Data Transmission with FSK Permutation Modulation

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Performance characteristics are derived for an FSK data transmission scheme in which M frequencies out of N are used simultaneously. Noncoherent matched filters are applied at the N frequencies, and the filter outputs are compared as in a permutation modulation system.

It is shown that many permutation alphabets provide energy per bit advantage over binary FSK, although the best results are obtained with one-out-of-N alphabets. Considering bits per unit bandwidth, many permutation alphabets perform as well as or better than binary; however, one-out-of-N alphabets carry less information per unit bandwidth when $N > 4$.

I. INTRODUCTION

The technique of N-ary frequency modulation in which energy is transmitted on 1 out of N frequencies to convey $\log_2 N$ bits of information per character has been known for some years.[1,2] David Slepian[3] has recently described a general modulation system, permutation modulation, which is applied here to a multifrequency modulation scheme in which energy is transmitted simultaneously on M frequencies out of N, thus conveying $\log_2 \binom{N}{M}$ bits of information per character. Binary and one out of N FSK modulation are special cases of FSK permutation modulation.

Such a transmission scheme is basically not new; it has been used for many years for transmitting decimal digits, address, and other supervisory information in the telephone plant. This work was motivated by a requirement to compare the information transmission capability of these alphabets. However, the application analyzed here is, in fact, different because we assume a baud synchronous matched filter receiver with a mutually orthogonal set of signals. The channel is assumed to be nonfading, frequency flat, with white gaussian additive noise.

II. GENERAL DESCRIPTION

For convenience, we shall refer to this modulation scheme as PFSK. The PFSK alphabet has $\binom{N}{M}$ characters,

$$\binom{N}{M} = \frac{N!}{M!\,(N-M)!}.$$

The $\binom{2}{1}$ alphabet is the binary FSK modulation with which other $\binom{N}{M}$ alphabets will be compared. The $\binom{N}{1}$ alphabet is commonly referred to as N-ary or MFSK (multiple frequency shift keyed).

PFSK transmission operates in a manner shown for the $\binom{N}{2}$ alphabet in Figure 1. One of the $\binom{N}{M}$ characters is input to the transmitter; the

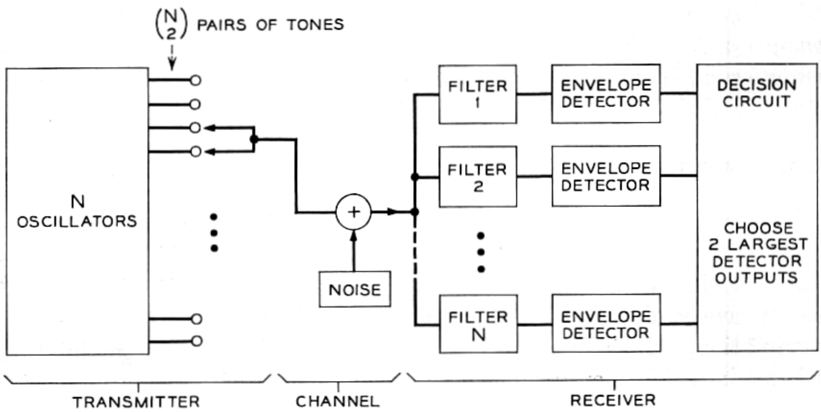

Fig. 1 — Transmission system for $\binom{N}{2}$ alphabets.

signal out is M simultaneous pulses of energy, one pulse on each of M distinct frequencies, lasting for T seconds. White gaussian noise is added in the channel. Filters, matched to the pulse shape, are tuned to each of the N possible frequencies. The filter outputs are envelope-detected and all N envelope samples are intercompared at the end of the pulse period. The largest M of these outputs determine the transmitted character.*

* Slepian has shown that this technique of amplitude comparisons minimizes the error probability. See Ref. 3.

III. ANALYSIS

An error is made in this process when any of the noise samples exceeds any of the signal plus noise samples. The error probability is P_e and is one minus the probability of making a correct decision:

$$P_e = 1 - \int_0^\infty p_M(s) P_{N-M}(s)\, ds \tag{1}$$

where

$p_M(s)$ is the p.d.f. of the smallest signal plus noise sample
$P_{N-M}(s)$ is the distribution function of the largest noise sample.

The p.d.f. of the smallest signal plus noise sample is determined as follows. The p.d.f. of the output sample of a matched filter detector can be written as*

$$p(y) = yI_0(y\sqrt{2R}) \exp\left(-\frac{y^2 + 2R}{2}\right) \tag{2}$$

where

y is the output envelope sample amplitude normalized to the rms noise
$I_0(\ )$ is the modified Bessel function
R is $\mathcal{E}_M/\mathfrak{N}_0$
$\mathcal{E}_M$ is the received signal energy in joules at each of the M transmitted frequencies
$\mathfrak{N}_0$ is the noise density, in watts per Hz.

The probability of the smallest of M samples exceeding a value s is the same as the probability that all M samples exceed the value s. This probability is expressed by equation (3), with independence of the M samples following from orthogonality.

$$1 - P_M(s) = \left[\int_s^\infty p(y)\, dy\right]^M = Q^M(\sqrt{2R}, s) \tag{3}$$

where

$P_M(s)$ is the distribution function of the smallest signal plus noise sample
$Q(\cdot, \cdot)$ is the Q function and is tabulated by Marcum[6].

* We view equation 2 as a renormalization of an expression by Helstrom[4] for matched filter detection, although it was originally derived by Rice[5] in a different context.

Thus, the p.d.f. of the smallest output is simply

$$p_M(s) = \frac{d}{ds} P_M(s) = Q^{M-1}(\sqrt{2R}, s)p(s). \tag{4}$$

Similarly, with independent noise samples, we find the distribution function of the largest noise sample.

$$\begin{aligned} P_{N-M}(s) &= \left[\int_0^s y \exp(-y^2/2)\, dy\right]^{N-M} \\ &= \sum_{r=0}^{N-M} (-1)^r \binom{N-M}{r} \exp\left(-\frac{rs^2}{2}\right) \end{aligned} \tag{5}$$

Substitution of equations (4) and (5) into equation (1) yields (after some labor) the character error probability:

$$\begin{aligned} P_e = M \sum_{r=1}^{N-M} (-1)^{r+1} \binom{N-M}{r} \int_0^\infty [Q(\sqrt{2R}, s)]^{M-1} \\ \times sI_0(s\sqrt{2R}) \exp\left[-\frac{(r+1)s^2 + 2R}{2}\right] ds \end{aligned} \tag{6}$$

A closed form expression for the case $M = 1$ was found by Reiger.[2]

$$P_e(M = 1) = \frac{1}{N} \sum_{r=2}^{N} (-1)^r \binom{N}{r} \exp\left[-R\left(1 - \frac{1}{r}\right)\right]. \tag{7}$$

A closed form expression for the case $M = 2$ is obtained from equation (6) using integration forms, having Q function integrands, given by Stein:[7]

$$\begin{aligned} P_e(M = 2) = \frac{2}{N-1} \sum_{r=2}^{N-1} (-1)^r \binom{N-1}{r} \left(\frac{1}{r+1}\right) \\ \times [1 + rQ(\alpha, \beta) - Q(\beta, \alpha)] \exp\left[-R\left(1 - \frac{1}{r}\right)\right] \end{aligned} \tag{8}$$

where

$$\alpha = \left(\frac{2Rr}{r+1}\right)^{\frac{1}{2}}, \qquad \beta = \frac{\alpha}{r}.$$

Closed form expressions for cases of $M > 2$ are not known; however, an asymptotic form for large R is obtained following arguments by Helstrom[8] for approximating the Q function:

$$P_e(M) \approx \frac{M}{N - M + 1} \sum_{r=2}^{N-M+1} (-1)^r \binom{N - M + 1}{r} \cdot \exp\left[-R\left(1 - \frac{1}{r}\right)\right] \qquad (9)$$

or taking the predominant first term of equation (9) we have*

$$P_e(M) \approx \frac{M(N - M)}{2} \exp\left(-\frac{R}{2}\right). \qquad (10)$$

Equation (10) can also be obtained heuristically. At high signal-to-noise ratios, character errors occur because of a binary decision error; that is, one of the noise samples is mistaken for one of the signal plus noise samples. The probability of a binary decision error is

$$P_e = 1/2 \exp(-R/2).$$

In the multifrequency situation, there are $M(N-M)$ ways for this to happen; the product of these two factors yields equation (10).

IV. COMPARISON OF ALPHABETS

We interrelate the performances of the PFSK alphabets to those of binary FSK using two criteria: energy per bit required for an equivalent error rate, and bits per unit bandwidth.† First, the per character information of these alphabets is defined as k:

$$k \equiv \log_2 \binom{N}{M}.$$

The normalized energy per bit $\mathcal{E}/\mathfrak{N}_0$ is related to the ratio R, defined in equation (2), by

$$R = \left(\frac{k}{M}\right)\left(\frac{\mathcal{E}}{\mathfrak{N}_0}\right). \qquad (11)$$

Since the quantity R appears in the exponent of the error rate expression, it is apparent that, for low error rates, the power advantage (over binary FSK) of a PFSK alphabet approaches k/M. We can observe this numerically by comparing error rates on the basis

* It is easy to show that the first term is always an upper bound to P_e.

† The reader can compare the results of the work here with recent work of I. Jacobs,[9] who intercompares coherent modulation systems using virtually the same criteria.

of "equivalent error probability,"[10] which is the binary error probability for which the probability of one or more errors in a binary sequence of k bits is equal to the probability of error in the PFSK case. This equivalent error probability is defined as P_{eq}:

$$P_{eq} \equiv 1 - [1 - P_e(M)]^{1/k} \approx \frac{1}{k} P_e(M). \tag{12}$$

Figure 2 illustrates P_{eq} as a function of $\mathcal{E}/\mathfrak{N}_0$ for several alphabets. At error rates of 10^{-3}, the power advantage is within 0.6 dB of the k/M value for the $\binom{16}{1}$ and $\binom{16}{2}$ alphabets, and closer for the other examples.

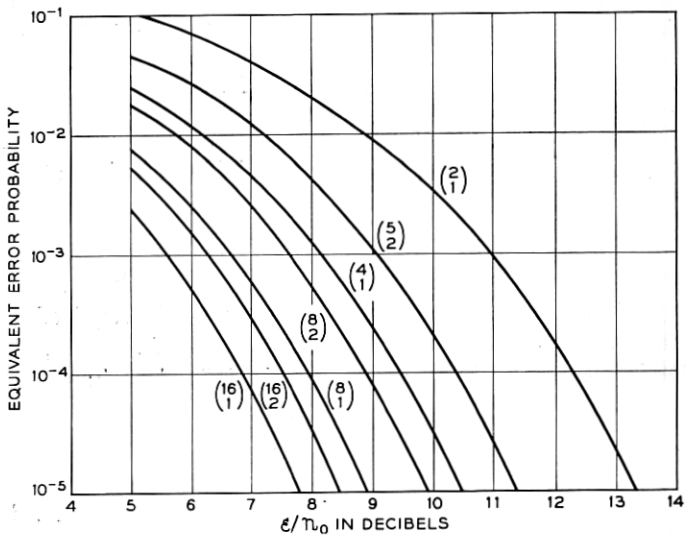

Fig. 2 — PFSK error probabilities.

The number of bits per unit bandwidth for a PFSK alphabet is determined by estimating the bandwidth as N times the frequency separation, which is $1/T$ for noncoherent orthogonal signals with minimum frequency spacing. Since the information rate is k/T, the desired bits per cycle ratio is simply k/N.* Figure 3 shows paired values of $10 \log_{10}(k/M)$ and k/N for illustrative alphabets.

* It is easy to show that this ratio for PFSK alphabets approaches a maximum value of 1 for large N, with $M = N/2$. At this point $k/M = 2$ for a 3 dB advantage.

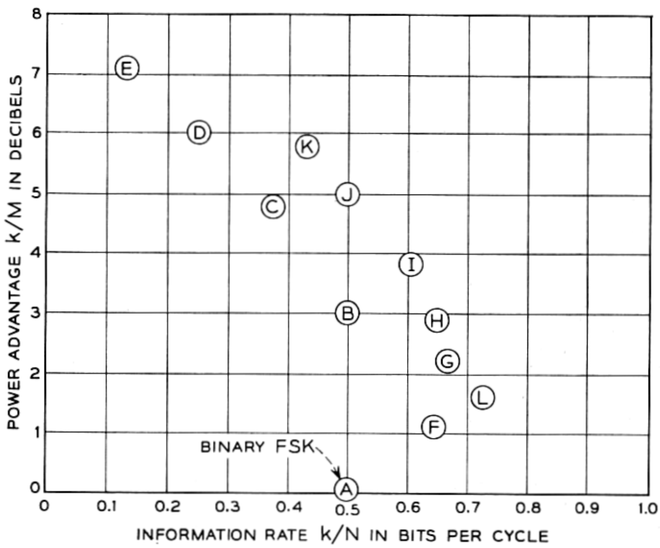

Fig. 3 — Performance comparison of PFSK alphabets.

Symbol	A	B	C	D	E	F	G	H	I	J	K	L
Alphabet	$\binom{2}{1}$	$\binom{4}{1}$	$\binom{8}{1}$	$\binom{16}{1}$	$\binom{32}{1}$	$\binom{4}{2}$	$\binom{5}{2}$	$\binom{6}{2}$	$\binom{8}{2}$	$\binom{12}{2}$	$\binom{16}{2}$	$\binom{6}{3}$

V. SUMMARY AND CONCLUSIONS

It has been shown that the PFSK technique gives significant power advantage over binary FSK. In addition, bandwidth can be controlled by the proper choice of alphabet.

Disadvantages of the technique are practical ones. Implementation of the decision function is relatively complicated. In some applications peak power limitations might make the average power calculations inapplicable.

A generally large number of characters in the alphabet is not suited to all applications, but can be very efficient in some. For example, the $\binom{5}{2}$ alphabet, containing 10 characters, is well suited to decimal digits.

ACKNOWLEDGEMENT

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