Analysis of Thermal and Shot Noise in Pumped Resistive Diodes

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This paper discusses certain important aspects of the noise behavior of a pumped resistive diode containing shot and thermal noise sources. The derivation of the following result has a central role in the discussion. It is shown that the noise behavior of a pumped diode which does not contain 1/f *noise sources can be derived in a very simple way from Nyquist's theorem. This follows from the fact that the small-signal terminal behavior of such a diode can always be represented, in the frequency range of practical interest, by means of a connection of two linear and time-invariant networks of which one is noiseless and the other is dissipative, contains only thermal noise sources and is held at a uniform temperature.*

I. INTRODUCTION

The process of frequency conversion and its applications are well known and are extensively treated in the literature.[1–22] This paper considers the special case of a resistive diode frequency converter. An important limitation on the minimum noise figure of such a frequency converter is imposed by the noise generated by the diode, and it is the main purpose of this paper to study the properties of this noise.

Until a few years ago, much of the noise generated by the diode was $1/f$ noise. Therefore, since very little was known about this type of noise, the early theories of frequency converters using positive resistance diodes paid little attention to the noise performance, and somewhat later theories accounted for noise only in a very approximate way. However, as the semiconductor craft has developed, $1/f$ noise has been subject to considerable reduction and, even though its exact mechanism has not yet been completely established, in present diodes it appears to be important only at very low frequencies.[23] Therefore, the study of shot and thermal noise in pumped diodes is of great practical importance.

Strutt showed the method of treating shot and thermal noise in a pumped diode many years ago.[6] Since that time the method has been applied to tunnel diode frequency converters by a number of authors[16–20] However, in the case of a frequency converter using a positive resistance diode, it is normally believed that, in order to calculate its noise figure, a detailed analysis of the noise behavior of the diode is not necessary.[10, 12–14]

Consider a positive resistance diode which does not contain frequency dependent noise sources. That is, assume that, for any fixed voltage v applied to its terminals, the small-signal terminal behavior of the diode is equivalent to that of an ordinary resistor held at a uniform temperature T. The conductance g of this resistor is equal to the differential conductance of the diode and T is the so-called equivalent noise temperature of the diode. Since in a frequency converter the diode is pumped periodically by the pump, v varies with time. Therefore, g and T also vary with time, because they both depend upon v, and one can write $g = g(t)$ and $T = T(t)$.

Normally it is convenient to represent the small-signal terminal behavior of the diode by means of a linear and time-invariant network with several separate terminal pairs, one for each frequency of interest. A study of the noise behavior of this equivalent network generally requires that the self- and cross-power spectral densities of its short-circuit terminal currents, or of its open-circuit terminal voltages, be determined. Normally, however, the difficulty in determining the statistics of these noise terms is overcome by making the assumption that the equivalent network may be treated as an ordinary time-invariant dissipative system which contains only thermal noise sources and is held at a uniform temperature T_x. T_x is normally assumed to be equal to a certain time average of $T(t)$.

Even though no general proof has yet been given for this representation, it is widely used, mainly because it greatly simplifies the treatment of the noise performance of a frequency converter. However, it is often viewed with reservations for several reasons.[15] One very important reason is that it is generally applied to cases, in which one can easily show that it is not applicable, such as cases in which significant $1/f$ noise is generated by the diode. Another reason is that its validity is not obvious even in the limiting case where the noise power available from the diode is frequency independent and does not vary with the applied voltage. In fact, even in this limiting case, it is often considered to be not strictly valid.

However, in this paper it is shown that, besides being valid under certain limiting conditions, such a representation can also be used for formulating and interpreting in a very simple way the noise behavior of a pumped diode under quite general conditions, including a negative resistance diode.

In the following discussion it is assumed that the diode does not contain frequency-dependent noise sources, so that its small-signal terminal behavior may be completely specified by the two time-varying parameters $g(t)$ and $T(t)$. Then it is shown that, in the limiting case where T is a constant, the following theorem is true:

Theorem 1: If a pumped resistive diode is characterized by a time-invariant equivalent noise temperature T, *then its small-signal terminal behavior can be represented by means of a time-invariant equivalent network which contains only thermal noise sources and is held at a uniform temperature* $T_x = T$.

From this general theorem, which is already known to be valid under certain particular circuit conditions,[22] a number of interesting results can be derived. One important result is of course that, in a frequency converter which is bilateral and in which the noise temperature of the diode has negligible variations with time, the noise figure can be readily calculated. In fact, under these limiting conditions the noise figure can be related in a very simple way to T and to the dissipation characteristics of the circuit.[10, 12–14]

Another important result is that, also in the general case where $T(t)$ is not a constant, the terminal behavior of the diode can be readily derived from theorem 1. This is a consequence of the following general property, which follows directly from the definition of $T(t)$ and is stated as a theorem for emphasis:

Theorem 2: Consider a pumped diode characterized by the time-varying parameters $T(t)$ *and* $g(t)$. *Its short-circuit noise current* $\delta n(t)$ *is identical to that of a second diode characterized by a time-invariant temperature* T_2 *and a differential conductance* $|g(t)|T(t)/T_2$.

According to theorem 1, this second diode can be represented by an equivalent network held at a uniform temperature T_2. Therefore, by applying to this equivalent network the generalized form of Nyquist's theorem derived by Twiss,[24] the correlations between the various frequency components of $\delta n(t)$ can be readily determined. One finds that these correlations are simply equal to the Fourier coefficients of $g(t)$.

This property is already known to be valid under certain particular circuit conditions.[16–20]

Now, consider a linear, reciprocal, passive and time-invariant one-terminal pair network containing different elements held at different temperatures. It is well known[25] that at a given frequency ω_1 the effective noise temperature of this network can be expressed as a weighted average of the various temperatures of the lossy elements. The weighting factors in this weighted average are simply equal to the amounts of power that are dissipated by the various lossy elements when the network is connected, at its two terminals, to a generator delivering a unit amount of power at the considered frequency ω_1 . This result is extended, in Section VIII, to a reciprocal and linear network containing a time-varying resistance, by introducing the concept of average temperature T_{av} of a pumped resistive diode. The significance of this parameter is best illustrated by the following example.

Suppose that one wants to calculate the noise power available from the output terminals of a frequency down-converter. It is shown that, if the frequency converter is bilateral, this power can be calculated by replacing the diode with one having the same i-v characteristic and a temperature equal to T_{av}, where T_{av} is given by the relation

$$T_{av} = \frac{\langle P(t)T(t)\rangle_{av}}{\langle P(t)\rangle_{av}}, \tag{1}$$

where $\langle \quad \rangle_{av}$ indicates the time average and $P(t)$ is the instantaneous small-signal power dissipated by the differential conductance of the diode when a small-signal generator is applied to the output terminals of the frequency converter. It is important to point out that T_{av} depends, in general, both on the characteristics of the diode and on those of the circuit connected to it.

II. SMALL SIGNAL EQUATIONS OF A NOISELESS PUMPED DIODE

Let the diode current i be a nonlinear function $f(v)$ of the terminal voltage v. It is assumed that the diode is pumped by a strong periodic source at a frequency ω_o and its harmonics. Therefore v and i contain large components $v_c(t)$ and $i_c(t)$ of the type:

$$v_c(t) = \sum_{k=-\infty}^{\infty} V_k \exp jk\omega_o t \tag{2}$$

$$i_c(t) = \sum_{k=-\infty}^{\infty} I_k \exp jk\omega_o t. \tag{3}$$

It is assumed that v and i contain, in addition, small components $\delta v(t)$ and $\delta i(t)$ and it is desired to derive the relation between $\delta v(t)$ and $\delta i(t)$, for the limiting case $\delta v(t) \to 0$ and $\delta i(t) \to 0$. Thus let

$$v = v(t) = v_c(t) + \delta v(t), \tag{4}$$

$$i = i(t) = i_c(t) + \delta i(t). \tag{5}$$

The differential conductance of the diode is equal to the derivative of $f(v)$. Let it be denoted by $g_d(v)$ and let

$$g(t) = g_d[v_c(t)]. \tag{6}$$

Since $v_c(t)$ is periodic, also $g(t)$ is periodic and therefore it can be written in the form

$$g(t) = \sum_{k=-\infty}^{\infty} g_k \exp jk\omega_o t. \tag{7}$$

Since $i = f(v)$ and $g_d(v)$ is the derivative of $f(v)$, from equations 4, 5, and 6 one has:

$$\delta i(t) = g(t)\ \delta v(t) \tag{8}$$

in the limiting case $\delta v(t) \to 0$. This relation completely describes the small-signal terminal behavior of the diode, in the absence of internal noise sources.

III. SMALL SIGNAL EQUATIONS OF A NOISELESS DIODE IN THE FREQUENCY DOMAIN

From equations 7 and 8 the relations between the different frequency components of $\delta v(t)$ and $\delta i(t)$ can be readily derived[4,7]. In fact, assume that both $\delta v(t)$ and $\delta i(t)$ contain components at only the pairs of side-frequencies $k\omega_o + p$ and $k\omega_o - p$ ($|k| = 0, 1, 2$, etc.; $2p < \omega_o$). Then $\delta v(t)$ and $\delta i(t)$ can be expressed as follows:

$$\delta v(t) = 2(\mathrm{Re})\left[\sum_{k=0}^{\infty} V_{\alpha k} \exp j(p + k\omega_o)t + \sum_{k=1}^{\infty} V_{\beta k} \exp j(p - k\omega_o)t\right] \tag{9}$$

$$\delta i(t) = 2(\mathrm{Re})\left[\sum_{k=0}^{\infty} I_{\alpha k} \exp j(p + k\omega_o)t + \sum_{k=1}^{\infty} I_{\beta k} \exp j(p - k\omega_o)t\right] \tag{10}$$

and, on substituting equations 9, 10, and 7 into equation 8, one obtains the following relations between the Fourier coefficients of the various frequency components of $\delta v(t)$ and $\delta i(t)$:

$$I_{\alpha r} = \sum_{k=0}^{\infty} g_{r-k} V_{\alpha k} + \sum_{k=1}^{\infty} g_{r+k} V_{\beta k} \qquad (r = 0, 1, \quad \text{etc.}) \tag{11}$$

$$I_{\beta r} = \sum_{k=1}^{\infty} g_{k-r} V_{\beta k} + \sum_{k=0}^{\infty} g_{-r-k} V_{\alpha k} \qquad (r = 1, 2, \quad \text{etc.}) \tag{12}$$

which can be written in the form:

$$\begin{bmatrix} I_{\alpha}] \\ I_{\beta}] \end{bmatrix} = \begin{bmatrix} [G_{\alpha\alpha}] & [G_{\alpha\beta}] \\ [G_{\beta\alpha}] & [G_{\beta\beta}] \end{bmatrix} \begin{bmatrix} V_{\alpha}] \\ V_{\beta}] \end{bmatrix} \tag{13}$$

where the matrix notation is defined as follows:

$$I_{\alpha}] = \begin{bmatrix} I_{\alpha 0} \\ I_{\alpha 1} \\ I_{\alpha 2} \\ \vdots \end{bmatrix}, \quad I_{\beta}] = \begin{bmatrix} I_{\beta 1} \\ I_{\beta 2} \\ I_{\beta 3} \\ \vdots \end{bmatrix}, \quad V_{\alpha}] = \begin{bmatrix} V_{\alpha 0} \\ V_{\alpha 1} \\ V_{\alpha 2} \\ \vdots \end{bmatrix}, \quad V_{\beta}] = \begin{bmatrix} V_{\beta 1} \\ V_{\beta 2} \\ V_{\beta 3} \\ \vdots \end{bmatrix} \tag{14}$$

and the elements of the matrices $[G\alpha\alpha]$, $[G\alpha\beta]$, etc., are

$$(G_{\alpha\alpha})_{r,k} = g_{r-k} \qquad (r, k = 0, 1, \quad \text{etc.}) \tag{15}$$

$$(G_{\beta\beta})_{r,k} = g_{k-r} \qquad (r, k = 1, 2, \quad \text{etc.}) \tag{16}$$

$$(G_{\alpha\beta})_{r,k} = g_{r+k} \qquad (r, k - 1 = 0, 1, \quad \text{etc.}) \tag{17}$$

$$(G_{\beta\alpha})_{r,k} = g_{-r-k} \qquad (r - 1, k = 0, 1, \quad \text{etc.}). \tag{18}$$

Equations 13 through 18 completely specify the terminal behavior of the diode at the frequencies $p \pm k\omega_0$ in the absence of internal noise sources.

IV. SMALL SIGNAL TERMINAL BEHAVIOR OF A NOISY DIODE

Up to this point the noise generated by the diode has been ignored. In the general case of a noisy diode equation 8 has to be modified as follows:

$$\delta i(t) = g(t)\, \delta v(t) + \delta n(t) \tag{19}$$

where $\delta n(t)$ is the equivalent short-circuit noise current of the diode. Equation 19 corresponds to the equivalent circuit shown in Fig. 1 in which the spontaneous fluctuations of the diode are ascribed to a current generator of infinite internal impedence, acting in parallel to the differential conductance of the diode.

Now, consider the components of $\delta n(t)$ occurring in an infinitesimal frequency range between $\omega - (d\omega)/2$ and $\omega + (d\omega)/2$. It is convenient to account for these components by means of a single pseudosinusoid

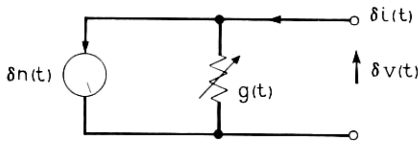

Fig. 1 — Equivalent circuit of a time-varying conductance containing noise sources.

with random complex amplitude[26-28]. Then, let $N_{\alpha k}$ and $N_{\beta r}$ (k = 0, 1, 2, etc.; r = 1, 2, etc.) be the complex Fourier amplitudes of the pseudosinusoids relative to the frequencies $p + k\omega_o$ and $p - k\omega_o$, respectively. Noise components occurring at frequencies different from these will be neglected since they have no effect on the small-signal terminal behavior of the diode at the frequencies $p \pm k\omega_o$. Then

$$\delta n(t) = 2(\mathrm{Re})\left\{\sum_{k=0}^{\infty} N_{\alpha k} \exp j(p + k\omega_o)t + \sum_{k=1}^{\infty} N_{\beta k} \exp j(p - k\omega_o)t\right\} \tag{20}$$

and from equations 13 and 19 one obtains:

$$\begin{bmatrix} I_\alpha] \\ I_\beta] \end{bmatrix} = \begin{bmatrix} [G_{\alpha\alpha}] & [G_{\alpha\beta}] \\ [G_{\beta\alpha}] & [G_{\beta\beta}] \end{bmatrix} \begin{bmatrix} V_\alpha] \\ V_\beta] \end{bmatrix} + \begin{bmatrix} N_\alpha] \\ N_\beta] \end{bmatrix} \tag{21}$$

where

$$N_\alpha] = \begin{bmatrix} N_{\alpha 0} \\ N_{\alpha 1} \\ \vdots \end{bmatrix} \quad \text{and} \quad N_\beta] = \begin{bmatrix} N_{\beta 1} \\ N_{\beta 2} \\ \vdots \end{bmatrix}. \tag{22}$$

Equation 21 completely specifies the small-signal terminal behavior of a pumped diode containing noise sources. Its physical interpretation is often facilitated by introducing the equivalent circuit of Fig. 2. In this equivalent circuit the diode is represented by a linear and time-invariant network in which the terminal voltages and currents occur at the same frequency. Their Fourier coefficients are equal to those of the various frequency components of $\delta v(t)$ and $\delta i(t)$.

The network of Fig. 2 is completely specified with respect to its terminal pairs by its admittance matrix

$$[G] = \begin{bmatrix} [G_{\alpha\alpha}] & [G_{\alpha\beta}] \\ [G_{\beta\alpha}] & [G_{\beta\beta}] \end{bmatrix} \tag{23}$$

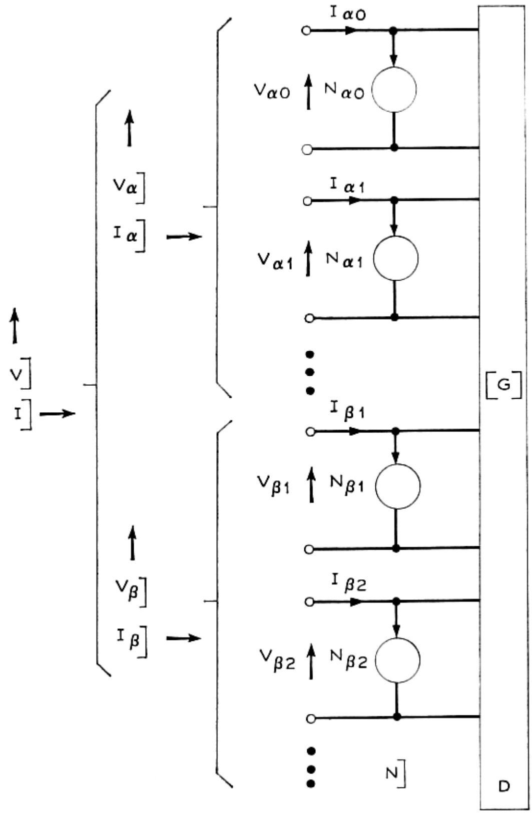

Fig. 2 — Time-invariant equivalent network of a pumped diode.

and by the noise column matrix

$$N] = \begin{bmatrix} N_\alpha] \\ N_\beta] \end{bmatrix} \tag{24}$$

which represents the complex Fourier amplitudes of its short-circuit terminal currents. The self- and cross-power spectral densities of these noise currents are

$$\frac{\langle N_{\alpha k} N_{\alpha r}^* \rangle}{df}, \quad \frac{\langle N_{\beta k} N_{\beta r}^* \rangle}{df}, \quad \text{and} \quad \frac{\langle N_{\alpha k} N_{\beta r}^* \rangle}{df} \tag{25}$$

where $\langle\ \rangle$ indicates the statistical average. They are conveniently represented by the matrix

$$\frac{1}{df} \langle N]N]^{\dagger}\rangle, \tag{26}$$

where superscript† denotes the Hermitian conjugate.

Now let the properties of the equivalent network of Fig. 2 be briefly examined. Since $g_k = g_{-k}^*$ ($k = 0, 1, 2,$ etc.), from equations 15 through 18 one has that the admittance matrix $[G]$ is Hermitian, that is:

$$[G] = [G]^{\dagger}. \tag{27}$$

It is interesting to note that this condition is equivalent to the condition that

$$(IM)(V]^{\dagger}[G]V]) = 0 \tag{28}$$

for all $V]$, which requires that the total reactive power flowing into the nonlinear resistance at the various side frequencies $p \pm k\omega_0$ ($k =$ 0, 1, etc.) be always zero. This property is a direct consequence of the general energy relations derived by Manley and Rowe for nonlinear resistors.[29] Because of equation 27 the average small-signal power dissipated in the admittance $g(t)$ can be expressed as

$$\langle \delta v(t)^2 g(t) \rangle_{av} = V]^{\dagger}([G] + [G]^{\dagger})V] = 2V]^{\dagger}[G]V]. \tag{29}$$

Therefore, if

$$g(t) > 0 \tag{30}$$

at all times, then $[G]$ is both Hermitian and positive definite and the equivalent network is dissipative.

Now, consider a linear and dissipative network which contains only thermal noise sources and is characterized by an admittance matrix equal to $[G]$. If such a network is held at a uniform temperature T, then the various spectral densities of its short-circuit terminal currents are simply given by the elements of the matrix

$$kT([G] + [G]^{\dagger}).$$

From this generalized form of Nyquist's Theorem, proved by Twiss,[24] and from equation 27 one has that, if condition 30 is satisfied and the matrix 26 satisfies the relation

$$\langle N]N]^{\dagger}\rangle = 2kTdf[G], \tag{31}$$

then the small-signal terminal behavior of the diode can be represented by means of an equivalent network which contains only thermal noise sources and is held at a uniform temperature T.

Of special importance is the particular case in which the circuit connected to the diode is resistive at the harmonics $2\omega_o$, $3\omega_o$, etc., of the pump frequency and, at these frequencies, does not contain generators. Under these conditions it is always possible to choose the origin of time in such a way as to make $v_c(t)$, $i_c(t)$ and $g(t)$ even functions of time.[7] In this case, since all of the coefficients g_k ($k = 0, 1$, etc.) become real,

$$g_k = g_{-k} \; , \tag{32}$$

and therefore $[G]$ becomes a real symmetric matrix, because of equations 15 through 18. If, in addition to equation 32, condition 30 is satisfied, then the equivalent network of Fig. 2 can be realized by means of an ordinary resistive network.[2]

Of course, in the general case where the origin of time cannot be chosen to make all the coefficients g_k real, the diode cannot be represented by a reciprocal (bilateral) network.

Condition 31 is never satisfied if $g(t)$ becomes negative for some values of t. In fact in this case $[G]$ is indefinite, while $\langle N]N]^\dagger\rangle$ is always a positive definite or semidefinite matrix. On the other hand, if

$$g(t) < 0 \tag{33}$$

for all values of t, then $[G]$ is negative definite and of special interest becomes the condition

$$\langle N]N]^\dagger\rangle = -2kT \, df[G]. \tag{34}$$

In fact, consider a pumped negative resistance diode which satisfies this condition. If a frequency converter is made from such a diode by imbedding it in a lossless network, then its noise measure, defined by Hauss and Adler,[30] is independent of the characteristics of the lossless network and is simply equal to T/T_o, where T_o is standard temperature, 290°K. Therefore, if G_e is the exchangeable gain of such a frequency converter, its noise figure F is simply equal to

$$F = 1 + T/T_o \, (1 - 1/G_e). \tag{35}$$

V. SHOT NOISE IN A PUMPED DIODE

Assume that the diode only contains shot noise sources and that in the frequency range of interest transit time effects can be neglected.[6, 11, 26, 31, 32]

Assume for the moment that the voltage v applied to the diode is time-invariant. Then $\delta n(t)$ can be treated as white noise over the frequency range of practical interest. Therefore, if S denotes its spectral density, it can be expressed as

$$\delta n(t) = S^{\frac{1}{2}} x(t), \tag{36}$$

where $x(t)$ is white noise with unit spectral density. S will in general depend upon the voltage v applied to the diode and it is convenient to express this voltage dependence in the following form:

$$S = S_d(v) = 2kT_d(v) \mid g_d(v) \mid \tag{37}$$

where $kT_d(v)/2$ represents the exchangeable noise power at the diode terminals, per unit bandwidth. In equation 37 the occurrence of the factor 2, in place of the usual factor 4, results from the fact that here both positive and negative frequencies are considered.

Now, consider the general case where v is not a constant and let the concise notation

$$\begin{aligned} T(t) &= T_d[v_c(t)] \\ S(t) &= S_d[v_c(t)] = 2kT(t) \mid g(t) \mid \end{aligned} \tag{38}$$

be introduced. Then $\delta n(t)$ results from the superposition of statistically independent random disturbances whose probability of occurrence is proportional to the deterministic and periodic function

$$h(t) = [S(t)].^{\frac{1}{2}} \tag{39}$$

Since it is assumed that the duration of these disturbances is much smaller than the reciprocal of the highest significant frequency of $h(t)$, equation 36 is still applicable and therefore

$$\delta n(t) = h(t)x(t). \tag{40}$$

Now let consideration be restricted to the fluctuation components occurring in infinitesimal frequency intervals of width df, centered at the frequencies $p \pm k\omega_o$. Then, since $x(t)$ is white noise with unit spectral density, from equation 40 one has that $\delta n(t)$ can be expressed as follows:[6,26]

$$\delta n(t) = h(t) \sum_{s=-\infty}^{\infty} 2(df)^{\frac{1}{2}} \cos\left[(s\omega_o + pt) + \varphi_s\right] \tag{41}$$

where φ_s are statistically independent random phase angles distributed uniformly over the range $(0, 2\pi)$.

Let $h(t)$ and $S(t)$ be represented by the Fourier series

$$S(t) = \sum_{k=-\infty}^{\infty} S_k \exp jk\omega_o t \tag{42}$$

$$h(t) = \sum_{k=-\infty}^{\infty} H_k \exp jk\omega_o t. \tag{43}$$

From equation 39 one has that S_k and H_k are related through the relations:

$$S_r = \sum_{k=-\infty}^{\infty} H_k H_{r-k} \ . \tag{44}$$

Introduction of equation 43 in equation 41 gives:

$$\begin{aligned}
\delta n(t) &= (df)^{\frac{1}{2}} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} H_r \{\exp j[(s+r)\omega_o t + pt + \varphi_s] \\
&\qquad + \exp j[(r-s)\omega_o t - pt - \varphi_s]\} \\
&= (df)^{\frac{1}{2}} \sum_{s=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} H_{k-s} \exp j\varphi_s \exp j(k\omega_o + p)t \\
&\qquad + H_{s-k} \exp -j\varphi_s \exp -j(k\omega_o + p)t \\
&= 2(\mathrm{Re})\left\{(df)^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} H_{k-s} \exp j\varphi_s \exp j(k\omega_o + p)t\right\}.
\end{aligned} \tag{45}$$

From this last relation and from equation 20 one obtains

$$N_{\alpha k} = (df)^{\frac{1}{2}} \sum_{s=-\infty}^{\infty} H_{k-s} \exp j\varphi_s \tag{46}$$

$$N_{\beta k} = (df)^{\frac{1}{2}} \sum_{s=-\infty}^{\infty} H_{-k-s} \exp j\varphi_s \ . \tag{47}$$

Hence, since

$$\langle \exp j\varphi_s \exp -j\varphi_r \rangle = \begin{cases} 1, & r = s \\ 0, & r \neq s \end{cases},$$

from equations 46 and 47 one obtains:

$$\begin{aligned}
\frac{\langle N_{\alpha k} N_{\alpha r}^* \rangle}{df} &= \sum_{s=-\infty}^{\infty} H_{k-s} H_{s-r} \\
&= S_{k-r}
\end{aligned} \tag{48}$$

and

$$\frac{\langle N_{\alpha k} N_{\beta r}^* \rangle}{df} = \sum_{s=-\infty}^{\infty} H_{k-s} H_{r+s} \qquad (49)$$
$$= S_{k+r} \, .$$

Now, consider a time-varying conductance equal to $S(t)$ and let $[S]$ denote its admittance matrix. Then $[S]$ is obtained from equations 15 through 18, and 23 by formally replacing g_k and G with S_k and S throughout. Therefore the elements of $[S]$ are equal to the various Fourier coefficients S_k and, from equations 48 and 49, one obtains the final result

$$\langle N]N]^\dagger \rangle = df[S]. \qquad (50)$$

VI. NOISE BEHAVIOR OF THE DIODE

The preceding section showed that if a diode contains only shot noise sources, then the self- and cross-power spectral densities of its short-circuit terminal currents are simply equal to the Fourier coefficients of $2kT(t) \; |g(t)|$, over the frequency range of practical interest. Let us examine the significance of these relations, which are valid even if the diode contains thermal noise sources.

First, consider the special case of a positive resistance diode characterized by an equivalent noise temperature $T_d(v)$ which is approximately constant over the range of voltages of interest, so that the approximation

$$T_d(v) = T = \text{constant} \qquad (51)$$

can be made. In this case since equations 38 give

$$S_k = 2kTg_k \, , \qquad (52)$$

one has

$$[S] = 2kT[G] \qquad (53)$$

and therefore from equation 50 it follows that the spectral density matrix satisfies condition 31. One concludes that, if $T_d(v)$ is independent of v and condition 30 is satisfied, then the small-signal terminal behavior of the diode can be represented by a time-invariant dissipative network held at a uniform temperature T, as stated in theorem 1.

Thus, in the limiting case (51) and under the restriction $g(t) > 0$, equation 50 can be interpreted as a direct consequence of theorem 1 and Nyquist's theorem. Now, a little reflection shows why equation 50 also is valid in the general case where $T(t)$ is not a constant and the restriction $g(t) > 0$ is removed. In fact, two diodes having the same $S(t)$ have the same short-circuit terminal currents, no matter what their differential admittances may be. Notice that from this rather obvious property theorem 2 follows at once. That is, the short-circuit terminal currents of a diode characterized by an equivalent noise temperature $T(t)$ and by a differential conductance $g(t)$ are identical to those of a diode characterized by a constant temperature T_2 and a differential conductance

$$| g(t) | [T(t)]/T_2, \tag{54}$$

where T_2 is an arbitrary temperature. It is important to point out that, even though the foregoing two diodes have the same short-circuit terminal currents, they are not equivalent since they have different conductances. On the other hand, the terminal behavior of a diode characterized by a voltage-dependent temperature $T_d(v)$ and a conductance $g_d(v)$ is equal to that of the parallel connection of the two diodes (see Fig. 3) with voltage-independent temperatures T_1 and T_2 and with the differential conductances $g_{d1}(v)$ and $g_{d2}(v)$ defined by the following equations:

$$g_{d1}(v) + g_{d2}(v) = g_d(v) \tag{55}$$

$$g_{d1}(v)T_1 + g_{d2}(v)T_2 = g_d(v)T_d(v) \tag{56}$$

where T_1 and T_2 are subject to the only condition

$$T_1 < T_d(v) < T_2 \tag{57}$$

which guarantees that $g_{d1}(v)$, $g_{d2}(v)$ and $g_d(v)$ have all the same sign. Notice that the equivalent circuit of Fig. 3 and the original diode have the same short-circuit terminal currents because of equa-

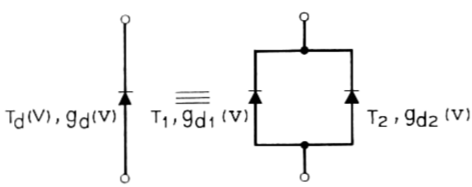

Fig. 3 — Representation of an arbitrary noisy resistive diode by means of two diodes with voltage-independent noise temperatures T_1 and T_2.

tion 56, and have the same differential conductances because of equation 55. An important feature of this equivalent circuit is that theorem 1 is applicable to both diodes and it can therefore be studied by standard techniques. Particularly interesting is the limiting case $T_1 = 0$. In fact in this case one of the two diodes becomes noiseless and the other has the time-varying conductance

$$g_2(t) = g_{d2}[v_c(t)] = g(t)[T(t)]/T_2 , \tag{58}$$

when the pump voltage $v_c(t)$ is applied to it. Hence, by comparing equation 54 with equation 58 one obtains the following result. If $g(t) > 0$ and $T_2 > T(t)$ for all values of t, then, by connecting a noiseless diode having the conductance

$$g_1(t) = g_{d1}[v_c(t)] = g(t)[T_2 - T(t)]/T_2 \tag{59}$$

in parallel with the second diode of theorem 2, one obtains a circuit completely equivalent to the original diode.

Now, consider the case where $g(t) < 0$ for all values of t and suppose that condition 51 is satisfied. Then

$$[S] = -2kT[G] \tag{60}$$

and from equation 50 one has that condition 34 is satisfied. Hence, the remarks about this possibility at the end of Section III apply. In general, where $T(t)$ is not a constant, equation 35 is not valid. However, if T_1 and T_2 are the minimum and maximum values of $T(t)$, so that

$$T_1 \leqq T(t) \leqq T_2 \tag{61}$$

then one can say that the noise performance of the diode will be bounded by the two limiting values obtained from equation 35 for the two limiting cases $T = T_1$ and $T = T_2$.

VII. TERMINAL BEHAVIOR OF THE DIODE IN THE IMPEDANCE-MATRIX REPRESENTATION

In some cases it is convenient to use the impedance-matrix representation, rather than the admittance-matrix representation, for describing the terminal behavior of the diode. Let

$$r(t) = 1/g(t) = \sum_{k=-\infty}^{\infty} r_k \exp jk\omega_o t \tag{62}$$

be the differential resistance of the diode. Then the impedance-matrix representation of the small-signal terminal behavior of the diode can be written in the form

$$V] = [R]I] + N_v] \tag{63}$$

where the relations between the elements of the impedance-matrix $[R]$ and the coefficients r_k are identical to those between the elements of $[G]$ and g_k (see equations 15 through 18). The column matrix $N_v]$ consists of the amplitudes of the open-circuit terminal voltages of the diode. If

$$0(t) = 2kT(t) \mid r(t) \mid = \sum_{k=-\infty}^{\infty} 0_k \exp jk\omega_o t \tag{64}$$

then one has

$$\langle N_v]N_v]^\dagger \rangle = df[0] \tag{65}$$

where $[0]$ can be obtained from equations 15 through 18, and 23 by replacing G and g_k with 0 and 0_k throughout. Notice that equation 65, which is analogous to equation 50, follows from theorem 1, Nyquist's theorem[24] and the fact that two diodes having the same $0(t)$ have the same $N_v]$, no matter what their differential resistances may be.

VIII. AVERAGE TEMPERATURE OF A PUMPED DIODE

Consider a linear, reciprocal, passive and time-invariant one-terminal pair network containing different elements held at different temperatures. It is well known[25] that at a given frequency ω_1 the effective noise temperature of this network can be expressed as a weighted average of the various temperatures of the lossy elements. The weighting factors in this weighted average are simply equal to the amounts of power that are dissipated by the various lossy elements when the network is connected, at its two terminals, to a generator delivering a unit amount of power at the considered frequency ω_1. This result is extended, in this section, to a pumped diode.

The concept of average noise temperature T_{av} of a pumped diode is introduced in this section. Consideration is restricted to the case where $g(t) \geqq 0$ and condition 32 is satisfied, so that the equivalent circuit of the diode is passive and bilateral. It is shown that T_{av} depends, in general, both on the characteristics of the diode and on those of the linear and time-invariant circuit connected to it. However, if certain conditions are satisfied, then it only depends on the diode characteristics.

Consider a one-terminal-pair network N consisting of a pumped diode imbedded in a linear, time-invariant and bilateral two-terminal-pairs network N' (Fig. 4). Assume, furthermore, that the network N can exchange power at a single frequency ω_1, at its terminals. Then the average temperature T_{av} of the diode is defined in the following way:

T_{av} is such that the noise power available from the terminals of the network N does not change if the actual temperature-voltage characteristic $T_d(v)$ of the diode is replaced with a constant temperature equal to T_{av}.

Now let a small-signal generator of frequency ω_1 be connected to the terminals of the network N, and let $\delta i(t)$ and $\delta v(t)$ be the small signals produced at the diode terminals. It will be shown that

$$T_{av} = \frac{\displaystyle\int_0^{2\pi/\omega_0} \delta i(t)\ \delta v(t) T(t)\ dt}{\displaystyle\int_0^{2\pi/\omega_0} \delta i(t)\ \delta v(t)\ dt}. \tag{66}$$

Notice that this equation is equivalent to equation 1.

Proof: It is convenient to represent the circuit of Fig. 4 by means of the equivalent circuit of Fig. 5, where the network D represents the small-signal terminal behavior of the diode and each terminal pair of D exchanges power at only one frequency. Notice that in Fig. 5 the network N' has been represented by means of several separate equivalent circuits, N_1', N_2', etc., one for each frequency of interest.

The network D can be decomposed into two separate networks D_1 and D_2, each held at a uniform temperature.

In fact, let the diode of Fig. 4 be replaced by the two diodes shown in Fig. 3 and let $\delta i_1(t)$ and $\delta i_2(t)$ be the small-signal currents of the

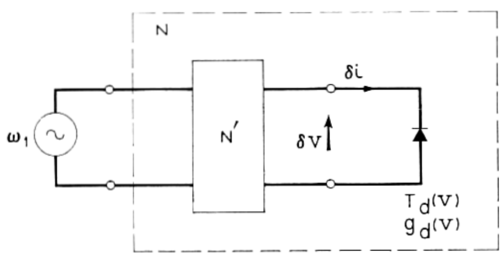

Fig. 4 — Diode imbedded in a linear, time-invariant and bilateral network N'.

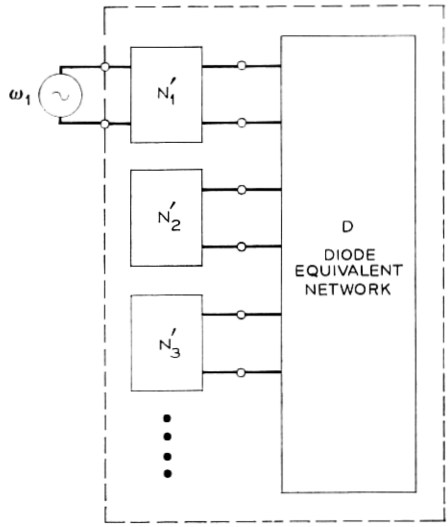

Fig. 5 — Equivalent circuit of the network N of Fig. 4.

two diodes. Then, from equation 56 one has:

$$\delta v(t)\ \delta i(t)\, T(t) = \delta v(t)\ \delta i_1(t)\, T_1 + \delta v(t)\ \delta i_2(t)\, T_2$$

which gives:

$$\frac{\langle \delta v(t)\ \delta i(t) T(t)\rangle_{\text{av}}}{\langle \delta v(t)\ \delta i(t)\rangle_{\text{av}}} = \frac{\langle \delta v(t)\ \delta i_1(t)\rangle_{\text{av}} T_1 + \langle \delta v(t)\ \delta i_2(t)\rangle_{\text{av}} T_2}{\langle \delta v(t)\ \delta i(t)\rangle_{\text{av}}}. \tag{67}$$

Since theorem 1 is applicable to both diodes of Fig. 3, it is clear that the network D of Fig. 5 can be represented by the parallel connection of two networks (D_1 and D_2) of which one is held at a uniform temperature T_1 and dissipates an average power equal to $\langle \delta v(t) \delta i_1(t)\rangle_{\text{av}}$, and the other is at T_2 and dissipates $\langle \delta v(t) \delta i_2(t)\rangle_{\text{av}}$.

Now, since both D_1 and D_2 are bilateral, the noise power available from the network N of Fig. 5 does not vary[25] if the temperatures of D_1 and D_2 are changed so that they become equal to

$$\frac{\langle \delta v(t)\ \delta i_1(t)\rangle_{\text{av}} T_1 + \langle \delta v(t)\ \delta i_2(t)\rangle_{\text{av}} T_2}{\langle \delta v(t)\ \delta i_1(t)\rangle_{\text{av}} + \langle \delta v(t)\ \delta i_2(t)\rangle_{\text{av}}}$$

which, together with eq. 67 gives eq. 66.

Equation 66 is of particular interest when $\omega_1 = p$, since in this case it corresponds to the example considered in the introduction. Notice that T_{av} is not, in general, a function of the diode characteristics alone.

In fact, it also depends both on the particular characteristics of the network N' in which the diode is imbedded and on the value of the frequency ω_1 at which T_{av} is defined, unless $T(t)$ is constant.

Under conditions of practical interest T always varies with time and consequently a direct application of theorem 1 is never strictly valid. Equation 66 shows, however, that if certain conditions are satisfied, then T_{av} is little affected by the particular choice of ω_1 and N', and consequently a direct application of theorem 1 may not introduce significant errors. More precisely, suppose that either

$$g_d(v) \approx 0 \qquad \text{or} \qquad \frac{1}{g_d(v)} \approx 0 \tag{68}$$

for some values of v and that

$$T_d(v) \approx T' = \text{constant} \tag{69}$$

over the range of voltages for which conditions 68 are not satisfied. Under these conditions either $T(t) \approx T'$ or $\delta v(t)\ \delta i(t) \approx 0$, for all values of t, and consequently from equation 66 one obtains $T_{av} \approx T'$. Therefore in this case, and only in this case, T_{av} can be regarded as a function of the diode characteristics alone and theorem 1 is applicable, with T replaced by T'.

An important application of the preceding result is given by an ideal Schottky barrier diode. In fact, the relations derived in Ref. 22 between the noise figure and the conversion loss of such a diode imply that its junction can be represented, under certain particular conditions, by means of an ordinary resistive network held at half the temperature T_o of the junction.

REFERENCES

1. Strutt, M. J. O., "Mixing Valves," Wireless Eng., *12,* No. 137 (February 1935, pp. 59–64.
2. Peterson, E. and Hussey, L. W., "Equivalent Modulator Circuits," B.S.T.J., *18,* No. 1 (January 1939), pp. 32–48.
3. Herald, E. W. and Malter, L., "Some Aspects of Radio Reception at Ultra-High Frequencies," Proc. IRE, *31,* No. 10 (October 1943), pp. 575–582.
4. Peterson, L. C. and Llewellyn, F. B., "The Performance of Mixers in Terms of Linear-Network Theory," Proc. IRE, *33,* No. 7 (July 1945), pp. 458–476.
5. Herold, E. W., Bush, R. R., and Ferris, W. R., "Conversion Loss of Diode Mixers Having Image-Frequency Impedance," Proc. IRE, *33,* No. 9 (September 1945), pp. 603–609.
6. Strutt, M. J. O., "Noise Figure Reduction in Mixer Stages," Proc. IRE, *34,* No. 12 (December 1946), pp. 942–950.
7. Torrey, H. C. and Whitmer, C. A., *Crystal Rectifiers,* Radiation Laboratory Series, *15,* New York: McGraw-Hill, 1948.
8. Pound, R. V., *Microwave Mixers,* Radiation Laboratory Series, *16,* New York: McGraw-Hill, 1948.

9. Crawford, A. B., material included in G. C. Southworth's *Principles and Applications of Waveguide Transmission,* Princeton, N. J.: van Nostrand, 1950, pp. 626–636.
10. Strum, P. D., "Some Aspects of Crystal Mixer Performance," Proc. IRE, *41,* No. 7 (July 1953), pp. 876–889.
11. van der Ziel, A., *Noise,* New York: Prentice-Hall, 1954.
12. Pritchard, W. L., "Notes on a Crystal Mixer Performance," Trans. IRE, *MTT-3,* No. 11 (January 1955), pp. 37–39.
13. Strum, P. D., "A Note on Noise Temperature," IRE Trans., Microwave Theory and Technique, *MTT-4,* No. 3 (July 1956), pp. 145–151.
14. Messenger, G. C. and McCoy, C. T., "Theory and Operation of Crystal Diodes as Mixers," Proc. IRE, *45,* No. 9 (September 1957), pp. 1269–1283.
15. Houlding, N., "Mixer Crystal Noise," Proc. IRE, *46,* No. 5, Part 1 (May 1958), pp. 917–918.
16. Kim, C. S., "Tunnel Diode Converter Analysis," IRE Trans. Electron Devices, *ED-8,* No. 5 (September 1961), pp. 394–405.
17. Pucel, R. A., "Theory of the Esaky Diode Frequency Converter," Solid-State Electronics, *3,* No. 3 (November 1961), pp. 167–207.
18. Lo, Shih-Fang, "Noise in Tunnel Diode Mixers," Proc. IRE, *49,* No. 11 (November 1961), pp. 1688–1689.
19. Sterzer, F. and Presser A., "Stable Low Noise Tunnel-Diode Frequency Converters," RCA Rev., *23,* No. 1 (March 1962), pp. 3–28.
20. Barber, M. R., "A Numerical Analysis of the Tunnel-Diode Frequency Converter," IEE Trans., Microwave Theory and Technique, *MTT-13,* No. 5 (September 1965), pp. 663–665.
21. Barber, M. R. and Ryder, R. M., "Ultimate Noise Figure and Conversion Loss of the Schottky Barrier Mixer Diode," Int. Microwave Symp. Digest, May 1966, pp. 13–17.
22. Barber, M. R., "Noise Figure and Conversion Loss of the Schottky Barrier Mixer Diode," IRE Trans., Microwave Theory and Technique, *MTT-15,* No. 11 (November 1967), pp. 629–635.
23. Eng, S. T., "A New Low 1/f Noise Mixer Diode; Experiments, Theory and Performance," Solid State Elec., *8,* No. 1 (January 1965), pp. 59–77.
24. Twiss, R. Q., "Nyquist's and Thevin's Theorems Generalized for Non-reciprocal Linear Networks," J. Appl. Phys., *26,* No. 5 (May 1955), pp. 599–602.
25. Siegman, A. E., "Thermal Noise in Microwave Systems," Microwave J., *4,* No. 3 (March 1961), pp. 81–90.
26. Rice, S. O., "Mathematical Analysis of Random Noise," B.S.T.J., *23,* No. 3 (July 1944), pp. 282–332.
27. Bennett, V. R., *Electrical Noise,* New York: McGraw-Hill, 1960.
28. Rowe, H. E., *Signals and Noise in Communications Systems,* Princeton, N. J.: van Nostrand, 1965.
29. Manley, J. M. and Rowe, H. E., "Some General Properties of Nonlinear Elements, Part 1, General Energy Relations," Proc. IRE, *44,* No. 7 (July 1956), pp. 904–913.
30. Adler, R. B. and Haus, H. A., *Circuit Theory of Linear Noisy Networks,* New York: John Wiley & Sons, 1959.
31. Uhlir, Jr., A., "Shot Noise in p-n Junction Frequency Converters," B.S.T.J., *37,* No. 4 (July 1958), pp. 951–988.
32. Davenport, W. B. and Root, W. L., *Random Signals and Noise,* New York: McGraw-Hill, 1958.