Gain of Antennas with Random Surface Deviations

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On-axis gain of antennas with rough reflecting surfaces is computed as a function of rms surface deviation ϵ, *correlation distance c, antenna area A and wavelength* λ. *Gaussian stationary surface deviations, Gaussian correlation functions, and uniform illumination are assumed. Antennas with rectangular and circular apertures are considered. It is shown that a normalized gain can be defined which has the same functional form for both. A principal result of this work is quantitative calculation of the on-axis antenna gain when the normalized variance* $(4\pi\epsilon/\lambda)^2$ *of the rough surface is larger than 4. The off-axis gain is also considered, and it is shown that in the asymptotic limit (as* $\lambda \rightarrow 0$*), the gain reduces to that obtained by using geometrical optics.*

I. INTRODUCTION

The gain of shallow paraboloid reflector antennas with random surface deviations has been derived by Ruze.[1,2] The deviation was based on scalar Kirchhoff approximation to the radiation from reflector antennas. The surface deviations were assumed to be gaussian stationary with gaussian correlation functions. On these bases, an approximate solution for the antenna gain was obtained in terms of an infinite series. The series has been evaluated for relatively small rms surface deviations, ϵ, in comparison to the wavelength, λ, namely $(4\pi\epsilon/\lambda)^2 \leqq 4$. Asymptotic limits (as $\lambda \rightarrow 0$) for the gain were also given by Ruze[2] based on a similar analysis by Scheffler[3]. On-axis gain measurements of large reflector antennas as a function of frequency, exhibit the characteristics as predicted theoretically by Ruze.

The present work was motivated primarily to determine the gain in the intermediate region between very long and very short wavelengths and to establish a criterion for applicability of the asymptotic limit. Of primary interest was the near axis field distribution in the focal plane of a paraboloid reflector antenna illuminated by an incident plane wave. However, since both the far-field radiation pattern and the field distribution in the focal plane are Fourier transforms of the antenna aperture illumination, the derivations by Ruze are applicable for determining both the far-field and focal-plane distributions.

The series solution for the antenna gain obtained by Ruze does not seem to be suitable for numerical computations for large arguments. This is because some of the terms in the series will assume large values before the terms begin to decrease. However, the series for the on-axis gain is related to an exponential integral. The exponential integral also has an asymptotic series representation, which is particularly suitable for numerical computation for large arguments. On this basis, the on-axis gain has been computed as a function of the rms surface deviation to the wavelength ratio and for a range of correlation parameters. The asymptotic limit for the gain is evident from these computations.

The off-axis gain is also considered. Asymptotic representations of the series which may facilitate the off-axis gain computations are discussed. The limiting value $(\lambda \to 0)$ for the off-axis gain is obtained, and it is shown that in this limit, the gain reduces to that obtained from geometrical optics.[4]

The gain of antennas with rectangular apertures and gaussian stationary surface deviations is presented by assuming uniform illumination. A generalization to include certain types of nonuniform illuminations is discussed. The on-axis gain for antennas with circular apertures also is given. It is shown that the on-axis gain for antennas with rectangular and circular apertures can be normalized, such that the normalized gain is the same for both. The off-axis gain is expressed in terms of series with known asymptotic expansions.

II. ANTENNA GAIN

The far field gain, $G(\theta,\Phi)$, in the vicinity of the axis of a shallow paraboloid reflector antenna with surface deviations, $z(x,y)$ is, using the scalar Kirchhoff approximation[5]

$$G(\theta, \Phi) = \frac{4\pi}{\lambda^2} \cdot \frac{\iint_s \iint_s E_a(x, y)E_a^*(x_1, y_1) \exp(j\{\beta_x u + \beta_y v + 2k[z(x, y) - z(x_1, y_1)]\})\, ds\, ds_1}{\iint_s E_a(x, y)E_a^*(x, y)\, ds} \tag{1}$$

where E_a is the projected electric field on the antenna aperture and s is the aperture area.

$$k = \frac{2\pi}{\lambda} = \text{free space propogation constant}$$

$$\lambda = \text{wavelength}$$

$$\beta_x = k \sin \theta \cos \Phi \tag{2}$$

$$\beta_y = k \sin \theta \sin \Phi \tag{3}$$

θ and Φ are the spherical coordinates indicated in Fig. 1.

$$u = x - x_1 \tag{4}$$

$$v = y - y_1 \tag{5}$$

The Kirchhoff approximation is based on the assumption that the surface is locally plane, and hence equation (1) is applicable to surfaces for which the curvatures are small.

Equation (1) can also be used to determine the power distribution in the focal plane of shallow paraboloid reflector antennas in the vicinity of the focal point, in which case (referring to Fig. 1)

$$\beta_x = kx_f/f \tag{6}$$

$$\beta_y = ky_f/f \tag{7}$$

where x_f and y_f are the coordinates in the focal plane and f is the focal length.

If $z(x,y)$ is a Gaussian stationary random variable with zero mean it has been shown[1, 6] that by performing the statistical averaging, the expectation value for the gain, $\langle G(\theta, \Phi)\rangle_{av}$ is:

$$\langle G(\theta, \Phi)\rangle_{av} = \frac{4\pi}{\lambda^2} \exp(-\delta^2)$$

$$\cdot \frac{\iint_s \iint_s E_a(x, y)E_a^*(x_1, y_1) \exp[j(\beta_x u + \beta_y v)] \exp[\delta^2 r(u, v)]\, ds\, ds_1}{\iint E_a(x, y)E_a^*(x, y)\, ds} \tag{8}$$

where

$$\delta = \frac{4\pi}{\lambda} \epsilon \tag{9}$$

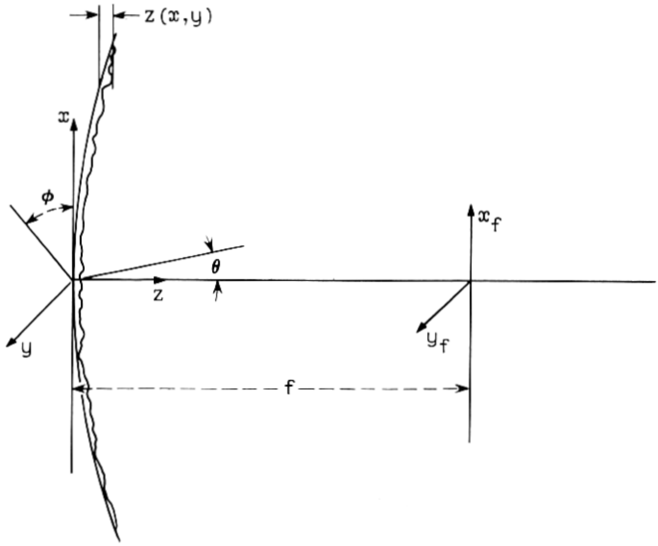

Fig. 1 — Antenna coordinates.

$$\epsilon = \text{rms surface deviation}$$

$$\delta^2 r(u, v) = \text{correlation function.}$$

To evaluate equation (8), four integrations have to be performed. It is shown in Appendix A that for antennas with rectangular apertures two integrations can be readily eliminated for certain types of illuminations, truncated cosine illuminations, for example.

In particular for uniform illuminations, $E_a(x, y) = 1$, and for a gaussian correlation function with

$$r(u, v) = \exp\left(-\frac{u^2 + v^2}{c^2}\right) \tag{10}$$

where c is the correlation length, it is shown in Appendix A that the expectation value of the gain is:

$$\langle G(\theta, \Phi)\rangle_{\text{av}} = \exp(-\delta^2) G_0(\theta, \Phi) + \left(\frac{2\pi c}{\lambda}\right)^2 \exp(-\delta^2) \sum_{n=1}^{\infty} \frac{\delta^{2n}}{n!\, n} \left[\exp\left(-\frac{\beta^2 c^2}{4n}\right) - \Delta_n\right] \tag{11}$$

where $G_0(\theta, \Phi)$ is the antenna gain in the absence of surface deviations.

For an antenna with a rectangular aperture

$$G_0(\theta, \Phi) = \frac{4\pi A}{\lambda^2}\left(\frac{\sin \beta_x a}{\beta_x a}\ \frac{\sin \beta_y b}{\beta_y b}\right)^2 \tag{12}$$

where $A = 4ab$ is the aperture area.

$$\beta = \frac{2\pi}{\lambda}\sin\theta \tag{13}$$

and

$$\Delta_n < \frac{c}{2(\pi n)^{\frac{1}{2}}}\left[\frac{1}{a} + \frac{1}{b}\right] \tag{14}$$

Equation (11) agrees with the gain derived by Ruze for antennas with circular apertures except for the term Δ_n. This term is small, if the correlation distance c is small compared with the linear dimensions of the antenna. This assumption is made in the subsequent computations.

For antennas with circular apertures the exact evaluation of equation (8) is in general more difficult. However, for uniform illumination, the on-axis gain $\langle G(0,\ 0)\rangle_{\text{av}}$ is evaluated exactly in Appendix B with the aid of Q functions. The gain has the same functional form as equation (11) with $\beta = 0$. In particular for $n/2(D/c)^2 \gg 1$

$$\Delta_n \approx \frac{2c}{D(\pi n)^{\frac{1}{2}}} \tag{15}$$

where D is the antenna aperture diameter.

III. ON-AXIS GAIN

Equation (11) can be readily computed for small values of δ^2. For large values of δ^2 the terms $\delta^{2n}/n!n$ will become very large; therefore, the series is not suitable for direct computation if δ^2 is large. Nevertheless, the gain on-axis can be readily computed by noticing that the series in equation (11) for $\beta = 0$ is related to an exponential integral, which also has an asymptotic representation.

The exponential integral, E_i, can be written[7]

$$E_i(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n!\,n} \tag{16}$$

where γ is Euler's constant. The asymptotic series ($x \to \infty$) for

$E_i(x)$ is

$$E_i(x) = \frac{\exp(x)}{x} \sum_{n=0}^{N-1} \left[\frac{n!}{x^n} + O\left(\frac{1}{x^N}\right) \right]. \tag{17}$$

Though the asymptotic series diverges for all finite values of x it can be used to evaluate $E_i(x)$ for large x by using up to N terms,[8] where N is an integer nearest to the value of x.

In terms of the exponential integral, the on-axis gain for both rectangular and circular aperture antennas is

$$\langle G(0, 0) \rangle_{av} = \left(\frac{D_0}{4\epsilon}\right)^2 \cdot \left\{ \delta^2 \exp(-\delta^2) + \left(\frac{2c}{D_0}\right)^2 \delta^2 \exp(-\delta^2)[E_i(\delta^2) - \ln \delta^2 - \gamma] \right\} \tag{18}$$

where D_0 is related to the antenna area, A, by

$$A = \frac{\pi D_0^2}{4}. \tag{19}$$

One parameter in (18) is readily eliminated by defining a normalized on-axis gain, $\langle g(0,0) \rangle_{av}$, by

$$\begin{aligned} \langle g(0, 0) \rangle_{av} &= \frac{\langle G(0, 0) \rangle_{av}}{(D_0/4\epsilon)^2} \\ &= \delta^2 \exp(-\delta^2) \left\{ 1 + \left[\frac{2c}{D_0}\right]^2 [E_i(\delta^2) - \ln \delta^2 - \gamma] \right\}. \end{aligned} \tag{20}$$

The normalized gain thus depends only on two parameters, δ^2 and $(c/D_0)^2$.

Equation (20) has been computed by using a SHARE program for the computation of the exponential integral*. This program computes $E_i(x)$ with at least four-decimal accuracy.

Computations have been performed for $10^{-4} \leqq \delta^2 \leqq 80$ and for $10^{-3} \leqq c/D_0 \leqq 0.1$. The computed normalized gain is shown in Fig. 2.

The computations show the normalized antenna gain has three distinct regions which are characterized by the normalized rms surface deviation to wavelength ratio, δ.

In the region $0 \leqq \delta^2 \leqq 1$ the normalized antenna gain is nearly independent of the correlation length c, and increases almost linearly with δ^2. In the region $1 \leqq \delta^2 \leqq 20$ the gain is dependent on both δ^2

* Contributed by D. S. Villars.

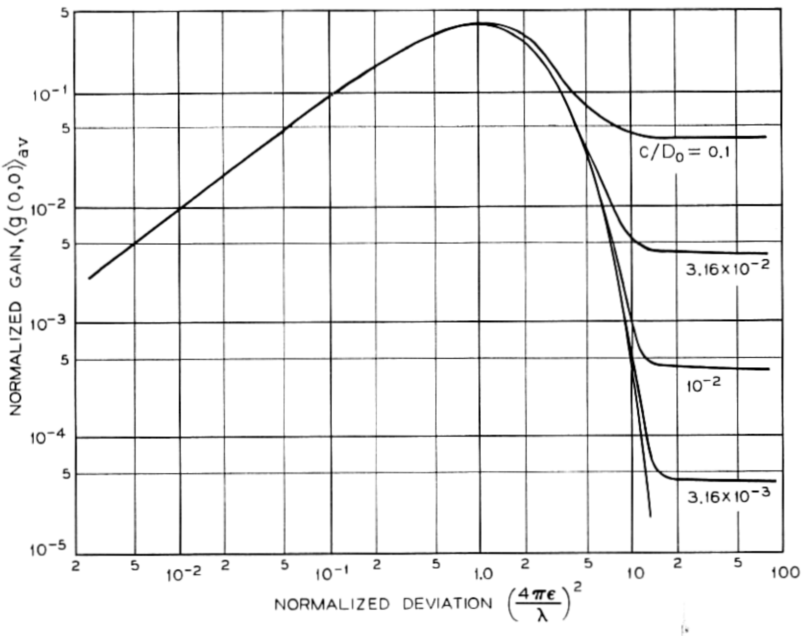

Fig. 2 — Normalized antenna gain.

and c. In the region $\delta^2 > 20$ the gain is almost independent of δ^2 and is a function of c/D_0 only. This region is the asymptotic region. For the range of parameters used in the computation, the gain in the asymptotic region, for a given c/D_0, deviates by less than 5 percent from the asymptotic value.

The curves shown in Fig. 2 seem to confirm the general characteristics of the measured gain as a function of frequency of large reflector antennas presented by Ruze.[2] The presented measurements extend only slightly from the first into the second region but not sufficiently far to determine qualitative agreement between the theory and experiment in much of the intermediate and all of the asymptotic regions. A detailed comparison of the measured and computed gain can not be made since uniform illumination has been assumed in the computation.

IV. ASYMPTOTIC VALUES FOR THE OFF-AXIS GAIN

Computation of the off-axis gain directly from equation (11) can only be readily performed for relatively small values of δ^2. An alternate representation for the off-axis gain for large values of δ is obtained by expanding the exponential in the second term of equation (11) in a power series. By using this expansion and neglecting Δ_n, (11) can be rewritten as follows:

$$\langle G(\theta, \Phi)\rangle_{\text{av}} = \exp(-\delta^2)G_0(\theta, \Phi) + \left(\frac{c}{2\epsilon}\right)^2 \sum_{m=0}^{\infty} (-1)^m \frac{A_m(\delta^2)}{m!} \left(\frac{\beta c}{2\delta}\right)^{2m} \tag{21}$$

where

$$A_m(\delta^2) = \exp(-\delta^2) \sum_{n=1}^{\infty} \frac{(\delta^2)^{n+m+1}}{n!\, n^{m+1}}. \tag{22}$$

The series for $A_m(\delta^2)$ are special cases of functions considered by Barnes[9] who obtained their asymptotic expansions. These functions were also studied by Ford[10] who also presented a recurrence relation for the coefficients of the asymptotic series. The above functions designated by $G_\beta(x,\Theta)$ are:

$$G_\beta(x, \Theta) = \sum_{n=0}^{\infty} \frac{x^n}{n!\,(n+\Theta)^\beta}. \tag{23}$$

The functions $A_m(\delta^2)$ can be written as:

$$A_m(\delta^2) = (\delta^2)^{m+2} \exp(-\delta^2) G_{m+2}(\delta^2, 1). \tag{24}$$

Only the asymptotic limit for the off-axis gain is considered. For $x \to \infty$

$$G_\beta(x, \Theta) = \frac{\exp(x)}{x^\beta}\left[1 + O\left(\frac{1}{x}\right)\right]$$

hence for $\delta^2 \to \infty$

$$A_m = 1 + O\left(\frac{1}{\delta^2}\right). \tag{25}$$

The off-axis asymptotic gain will be designated by $G(\theta, \Phi)_\infty$ and is given by:

$$G(\theta, \Phi)_\infty = \left(\frac{c}{2\epsilon}\right)^2 \exp\left[-\left(\frac{c}{4\epsilon}\sin\theta\right)^2\right] \tag{26}$$

The corresponding normalized gain is found as

$$g(\theta, \Phi)_\infty = \left(\frac{2c}{D_0}\right)^2 \exp\left[-\left(\frac{c}{4\epsilon}\sin\theta\right)^2\right]. \tag{27}$$

The asymptotic value for the gain, equation (26), is in agreement with the gain obtained, based on an approximation to the gaussian correlation functions,[3] and also with results obtained by using geometrical optics.[4] Equation (24) is independent of frequency but strongly dependent on the ratio ε/c. This ratio has been interpreted as an average surface slope.[2] The range of δ over which the above off-axis approximation applies has not been determined precisely, however, it is reasonable to assume that this range will correspond to the asymptotic region for the on-axis gain.

V. CONCLUSIONS

The on-axis gain of antennas with gaussian stationary random surface deviations and gaussian correlation functions has been determined for antennas with rectangular and circular apertures by assuming uniform illumination. For both types a normalized expression for the gain was derived which depends only on the normalized rms surface deviation ε, to wavelength λ ratio, $\delta(=4\pi\varepsilon/\lambda)$, and the ratio of the correlation length c to a defined linear antenna dimension D_o. For circular antennas, D_o is the diameter.

The antenna gain as a function of δ exhibits three distinct regions: (*i*) $0 \leqq \delta^2 \leqq 1$, (*ii*) $1 \leqq \delta^2 \leqq 20$, and (*iii*) $\delta^2 > 20$. The last is called the asymptotic region. In this region the gain is nearly independent of wavelength.

The computed gain exhibits in general the characteristics of the measured gain as a function of frequency of large reflector antennas reported in the literature. These measurements extend only partially into the second region and have not been obtained in the third (asymptotic) region.

For large values of δ^2 the off-axis gain can be expressed in terms of series with known asymptotic expansions. The limiting value for the off-axis gain has been obtained and reduces to that obtained by geometrical optics.

VI. ACKNOWLEDGMENTS

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APPENDIX A

Gain of Antennas with Rectangular Apertures

To evaluate equation (8) for antennas with rectangular apertures, consider the following integral

$$\langle I \rangle_{\text{av}} = \exp(-\delta^2) \int_{-a}^{a} \int_{-a}^{a} \int_{-b}^{b} \int_{-b}^{b} E_a(x, y) E_a(x_1, y_1) \cdot \exp[\delta^2 r(u, v)] \exp[j(\beta_x u + \beta_y v)] \, dx \, dx_1 \, dy \, dy_1 \tag{28}$$

with

$$u = x - x_1 \tag{29}$$

$$v = y - y_1. \tag{30}$$

Since equation (28) contains the correlation function in terms of u and v, it is preferable to introduce the u,v coordinate system.

In the x,x_1 coordinate system the integrations are over the square region shown in Fig. 3a. In the y, y_1 system the region is similar. With

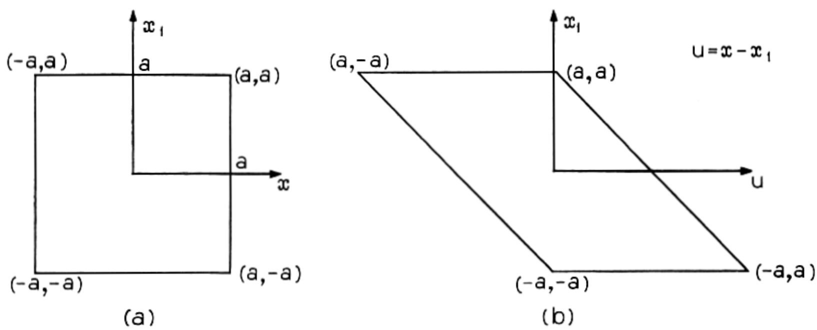

Fig. 3 — Coordinate transformation.

the coordinate transformation equation (29), the transformed region in the x_1,u coordinate system is also shown in Fig. 3b.

In the x_1,u plane the integration with respect to x_1 is readily performed for certain types of illumination functions.* In particular, let

$$E_a(x, y) = E_{ax}(x) E_{ay}(y) \tag{31}$$

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* A similar method has been used by Hoffman in his treatment of scattering of electromagnetic waves from a random surface.[6]

where

$$E_{ax}(x_1 + u) = \sum_{n=1}^{N} g_n(x_1)f_n(u) \tag{32}$$

with a similar equation for $E_{ay}(y_1 + v)$. An example of such illuminations are truncated cosine illuminations, where equation (32) will consist of two terms.

It is sufficient to consider the following integral

$$\langle I_1 \rangle_{av} = \int_{-a}^{a} \int_{-a-x_1}^{a-x_1} g(x_1)f(u, v)\, du\, dx_1 \,. \tag{33}$$

Referring to Fig. 3, equation (33) can be written as

$$\langle I_1 \rangle_{av} = \int_0^{2a} \int_{-a}^{a-u} g(x_1)f(u, v)\, dx_1\, du + \int_{-2a}^{0} \int_{-a-u}^{a} g(x_1)f(u, v)\, dx_1\, du \tag{34}$$

let

$$G(x_1) = \int g(x_1)\, dx_1$$

then

$$\langle I_1 \rangle_{av} = \int_0^{2a} \{[G(a - u) - G(-a)]f(u, v) + [G(a) - G(-a + u)]f(-u, v)\}\, du. \tag{35}$$

Using equation (35) and assuming uniform illumination, $E_a(x,y) = 1$, two integrations are readily eliminated and equation (28) reduces to

$$\langle I \rangle_{av} = 4 \exp(-\delta^2) \int_0^{2a} \int_0^{2b} (2a - u)(2b - v) \cdot \exp[\delta^2 r(u, v)] \cos \beta_x u \cos \beta_y v\, du\, dv. \tag{36}$$

By expanding the exponential function in a power series, equation (36) can be divided into two parts corresponding to the coherent and incoherent parts of gain, as follows

$$\langle I \rangle_{av} = I_c + I_{inc} \tag{37}$$

where

$$I_c = A^2 \exp(-\delta^2)\left(\frac{\sin \beta_x a}{\beta_x a}\frac{\sin \beta_y b}{\beta_y b}\right)^2 \tag{38}$$

and

$$I_{inc} = 4A \sum_{n=1}^{\infty} \int_0^{2a} \int_0^{2b} \frac{[\delta^2 r(u, v)]^n}{n!} \cos \beta_x u \cos \beta_y v \, dv \, du - \Delta I_{inc} \quad (39)$$

where

$$\Delta I_{inc} = 4 \exp(-\delta^2) \sum_{n=1}^{\infty} \int_0^{2a} \int_0^{2b} [2(bu + av) - uv] \frac{[\delta^2 r(u, v)]^n}{n!} \cdot \cos \beta_x u \cos \beta_y v \, du \, dv \quad (40)$$

and $A = 4ab$ is the aperture area.

The coherent part of the gain is the same as the antenna gain in the absence of surface deviations but multiplied by $\exp(-\delta^2)$. This follows from equation (28) by expanding the exponential function which contains the correlation function in a power series.

To obtain an estimate for the on-axis gain, equation (39) is evaluated for $\beta_x = \beta_y = 0$, and for a gaussian correlation function with

$$r(u, v) = \exp\left(-\frac{u^2 + v^2}{c^2}\right) \quad (41)$$

where c is the correlation distance.

On-axis

$$I_{inc}(0, 0) = \pi A c^2 \sum_{n=1}^{\infty} \frac{(\delta^2)^n}{n!\, n} \left[1 - \frac{c}{2(\pi n)^{\frac{1}{2}}} \left(\frac{1}{a} + \frac{1}{b}\right) + \frac{c^2}{\pi A n}\right]. \quad (42)$$

In equation (42) terms of order $\exp[-n(2a/c)^2]$ and $\exp[-n(2b/c)^2]$ were neglected.

By extending the limits of integrations in equation (39) to ∞, the integration of the first part of this equation can be performed and gives equation (11).

APPENDIX B

On-Axis Gain for Circular Aperture Antennas

For circular aperture antennas the on-axis gain for uniform illumination and a gaussian correlation function is obtained from equation (8) by expanding the exponential function and performing the integrations for the $n = 0$ term, and the integration with respect to the azimuthal coordinates for the remaining terms, resulting in

$$\langle G(0, 0)\rangle_{av} = \left(\frac{\pi D}{\lambda}\right)^2 \exp(-\delta^2) + \left(\frac{8\pi}{\lambda D}\right)^2 \sum_{n=1}^{\infty} \frac{(\delta^2)^n}{n!} I_{cn} \quad (43)$$

where D is aperture diameter, and

$$I_{cn} = \int_0^{D/2} \int_0^{D/2} \exp\left[-\frac{n(\rho^2 + \rho_1)}{c^2}\right] I_0\left(\frac{2n\rho\rho_1}{c^2}\right)\rho\, d\rho\rho_1\, d\rho_1 . \qquad (44)$$

I_o = Modified Bessel function of order zero.

The two integrations in equation (44) can be performed either with the aid of the $Q(y, a_n)$ function defined by[11]

$$Q(y, a_n) = \int_{a_n}^{\infty} \exp\left(-\frac{x^2 + y^2}{2}\right) I_0(xy) x\, dx \qquad (45)$$

or by means of recently evaluated integrals of products of Bessel functions.[12] Let us use the former method. Let

$$x = (2n)^{\frac{1}{2}}\rho/c \qquad (46)$$

$$y = (2n)^{\frac{1}{2}}\rho_1/c \qquad (47)$$

$$a_n = D/c(n/2)^{\frac{1}{2}}. \qquad (48)$$

With equations (45) through (48), equation (44) can be written

$$I_{cn} = \left(\frac{c^2}{2n}\right)^2 \int_0^{a_n} [1 - Q(y, a_n)] y\, dy. \qquad (49)$$

Integrating by parts results in

$$I_{cn} = \left(\frac{c^2}{2n}\right)\left\{\frac{a_n^2}{2}[1 - Q(a_n, a_n)] + \int_0^{a_n} \frac{y^2}{2}\frac{\partial Q}{\partial y}\, dy\right\}. \qquad (50)$$

The derivative in equation (50) can be expressed as

$$\frac{\partial Q}{\partial y} = a_n \exp\left(-\frac{a_n^2 + y^2}{2}\right) I_1(a_n y). \qquad (51)$$

Equation (51) is readily derived from (45) and the following integral[13]

$$\int_0^{\infty} \exp(-t^2/2) J_v(xt) J_v(yt) t\, dt = \exp\left(-\frac{x^2 + y^2}{2}\right) I_v(xy) \qquad (52)$$

where J_v is a Bessel function of order v. Substituting equation (51) into (50) and integrating by parts yields the result

$$I_{cn} = \left(\frac{c^2}{2n}\right)^2 \left\{\frac{a_n^2}{2}[1 - Q(a_n, a_n) - \exp(-a_n^2) I_1(a_n^2)] + \frac{a_n}{2}\int_0^{a_n} \exp\left[-\tfrac{1}{2}(a_n^2 + y^2)\right] \frac{d}{dy}[y I_1(a_n y)]\, dy\right\}. \qquad (53)$$

Using the relation

$$\frac{d}{dy}[yI_1(a_ny)] = a_ny\, I_0(a_ny) \tag{54}$$

and the definition of the Q function, equation (45), yields

$$I_{cn} = \left(\frac{c^2}{2n}\right)\left\{a_n^2[1 - Q(a_n, a_n)] - \frac{a_n^2}{2}\exp(-a_n^2)I_1(a_n^2)\right\}. \tag{55}$$

To evaluate $Q(a_n,a_n)$, the following relation, readily derived from (45) and (52), is used.

$$Q(\alpha, \beta) + Q(\beta, \alpha) = 2 + \int_0^\infty \exp(-\tfrac{1}{2}t^2)\left\{\frac{d}{dt}[J_0(\alpha t)J_0(\beta t)]\right\}dt. \tag{56}$$

Integrating (56) by parts and using (52), gives the known relation

$$Q(y, x) + Q(x, y) = 1 + \exp\left(-\frac{x^2 + y^2}{2}\right)I_0(xy). \tag{57}$$

With equations (55), (57), and (48), (44) is given by

$$I_{cn} = \left(\frac{cD}{4}\right)^2\frac{1}{n}\left\{1 - \exp\left[-\frac{n}{2}\left(\frac{D}{c}\right)^2\right]\left[I_0\left(\frac{n}{2}\frac{D^2}{c^2}\right) + I_1\left(\frac{n}{2}\frac{D^2}{c^2}\right)\right]\right\}. \tag{58}$$

The gain on axis (43) can therefore be written by using equation (58) as:

$$G(0, 0) = \left(\frac{D}{4\epsilon}\right)^2\left[\delta^2\exp(-\delta^2) + \left(\frac{2c}{D}\right)^2\delta^2\exp(-\delta^2)\sum_{n=1}^{\infty}\frac{\delta^{2n}}{n!\,n}[1 - \Delta_n]\right] \tag{59}$$

with

$$\Delta_n = \exp\left[-\frac{n}{2}\left(\frac{D}{c}\right)^2\right]\left[I_0\left(\frac{n}{2}\frac{D^2}{c^2}\right) + I_1\left(\frac{n}{2}\frac{D^2}{c^2}\right)\right]. \tag{60}$$

For $n/2(D/c)^2 \gg 1$, when the modified Bessel functions can be approximated by the first terms of the asymptotic series, Δ_n is then given by equation (15).

REFERENCES

1. Ruze, J., "The Effect of Aperture Errors on the Antenna Radiation Pattern," Suppl. al Nuovo Cimento, *9*, No. 3 (1952), pp. 364–380.
2. Ruze, J., "Antenna Tolerance Theory—A Review," Proc. IEEE, *54*, No. 4 (April 1966), pp. 633–640.
3. Scheffler, H., "Über die Genauigkeitsforderungen bei der Herstellung Opti-

scher Flachen fur Astronomiche Teleskope," Z. Astrophys (German), *55,* No. 1 (1962), pp. 1–20.
4. Chaevski, E. V., "Energy Characteristics of the Field Scattered by a Rough Surface," Izvestiya VUZ. Radiofizika, *8,* No. 6 (1965), pp. 1128–1133.
5. Silver, S., *Microwave Antenna Theory and Design,* McGraw-Hill, New York, 1949.
6. Hoffman, W. C., "Scattering of Electromagnetic Waves from A Random Surface," Quart. Appl. Math., *13,* No. 3 (1955), pp. 291–304.
7. A. Erdelyi et al, *Higher Transcendental Functions,* Vol. 2, McGraw-Hill, 1953, pp. 143–144.
8. Copson, E. T., *Asymptotic Expansions,* Cambridge: The University Press, 1965, pp. 19–21 and pp. 58–60.
9. Barnes, E. W., "On the Asymptotic Expansion of Integral Functions Defined by Taylor Series," Phil. Trans. Royal Soc. London, Series A, *206* (1906), pp. 249–297.
10. Ford, W. B., *Studies on Divergent Series and Asymptotic Developments,* New York: Chelsea Publishing Co., 1960, pp. 269–272.
11. Luke, Y. L., *Integrals of Bessels Functions* New York: McGraw-Hill, 1962, pp. 271–283.
12. Ford, F. A. J., "Some Infinite Integrals Involving Products of Bessel Functions," J. London Math. Soc., *41,* No. 164 (October 1966), pp. 728–730.
13. Watson, G. N., *A Treatise on the Theory of Bessel Functions,* 2nd ed., Cambridge: The University Press, 1962, pp. 393–395.