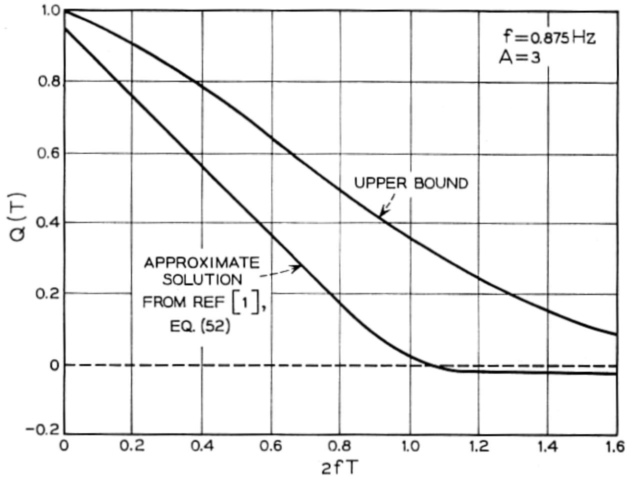

Fig. 2 — Comparison of upper bound and approximate solution of $Q(T)$ for zero crossings of sine wave plus gaussian noise.

In addition, the derivation of the general result can be modified slightly to obtain a useful upper bound on the conditional probability that $x(t)$ does not cross the zero axis for an interval of length T, given that $x(t) = 0$ at the start of the interval. Slepian has intensively investigated this latter probability. See Reference 5 for his discussion.

ACKNOWLEDGMENT

The research reported here was sponsored in part by the Doctoral Support Plan of Bell Telephone Laboratories, while the first-named author was on assignment from Bell Laboratories to New York University; and in part by the National Science Foundation under grant GK-1075. The paper is based on a portion of a dissertation submitted to the University's Graduate Division of The School of Engineering and Science in partial fulfillment of the requirements for the Ph.D. degree.

APPENDIX A

Derivation of $q_2^{(1)}$

We seek a simpler expression for the term

$$q_2^{(1)} = \frac{2}{\pi T^2} \int_0^T \int_0^T \int_{-\pi}^{\pi} \operatorname{erf}(a_1) \operatorname{erf}(a_2) \, d\varphi \, dt_1 \, dt_2 \tag{48}$$

which was first encountered in (39) and where

$$a_i = A \cos (2\pi f t_i + \varphi), \qquad i = 1, 2. \tag{49}$$

Substitution of the definition of erf (x) results in

$$q_2^{(1)} = \frac{1}{\pi^2 T^2} \int_0^T \int_0^T P(t_1, t_2)\, dt_1\, dt_2 \tag{50}$$

where

$$P(t_1, t_2) = \int_{-\pi}^{\pi} \int_0^{a_1} \int_0^{a_2} \exp[-\tfrac{1}{2}(x^2 + y^2)]\, dx\, dy\, d\varphi. \tag{51}$$

The first step is to notice the following three easily established properties of (51):

$$P(t_1, t_2) = P(t_1 - t_2) = P(t_2 - t_1) \tag{52}$$

$$P(\tau) = P\left(\tau + \frac{n}{f}\right), \qquad n = \pm 1, \pm 2, \cdots \tag{53}$$

$$P\left(\tau + \frac{1}{4f}\right) = -P\left(\frac{1}{4f} - \tau\right). \tag{54}$$

As a result of (52), (50) may be written as

$$q_2^{(1)} = \frac{2}{\pi^2 T^2} \int_0^T (T - \tau) P(\tau)\, d\tau. \tag{55}$$

It is a simple matter to demonstrate that, for any function, $H(\tau)$, satisfying the requirements of (52) through (54),

$$\int_{j/f}^{(j+1)/f} (T - \tau) H(\tau)\, d\tau = 0. \tag{56}$$

We next introduce an approximation to (51) that preserves properties (52) through (54). It is important to preserve these properties because, as a result of (56), if they are satisfied, $q_2^{(1)} = 0$ for $T = k/f$, $k = 1, 2, \cdots$; consequently, an approximation satisfying (52) through (54) will be accurate in a vicinity of these values of T. In addition, the three properties permit fast computation of (50).

The approximation chosen is given by

$$\int_0^{a_1} \int_0^{a_2} \exp[-\tfrac{1}{2}(x^2 + y^2)]\, dx\, dy \cong \int_0^{\pi/2} \int_0^{(a_1^2 + a_2^2)^{\frac{1}{2}}} e^{-r^2/2} r\, dr\, d\alpha \tag{57}$$

for sgn $[a_1]$ = sgn $[a_2]$, and

$$\int_0^{a_1} \int_0^{a_2} \exp\left[-\tfrac{1}{2}(x^2 + y)\right] dx\, dy \cong -\int_0^{\pi/2} \int_0^{(a_1^2+a_2^2)^{\frac{1}{2}}} e^{-r^2/2} r\, dr\, d\alpha \quad (58)$$

for sgn $[a_1]$ = $-$sgn $[a_2]$.

Essentially the approximation results in deforming the region of integration, as shown in Fig. 3. From this figure, it may be noticed that (57) is in reality an upper bound while (58) is a lower bound. Of course, it is easy to conceive of functions that give an upper bound to (58) and thus result in an upper bound for $q_2^{(1)}$. However, this results in a loss of properties (52) through (54).

Evaluating the integrals appearing in (57) and (58) and then substituting into (51) yields

$$\hat{P}(t_1, t_2) = \frac{\pi}{2} \int_{-\pi}^{\pi} p(\varphi, t_1, t_2)\left[1 - e^{-\frac{1}{2}(a_1^2+a_2^2)}\right] d\varphi \quad (59)$$

where

$$p(\varphi, t_1, t_2) = \text{sgn}\,[a_1\, a_2] \quad (60)$$

and

$$\hat{P}(t_1, t_2) \cong P(t_1, t_2).$$

Proving that (59) possesses the properties (52) through (54) only requires the use of elementary integration theory and will therefore be omitted. As a result of these properties, (59) may be written as

$$\hat{P}(\tau) = \frac{\pi}{2} \int_{-\pi}^{\pi} p(\theta, \tau)\left\{1 - e^{-(A^2/2)[\cos^2\theta + \cos^2(\omega\tau+\theta)]}\right\} d\theta \quad (61)$$

where $\omega = 2\pi f$

$$p(\theta, \tau) = \text{sgn}\,[\cos\theta \cos(\omega\tau + \theta)]; \quad (62)$$

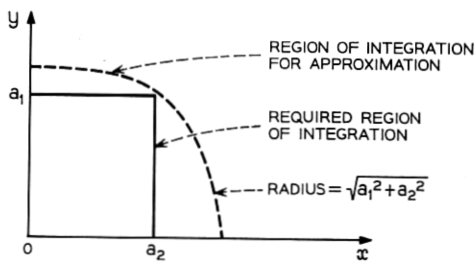

Fig. 3 — Deformation of region of integration for approximation in Appendix A.

furthermore, $\hat{P}(\tau)$ need only be evaluated for the range $0 < \tau < 1/(4f)$. Values beyond this range are related to values within the range by (52) through (54). To evaluate (61), an explicit expression for (62) is required. After studying this latter equation one finds that for

$$0 < \tau < 1/2f$$

$$p(\theta, \tau) = \begin{cases} 1 & \text{for} \quad \frac{\pi}{2} < \theta \leqq \frac{3\pi}{2} - \omega\tau \\ -1 & \text{for} \quad \frac{\pi}{2} - \omega\tau < \theta \leqq \frac{\pi}{2} \\ 1 & \text{for} \quad -\frac{\pi}{2} < \theta \leqq \frac{\pi}{2} - \omega\tau \\ -1 & \text{for} \quad -\frac{\pi}{2} - \omega\tau < \theta \leqq -\frac{\pi}{2} \end{cases} \tag{63}$$

with similar expressions for τ falling in the ranges

$$\frac{k}{2f} \leqq \tau \leqq \frac{k+1}{2f}, \qquad k = 1, 2, \cdots .$$

However, only the expression given by (63) is needed to evaluate (61) in the required range.

Substitution of (63) into (61) results in

$$\hat{P}(\tau) = \frac{\pi}{2}\left[\int_{-\pi/2}^{(\pi/2)-\omega\tau} F(\tau, \theta)\, d\theta + \int_{\pi/2}^{(3\pi/2)-\omega\tau} F(\tau, \theta)\, d\theta - \int_{-(\pi/2)-\omega\tau}^{-\pi/2} F(\tau, \theta)\, d\theta - \int_{(\pi/2)-\omega\tau}^{\pi/2} F(\tau, \theta)\, d\theta\right] \tag{64}$$

where

$$F(\tau, \theta) = 1 - e^{-(A^2/2)[\cos^2\theta + \cos^2(\omega\tau+\theta)]}.$$

With the help of some fundamental trigonometric identities it is easy to show that

$$F(\tau, \theta) = 1 - e^{-A^2/2}e^{-K(\tau)\cos(\omega\tau+2\theta)} \tag{65}$$

where

$$K(\tau) = \frac{A^2}{2}\cos\omega\tau. \tag{66}$$

Substituting (65) into (64) and performing obvious simplifications results in

$$\hat{P}(\tau) = \frac{\pi}{2} \{2\pi - 4\omega\tau - 2e^{-A^2/2}G(\tau)\} \tag{67}$$

where

$$G(\tau) = \int_0^{\pi-\omega\tau} e^{-K(\tau)\cos x}\, dx - \int_0^{\omega\tau} e^{K(\tau)\cos x}\, dx. \tag{68}$$

For the time being assume $n \leqq fT < n + \frac{1}{4}$ where $n = 0, 1, 2, \cdots$. Substituting (67) into (55) gives

$$q_2^{(1)} \cong \frac{2}{T^2} \int_0^T (T - \tau)S(\tau)\, d\tau$$

where

$$S(\tau) = 1 - 4f\tau - \frac{e^{-A^2/2}}{\pi} G(\tau). \tag{69}$$

Using the result (56), the latter equation equals

$$q_2^{(1)} \cong \frac{2}{T^2} \int_{n/f}^T (T - \tau)S(\tau)\, d\tau.$$

Setting $t = \tau - n/f$,

$$\begin{aligned} q_2^{(1)} &\cong \frac{2}{T^2} \int_0^{T-(n/f)} \left(T - \frac{n}{f} - t\right) S\left(t + \frac{n}{f}\right) dt \\ &= \frac{2}{T^2} \int_0^{T-(n/f)} \left(T - \frac{n}{f} - t\right) S(t)\, dt \end{aligned}$$

as a result of property (53). Now substitute $t = v/f$ to obtain

$$q_2^{(1)} \cong \frac{2}{b^2} \int_0^{b-n} (b - n - v)S(v)\, dv \quad \text{for} \quad n \leqq b < n + \tfrac{1}{4} \tag{70}$$

where $b = fT$.

This equation is the same as the first part of the final result stated in (28). The equations defined in (69), (68), and (66) are the same as (29), and (30), and (31), respectively, except for a convenient scale change. The rest of the results stated in (28), for various ranges of T, are derived in a similar manner as (70) was, using relations (52), (53), or (54), as required. Because only straightforward operations are used to obtain these results, they will not be derived.

APPENDIX B

Derivation of $q_2^{(2)}$

The first step in simplifying the expression for $q_2^{(2)}$, as defined in (40), is to interchange the order of integration of the two innermost integrals to yield

$$q_2^{(2)} = \frac{1}{\pi^2 T^2} \int_0^T \int_0^T \int_0^{\rho(\tau)} \frac{1}{(1-\alpha^2)^{\frac{1}{2}}} B(\alpha, t_1, t_2)\, d\alpha\, dt_1\, dt_2 \tag{71}$$

where

$$B(\alpha, t_1, t_2) = \int_{-\pi}^{\pi} \exp\left[-\frac{a_1^2 + a_2^2 - 2\alpha a_1 a_2}{2(1-\alpha^2)}\right] d\varphi. \tag{72}$$

Using the definitions of a_1 and a_2, (34), and some obvious trigonometric identities, it is easy to show that

$$a_1^2 + a_2^2 - 2\alpha a_1 a_2 = A^2\{\cos[\omega(t_1 + t_2) + 2\varphi][\alpha - \cos(\omega\tau)] + [1 - \alpha\cos(\omega\tau)]\}$$

where we have set $\omega = 2\pi f$ for convenience.

Substitution of this relationship into (72) results in

$$B(\alpha, t_1, t_2) = \exp[-J_1(\alpha, \tau)] \cdot \int_{-\pi}^{\pi} \exp\{-J_2(\alpha, \tau)\cos[\omega(t_1 + t_2) + 2\varphi]\}\, d\varphi \tag{73}$$

where

$$J_1(\alpha, \tau) = \frac{A^2}{2}\left[\frac{1 - \alpha\cos(\omega\tau)}{1-\alpha^2}\right] \tag{74}$$

$$J_2(\alpha, \tau) = \frac{A^2}{2}\left[\frac{\alpha - \cos(\omega\tau)}{1-\alpha^2}\right]. \tag{75}$$

Setting $\theta = \omega(t_1 + t_2) + 2\varphi$ in (73) and using the periodic properties of the integrand, yields

$$B(\alpha, \tau) = 2\exp[-J_1(\alpha, \tau)] \int_0^{\pi} \exp[-J_2(\alpha, \tau)\cos\theta]\, d\theta.$$

This integral is recognized as an expression for the modified Bessel function of the first kind (see Reference 11, page 181, Equation 4). And so

$$B(\alpha, \tau) = 2\pi I_o[J_2(\alpha, \tau)] \exp[-J_1(\alpha, \tau)] \tag{76}$$

where $I_0(x)$ is the modified Bessel function of the first kind, zero order.

Since (76) is only a function of τ, one may define

$$H(\tau) = \frac{2}{\pi} \int_0^{\rho(\tau)} \frac{1}{\sqrt{1 - \alpha^2}} B(\alpha, \tau) \, d\alpha. \tag{77}$$

Furthermore, it is easy to show that $H(\tau) = H(-\tau)$. Consequently, (71) can be written as

$$q_2^{(2)} = 2 \int_0^1 (1 - u) H(uT) \, du.$$

By setting $\tau = uT$ in (77) and by making the change of variable $\alpha = \sin \theta$, (26) and (27), which are the desired results, follow.

REFERENCES

1. Cobb, S. M., "The Distribution of Intervals Between Zero Crossings of Sine Wave Plus Random Noise and Allied Topics," IEEE Trans. Inform. Theory, *IT-11,* No. 2 (April 1965), pp. 220–231.
2. Newell, G. F., and Rosenblatt, M., "Zero-Crossing Probabilities for Gaussian Stationary Processes," Ann. Math. Stat., *33,* No. 4 (December 1962), pp. 1306–1313.
3. Rice, S. O., "Mathematical Analysis of Random Noise," B.S.T.J., *23,* No. 3 (July 1944), pp. 282–332; and *24,* No. 1 (January 1945), pp 46–156. See especially Sections 3.3 and 3.4.
4. Rice, S. O., "Distribution of the Duration of Fades in Radio Transmission," B.S.T.J., *37,* No. 3 (May 1958), pp. 581–636.
5. Slepian, D., "The One-Sided Barrier Problem for Gaussian Noise," B.S.T.J., *41,* No. 2 (March 1962), pp. 462–501.
6. Wong, E., "Some Results Concerning the Zero-Crossings of Gaussian Noise," SIAM J. Appl. Math., *14,* No. 6 (November 1966), pp. 1246–1254.
7. McFadden, J. A., "Urn Models of Correlation and a Comparison with the Multivariate Normal Integral," Ann. Math. Stat., *26,* No. 3 (September 1955), pp. 478–489.
8. Strakhov, N. A., "A Representation Theory for a Class of Non-Gaussian Distributions with Applications to Digital Data Transmission," New York: Doctoral Dissertation, New York University, October 1967.
9. Lawson, J. L., and Uhlenbeck, G. E., *Threshold Signals,* New York: McGraw-Hill Book Co., Inc., 1950.
10. Price, R., "A Useful Theorem for Nonlinear Devices Having Gaussian Inputs," IRE Trans. Inform. Theory, *IT-4,* No. 2 (June 1958), pp. 69–72.
11. Watson, G. N., *A Treatise on the Theory of Bessel Functions,* 2nd ed., Cambridge, England: University Press, 1962.