The Effects of Strain on Electromagnetic Modes of Anisotropic Dielectric Waveguides at p-n Junctions

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(Manuscript received May 22, 1968)

A first order perturbation expansion is carried out in order to analyze the effect of small spatially uniform strains on the lowest order (even) TE and TM modes in an anisotropic dielectric waveguide. This generalizes the results of an earlier paper in which the effects of certain special cases of uniform strain were calculated. Unlike in these special cases, the perturbed modes are, in general, neither purely TE or TM, and one effect of two of the offdiagonal components of the strain is to tilt the plane of polarization and change the relative phase of the two polarizations. To first order, the modes are not exponentially attenuated. Some numerical examples are considered in order to illustrate the results. It is found that, under appropriate conditions, the effect of the small strain may be quite large in relation to its magnitude.

I. INTRODUCTION

The concept of a multilayered dielectric waveguide is central to the theory of the GaP electro-optic diode modulator.[1–9] As part of a detailed study of the properties of electro-optic diode modulators, Nelson and McKenna[4] have investigated the possible discrete modes which can propagate in a number of such waveguides and have calculated the detailed properties of the lowest order mode of each polarization.

In the fabrication of a p-n junction a certain amount of strain is always introduced. Because of the photoelastic effect[10] this strain will induce a change in the dielectric matrix describing the unstrained p-n junction. In general the strain will be spatially nonuniform, making it extremely difficult to calculate modes in such a structure. However, a knowledge of the effect of a spatially uniform strain on the mode structure would provide insight into the effects of nonuniform strain. The effects of certain special cases of uniform strain on the modes of a simple model of a dielectric waveguide were calculated in Ref. 4. In the present paper we complete this investigation and calculate the modes in the same model dielectric waveguide when subjected to an arbitrary uniform strain. We use first order perturbation theory in a small parameter describing the magnitude of the strain.

Although the work presented in this paper was motivated by research on the theory of the electro-optic diode modulator, the results have considerable relevance to the theory of the GaAs injection laser. Here too, various dielectric waveguide models have been used to explain the light containment.[11–14] The same problems of strain exist, and the results of this paper give a qualitative picture of the effects of strain on modal structure. The effect of strain on completely different types of electro-optic light modulators have been studied by Kaminow[15] and by DiDomenico and Anderson.[16]

II. FORMULATION OF THE PROBLEM, AND RESULTS

In Ref. 4 the symmetric step model was used to study the effects of strain. This very simple model exhibits many of the main features of interest in dielectric waveguide models.

The model consists of an anisotropic crystalline slab bounded by the planes $x = \pm w$, whose refractive index is raised uniformly by some constant amount, the physical origin of which is still obscure, and which is embedded in an isotropic medium of relatively lower index of refraction. The central slab represents the junction region whose anisotropy is caused by the junction field $\mathbf{E}_J$ acting through the electro-optic effect.[9] The direction of the x-axis is always taken parallel to $\mathbf{E}_J$. The isotropic medium represents the normal GaP.

The model is determined by its dielectric matrix, which in the absence of strain and for certain orientations of $\mathbf{E}_J$ with respect to the crystal axes can be diagonal in a coordinate system having its x-axis parallel to $\mathbf{E}_J$. For such orientations of $\mathbf{E}_J$, the diagonal matrix elements of the dielectric matrix $K_\alpha^{(0)}(x)$, $\alpha = 1, 2, 3$, depend only on x. (We use x, y, z for the coordinates rather than x_1, x_2, x_3.) The matrix elements in the absence of strain are then defined by the equations

$$K_\alpha^{(0)}(x) = K_\alpha , \qquad | x | < w \tag{1}$$

$$K_\alpha^{(0)}(x) = K_0 , \qquad | x | > w \tag{2}$$

where $\alpha = 1, 2, 3$ (see Fig. 1).

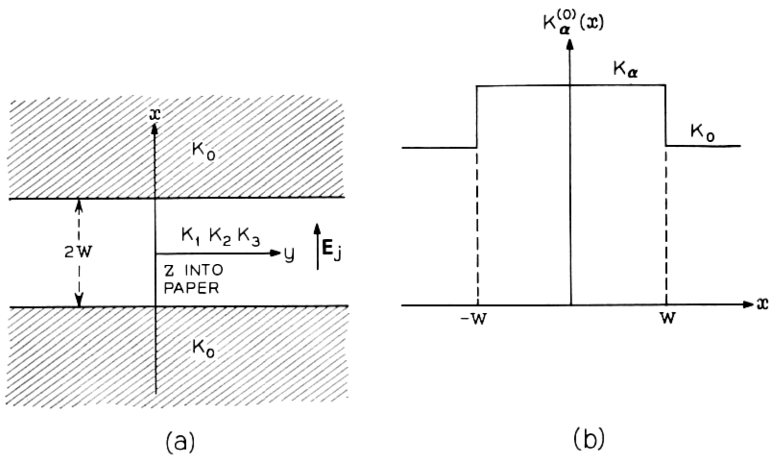

Fig. 1 — (a) The coordinate system used in the symmetric step model. (b) A graph of $K_a(x)$.

There are two orientations of $\mathbf{E}_J$ of particular interest which allow us to diagonalize the dielectric matrix in the desired coordinate system. If $\mathbf{E}_J$ is in the [111] direction, then the x, y, and z axes can be taken in the [111], $[\bar{1}10]$, and $[\bar{1}\bar{1}2]$ directions, while if $\mathbf{E}_J$ is in the [100] direction, the x, y, and z axes can be taken in the [100], $[01\bar{1}]$, and [011] directions. The set of axes determined by the unstrained model will be used in all the strain calculations and the dielectric matrix will always be referred to these axes. See Ref. 4 for further details of the model.

In the presence of a uniform strain, the dielectric matrix is in general no longer diagonal, and we can write for the dielectric matrix elements $K_{\alpha\beta}(x)$,

$$K_{\alpha\alpha}(x) = K_\alpha^{(0)}(x) + \eta S_{\alpha\alpha} , \qquad \alpha = 1, 2, 3 \tag{3}$$

$$K_{\alpha\beta}(x) = \eta S_{\alpha\beta} , \qquad \alpha \neq \beta. \tag{4}$$

The symmetric matrix $(\eta S_{\alpha\beta})$ is the contribution of the photoelastic effect[10] which we have written in this form for convenience in the perturbation analysis. The matrix elements $S_{\alpha\beta}$ are spatially constant. We assume that $n^{-2}S_{\alpha\beta}$ is of order unity, where n is the index of refraction of GaP and η is a small parameter. In Section II we express $\eta S_{\alpha\beta}$ in terms of the strain matrix and give estimates for the size of η.

We now seek solutions of the Maxwell curl equations

$$\nabla \times \mathbf{E} = -\mu_0 \dot{\mathbf{H}}, \tag{5}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \mathsf{K}(x) \cdot \dot{\mathbf{E}}, \tag{6}$$

of the form

$$\mathbf{E} = \mathbf{e}(x) \exp i(\omega t - \beta k z), \tag{7}$$

$$\mathbf{H} = \mathbf{h}(x) \exp i(\omega t - \beta k z). \tag{8}$$

These solutions correspond to waves travelling in the positive z direction, where $k = \omega(\mu_0 \epsilon_0)^{\frac{1}{2}} = 2\pi/\lambda$ is the free space wave number and λ the free space wavelength of the light.

In the strain free cases ($\eta = 0$), there are both TE and TM modes and these modes can be either even or odd functions of x. At most only a finite number of modes can exist, and Ref. 4 shows that for the typical parameter values encountered in GaP diode modulators only the lowest order even TE and TM modes can exist. For that reason we confine ourselves here to solutions which in the limit of zero strain ($\eta = 0$) reduce to even modes. However, the perturbation technique used here applies equally well to solutions which in the limit $\eta = 0$ reduce to odd modes.

When $\eta \neq 0$, we seek solutions of Maxwell's equations of the form

$$e_\alpha(x) = A_\alpha \exp -kp(x - w) + B_\alpha \exp -kq(x - w), \qquad x \geqq w \tag{9}$$

$$= C_\alpha \exp kr(x + w) + D_\alpha \exp ks(x + w), \qquad x \leqq -w \tag{10}$$

$$= F_\alpha \exp ikfx + G_\alpha \exp -ikgx + L_\alpha \exp iklx + M_\alpha \exp -ikmx, \qquad |x| \leqq w \tag{11}$$

for $\alpha = 1, 2, 3$. The general solution in each region is a sum of four linearly independent solutions, but in the regions $|x| > w$, the boundary conditions at infinity eliminate two of these solutions. The expressions for $h_\alpha(x)$ can be obtained from equation (5). The various coefficients and parameters $A_\alpha \ldots, p, \ldots$ can be expanded in powers of η

$$A_\alpha = A_\alpha^{(0)} + \eta A_\alpha^{(1)} + \cdots, \tag{12}$$

$$p = p_0 + \eta p_1 + \cdots, \quad \text{and so on.} \tag{13}$$

In Section III we list the terms in these expansions of order zero and one in η, and in Section IV we outline their derivation. In this section we merely discuss some of the features of the solutions.

We refer to solutions which in the limit as $\eta \to 0$ reduce to even TE modes as "perturbed TE modes"; similarly, we refer to "perturbed

TM modes." Expressions for the unperturbed TE and TM modes ($\eta = 0$) are given in equations (25) through (32).

Although the expressions we have obtained for the coefficients and parameters are quite complicated, several features of the perturbed modes stand out. If $S_{12} \neq 0$ or $S_{23} \neq 0$, then the modes cannot be purely TE or TM. In general an important effect of the strain is to tilt the plane of polarization. This tilt is in general a function of x but not of z. Since the coefficients A_α, ... are complex in general, the relative phase between E_y in the perturbed TE mode and E_x in the perturbed TM mode is a function of x. This relative phase at $x = 0$ cannot be determined unless the method of excitation is known, since all the A_α cannot be determined, as is shown in Section IV. Ref. 4 considers the special case where $S_{12} = S_{23} = 0$, $S_{13} \neq 0$ and shows that the modes are rigorously TE or TM. That paper calculates only such parameters as β and p, not such coefficients as A_α and B_α. The parameters are expanded in two small quantities δ and Δ describing the unstrained dielectric matrix. If we expand the expressions for the parameters in this paper to first order in the same small quantities δ and Δ (to second order for β) complete agreement is obtained with the Ref. 4 results.

In the absence of strain, the surfaces of constant phase for both TE and TM modes are the planes z = constant. However, in the presence of strain, the surfaces of constant phase are no longer planes, and are different for the perturbed TE and TM modes.[4]

Finally, $\beta_0 + \eta\beta_1$ is real in all cases. Thus at least to first order in η the modes experience no exponential attenuation as they propagate.

In order to get some feel for the magnitude of the effects involved, we consider several numerical examples. We first estimate the order of magnitude of η by relating it to observable phase differences. Consider a plane wave whose free space wavelength is λ propagating over a distance l in a medium of index of refraction $n + \Delta n$. The phase difference $\Delta\varphi$ which this wave would experience over the same wave if the index of refraction were n is

$$\Delta\varphi = \frac{2\pi}{\lambda}\, l(\Delta n). \tag{14}$$

If ηn^2 is the photoelastic contribution to the dielectric constant, then

$$\Delta n = (n^2 + \eta n^2)^{\frac{1}{2}} - n \cong \tfrac{1}{2} n\eta. \tag{15}$$

Therefore, we have

$$\eta \cong \lambda \Delta\varphi / nl\pi. \tag{16}$$

Now the upper limit of phase shifts observed[17] in GaP at $\lambda = 6328$ Å over a length $l = 0.6$ mm is about $\pi/4$. Taking[18] $n = 3.31$ this yields $\eta \cong 0.8 \times 10^{-4}$. This probably represents an extreme upper limit, and so we will assume $\eta = 10^{-6}$ in our examples. Recent X-ray measurements[19] of strain in P doped Si yield a value of η of about 10^{-6} when the concentration of the dope is $N_D \approx 10^{18}$ atoms per cubic centimeter.[9]

It can be shown that the matrix $(\eta S_{\alpha\beta})$ is approximately related to the strain matrix $(\epsilon_{\alpha\beta})$ by the equations[10]

$$\eta S_{\alpha\beta} = -n^4 \sum_{\mu,\nu=1}^{3} P_{\alpha\beta\mu\nu}\epsilon_{\mu\nu} \tag{17}$$

where n is the index of refraction and $P_{\alpha\beta\mu\nu}$ are the elasto-optical coefficients. Crystals of class $\bar{4}3$m have only three different elasto-optic coefficients when referred to the crystal axes. (See p. 251 of Ref. 10.) For GaP these are[20]

$$P_{11} = -0.151, \qquad P_{44} = -0.074, \qquad P_{12} = -0.082. \tag{18}$$

Since[18] $n = 3.31$ for GaP, and $n^{-2}S_{\alpha\beta}$ is at most of order one, it follows that the magnitude of the strain is roughly proportional to η. In order to obtain the values of the elasto-optic coefficients in the coordinate system used in this paper, it is necessary to make a transformation of the elasto-optic tensor from its representation in the crystal axes. We will not do that here; rather we take $(\eta S_{\alpha\beta})$ as given. In Table I we define three possible strain contributions to the dielectric matrix, labelled a, b and c. Matrices a and b were chosen to demonstrate the effect of the off-diagonal elements S_{12} and S_{23}, respectively (Ref. 4 considered the effect of S_{13} alone), while c was chosen to demonstrate a possible effect when all the off-diagonal elements are nonzero.

For a GaP diode modulator we can write[2, 4]

$$K_\alpha = n^2(1 - \delta_\alpha), \qquad \alpha = 0, 1, 2, 3, \tag{19}$$

where $n = 3.31$ is the index of refraction of GaP[17] and the quantities δ_α, $\alpha = 1, 2, 3$ are functions of the applied bias voltage V. In the original

Table I—Strain Contribution to the Dielectric Matrix*

Type	η	S_{11}	S_{22}	S_{33}	S_{12}	S_{23}	S_{13}
a	10^{-6}	0	0	0	10	0	0
b	10^{-6}	0	0	0	0	10	0
c	10^{-6}	7.07	7.07	7.07	7.07	7.07	7.07

* Components of the strain contribution, S_{ij}, and the magnitude parameter η.

symmetric step model δ_0 is independent of V. For $\mathbf{E}_J \,||\, [111]$ Ref. 4 showed that

$$\delta_1 = -2\delta, \qquad \delta_2 = \delta_3 = \delta, \tag{20}$$

where

$$\delta = -\,\overline{E}_J r_{41} n^2/(3)^{\frac{1}{2}}, \tag{21}$$

and where $\overline{E}_J$ is the (spatial) average junction field, r_{41} is the electro-optic coefficient, and n is the index of refraction. For a typical diode (diode KC46CA of Ref. 9), $\overline{E}_J$(measured in V/m) is related to the diode half width w(measured in m) and bias voltage V(measured in V) by[9]

$$\overline{E}_J = (2 - V)/(2w). \tag{22}$$

The half width can be determined by capacitance measurements[8, 9] and related to the bias voltage by

$$w(V) = 0.139 \times 10^{-6}(1 - V/1.8)^{0.380}. \tag{23}$$

Using the value $r_{41} = -0.86 \times 10^{-12}$ m/V,* we can now calculate δ_1, δ_2, and δ_3 as functions of V.

For this diode, $\delta_0 = 1.612 \times 10^{-3}$. However, it has been shown that the voltage dependence of the parameters of the symmetric step model is not correct, and the double walled waveguide much more closely describes the true voltage dependence.[4, 9] We have not used the double walled model because it is analytically complex. Instead, since the modes in the single and double walled guides are very similar in form because they both decay exponentially as functions of x outside the guide, we have used the single walled model but simulated the voltage dependence of the double walled model. This has been achieved by letting δ_0 vary with voltage. The voltage variation of δ_0 has been obtained by requiring the equality of expressions (2.33) and (3.18) in Ref. 4 for the decay constants p, and letting $w_1 = w(0)$ and $w_2 = w(V)$. This yields the relation

$$\delta_0 = (2.24 \times 10^{-10})/w. \tag{24}$$

In Table II we list these basic constants describing the unstrained diode as functions of V. Using these values, we can calculate from equations (33) through (37) the parameters of the unstrained TE

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* This is the unclamped value of r_{41} given in Ref. 18. After these calculations were made it was determined that the clamped value $r_{41} = -0.97 \times 10^{-12}$ m/V should be used. However, since our results supply only qualitative information about actual diodes, we have not redone the numerical example.

TABLE II—CHARACTERISTICS FOR A TYPICAL GaP DIODE*

Bias voltage (V)	w (10^{-5} cm)	$10^4\ \delta_1$	$10^4\ \delta_2$	$10^4\ \delta_3$	$10^4\ \delta_0$
−2	1.87	−1.17	0.58	0.58	12.00
−12	3.04	−2.51	1.25	1.25	7.38
−24	3.85	−3.67	1.84	1.84	5.82

* Given as functions of the applied reverse bias voltage V. The half width of the junction is w, and the components of the unstrained dielectric matrix are $K_j = n^2(1 - \delta_j)$, $j = 0, 1, 2, 3$, where $n = 3.31$ is the index of refraction of GaP.

and TM modes for $\lambda = 6328$ Å. These values are listed in Table III. Finally, in Tables IV and V we list the parameters of the corresponding perturbed TE and TM modes respectively. The accuracy of those terms less than 10^{-4} is uncertain in case c of Tables IV and V. In Figs. 2 through 7 we plot some of the components of the perturbed TE modes correct to first order in η. In Figs. 2, 3, 5, 6, and 7 the imaginary part of the component is negligible and is neglected, while in Fig. 4 the real part is negligible with respect to the imaginary part and is neglected. In all cases the e_3 component is negligible compared to the e_1 component. We have chosen the undetermined coefficients so that at $x = 0$, $z = 0$, e_2 in the perturbed TE mode and e_1 in the perturbed TM mode have zero phase to first order in η.

This example illustrates how much tilting of the plane of polarization, or coupling of the TE and TM modes, is to be expected. The S_{12} component produces the main effect, which from Figs. 2 and 3, is a maximum tilt of the plane of polarization of 3.5°. This effect

TABLE III—UNPERTURBED MODE PARAMETERS*

Type of mode	Bias voltage (V)	β_0	p_0	f_0	l_0
TE	−2	3.308	0.0226	0.1096	0.1180†
TE	−12	3.309	0.0195	0.0796	0.1022†
TE	−24	3.309	0.0160	0.0641	0.1007†
TM	−2	3.308	0.0259	0.1089†	0.1174
TM	−12	3.309	0.0307	0.0759†	0.0994
TM	−24	3.309	0.0363	0.0552†	0.0953

* Describing the unstrained TE modes, and the parameters β_0, p_0 and l_0 describing the unstrained TM modes as functions of the applied reverse bias voltage V. The wavelength of the light is 6328 Å.

† Derived parameters l_0 for the TE modes and f_0 for the TM modes. These derived parameters appear only in first and higher order corrections to the field.

TABLE IV—PARAMETERS FOR PERTURBED TE MODES*

Type of strain	Bias voltage (V)	β_1	p_1		q_1		l_1	m_1
			$Re(p_1)$	$Im(p_1)$	$Re(q_1)$	$Im(q_1)$		
a	−2	0	221		−221		0	0
	−12	0	257		−257		0	0
	−24	0	312		−312		0	0
b	−2	0		1.51		−1.51	0	0
	−12	0		1.51		−1.51	0	0
	−24	0		1.51		−1.51	0	0
c	−2	1.07	156	-0.3×10^{-7}	−156	−2.14	2.13	-0.4×10^{-8}
	−12	1.07	181	$+0.2\times10^{-7}$	−181	−2.14	2.13	-0.4×10^{-8}
	−24	1.07	221	-0.2×10^{-7}	−221	−2.14	2.13	-0.4×10^{-8}

* For all perturbed TE modes $f_1 = g_1 = 0$.

decreases with increasing reverse bias voltage. However, it should be noticed from Figs. 4 and 5 that the coupling effect resulting from S_{23} increases with reverse bias voltage. The e_1 component is roughly proportional to η, so a doubling of the strain would double the mode coupling. Mathematically, the existence of this relatively large effect results from the largeness of the factor c given in equation (64) for perturbed TE modes and in equation (84) for perturbed TM modes. The TM modes exhibit a similar behavior.

From Tables IV and V we see that the changes in the parameters, ηp_1, ηq_1, $\eta\beta_1$, and so on, are indeed small, which gives us confidence that the perturbation treatment is reasonable.

III. FORMULAS FOR THE SOLUTIONS

To list the formulas for the coefficients and parameters, $A_a, \ldots,$ $p, \ldots$ (which appear in the expressions (7) through (11) for the solutions in terms of the various parameters describing the symmetric step model and the strain matrix), we begin by writing down the solution for the strain free ($\eta = 0$) case for both the even TE and TM modes. When $\eta = 0$, we have for the even TE modes

$$e_1(x) = e_3(x) = 0, \qquad \text{all } x \tag{25}$$

$$e_2(x) = \cos(kf_0x), \qquad |x| \leqq w \tag{26}$$

$$= \cos(kf_0w)\exp kp_0(w - |x|), \qquad |x| \geqq w \tag{27}$$

while for the even TM modes

TABLE V—PARAMETERS FOR THE PERTURBED TM MODES

Type of strain	Bias voltage (V)	β_1	p_1		q_1		f_1	g_1	l_1	m_1
			$Re(p_1)$	$Im(p_1)$	$Re(q_1)$	$Im(q_1)$				
a	−2	0	193		−193		0	0	0	0
	−12	0	163		−163		0	0	0	0
	−24	0	138		−138		0	0	0	0
b	−2	0		1.51		−1.51	0	0	0	0
	−12	0		1.51		−1.51	0	0	0	0
	−24	0		1.51		−1.51	0	0	0	0
c	−2	1.07	136	-0.2×10^{-7}	−136	−2.13	-1.2×10^{-4}	-1.2×10^{-4}	2.13	−2.13
	−12	1.07	115	-0.2×10^{-7}	−115	−2.14	1.9×10^{-4}	1.9×10^{-4}	2.13	−2.13
	−24	1.07	97.3	-0.3×10^{-7}	−97.3	−2.14	0.72×10^{-4}	0.72×10^{-4}	2.13	−2.13

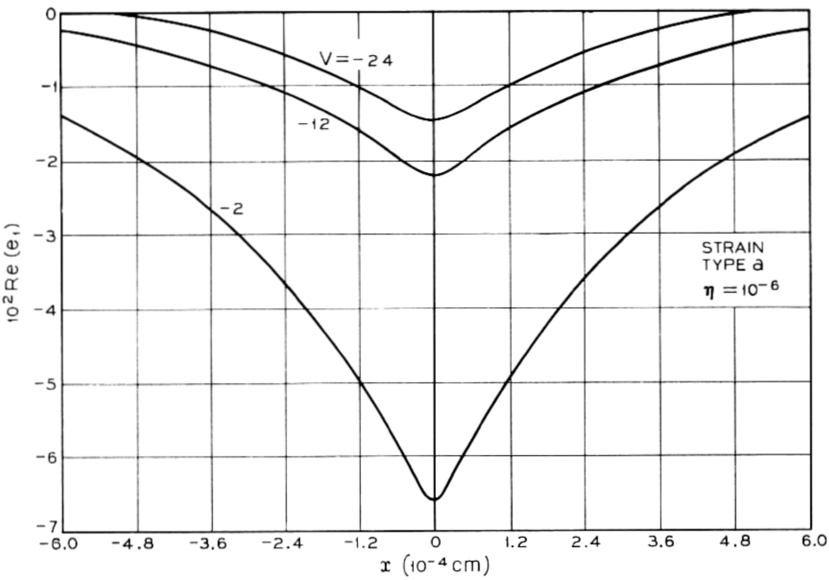

Fig. 2 — The relative amplitude of the real part of e_1 for the perturbed TE mode.

$$e_1(x) = \cos(kl_0x), \qquad |x| \leqq w \tag{28}$$

$$= \frac{K_1}{K_0}\cos(kl_0w)\exp kp_0(w - |x|), \qquad |x| \geq w \tag{29}$$

$$e_2(x) = 0, \qquad \text{all } x \tag{30}$$

$$e_3(x) = i\frac{l_0K_1}{\beta_0K_3}\sin(kl_0x), \qquad |x| \leqq w \tag{31}$$

$$= i\frac{p_0K_1}{\beta_0K_0}\cos(kl_0w)\operatorname{sgn}(x)\exp kp_0(w - |x|), \quad |x| \geqq w. \tag{32}$$

The parameters in these equations are given for the TE modes by the positive roots of the system of equations for p_0, β_0, and f_0

$$p_0^2 = \beta_0^2 - K_0, \tag{33}$$

$$f_0^2 = K_2 - \beta_0^2, \tag{34}$$

$$f_0\tan(kwf_0) = p_0 \tag{35}$$

while for the TM modes the parameters are the positive roots of the system of equations for p_0, β_0, and l_0, consisting of equation (33) and

$$l_0^2 = K_3 - (K_3/K_1)\beta_0^2, \tag{36}$$

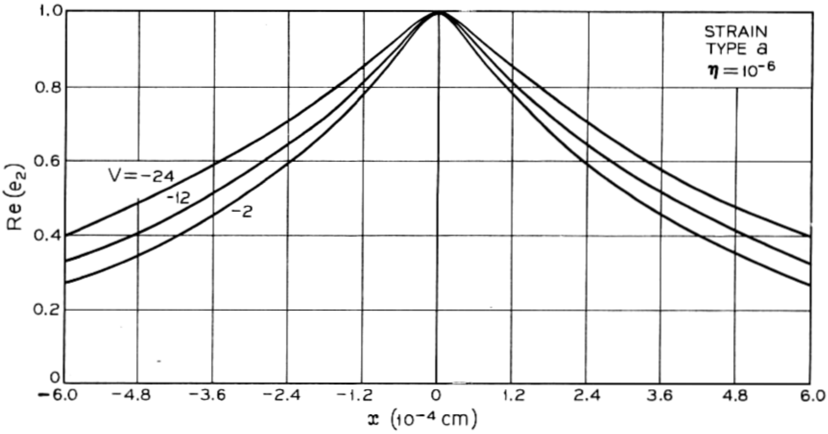

Fig. 3 — The relative amplitude of the real part of e_2 for the perturbed TE mode.

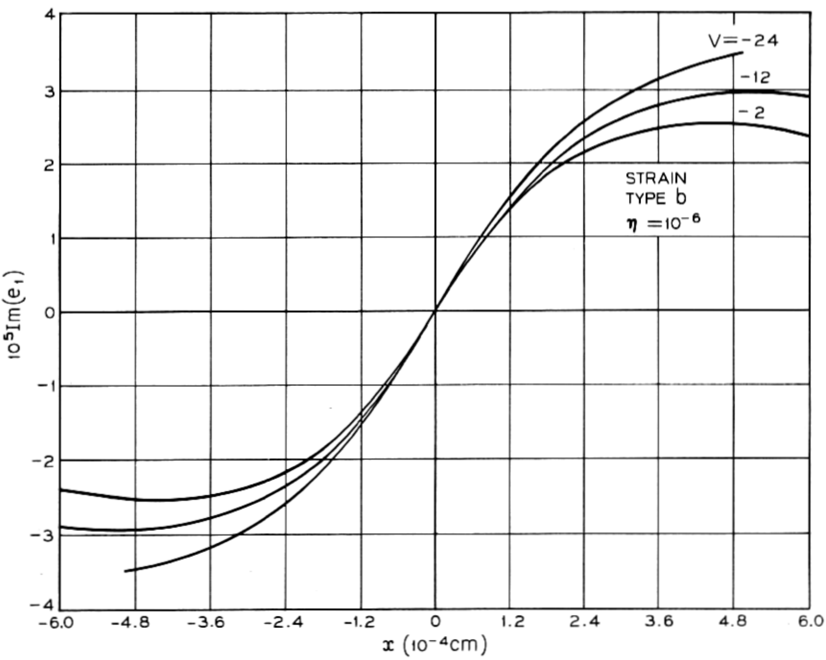

Fig. 4 — The relative amplitude of the imaginary part of e_1 for the perturbed TE mode.

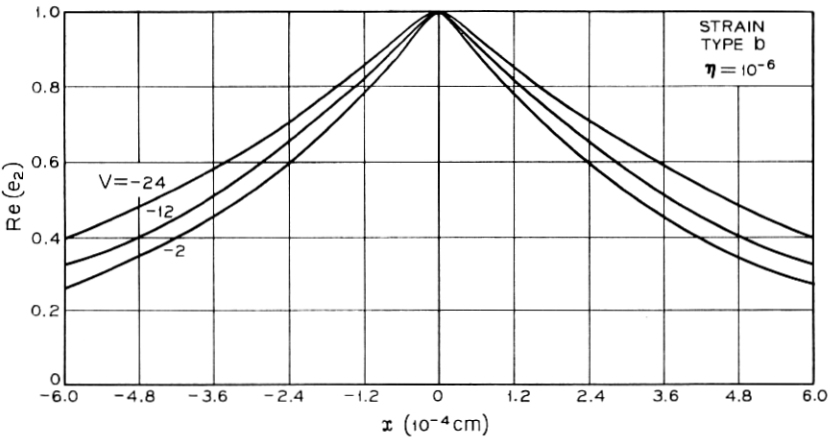

Fig. 5 — The relative amplitude of the real part of e_2 for the perturbed TE mode.

$$K_0 l_0 \tan(kwl_0) = K_3 p_0. \tag{37}$$

The expressions for $\mathbf{h}(x)$ can be obtained from equation (5).

We now turn to listing the formulas for the coefficients and parameters of the solutions for the perturbed TE and TM modes. For both the perturbed TE and TM modes we have the relations

$$\begin{aligned} \begin{matrix} p_1 \\ q_1 \end{matrix} = \left(\frac{\beta_0}{p_0}\right)\beta_1 - \left(\frac{1}{4p_0K_0}\right)[K_0(S_{22} + S_{33}) + \beta_0^2(S_{11} - S_{33}) + 2ip_0\beta_0S_{13}] \\ \pm \left(\frac{1}{4p_0K_0}\right)\{[K_0(S_{22} - S_{33}) - \beta_0^2(S_{11} - S_{33}) - 2ip_0\beta_0S_{13}]^2 \\ + 4K_0[\beta_0S_{12} + ip_0S_{23}]^2\}^{\frac{1}{2}}, \end{aligned} \tag{38}$$

where p_1 corresponds to the "+" sign and q_1 to the "−" sign. It is also true, at least to first order in η, that

$$e_\alpha(-x) = e_\alpha(x)^*, \qquad \alpha = 1, 2, 3, \tag{39}$$

hence we only list those parameters determining the solution in $x \geqq -w$.

For the perturbed TE modes p_0, f_0, and β_0 are the positive solutions of the system of equations (33) through (35), and l_0 is then given in terms of β_0 as the positive root in equation (36). The remaining parameters are

$$q_0 = p_0\ , \tag{40}$$

$$g_0 = f_0\ , \tag{41}$$

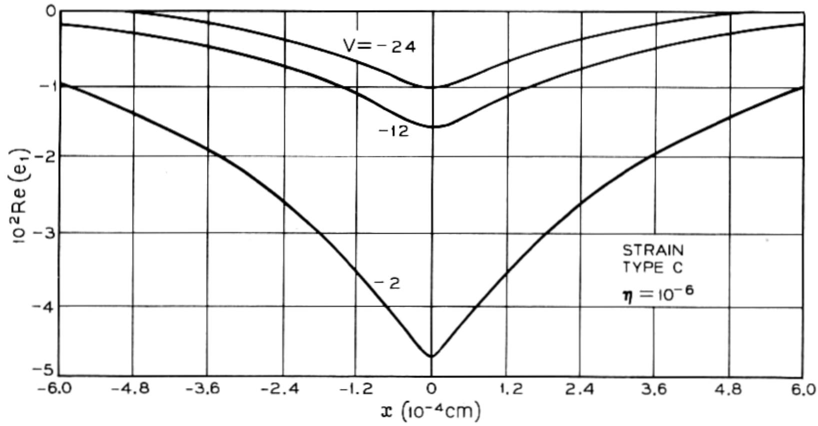

Fig. 6 — The relative amplitude of the real part of e_1 for the perturbed TE mode.

$$m_0 = l_0 , \tag{42}$$

$$\beta_1 = S_{22}/(2\beta_0). \tag{43}$$

The quantities p_1 and q_1 are now determined by equations (38) and (43). Next we have

$$f_1 = g_1 = (S_{22} - 2\beta_0\beta_1)/(2f_0), \tag{44}$$

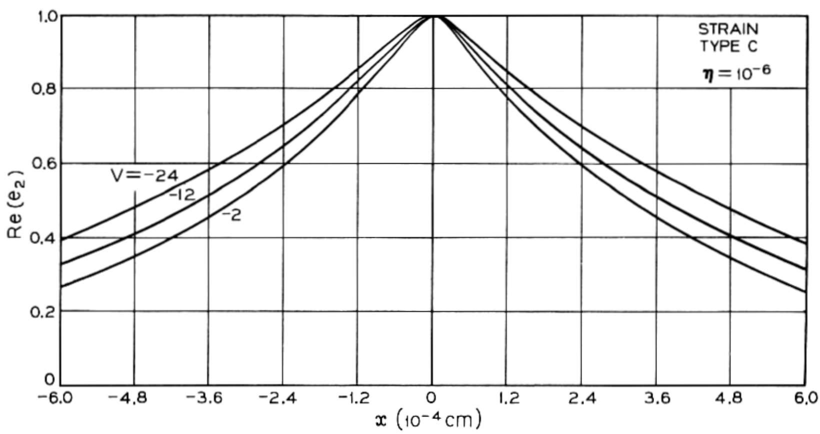

Fig. 7 — The relative amplitude of the real part of e_2 for the perturbed TE mode.

$$\begin{matrix} l_1 \\ m_1 \end{matrix} = \frac{1}{2K_1 l_0} [S_{11}(K_3 - l_0^2) - 2K_3\beta_0\beta_1 + S_{33}(K_1 - \beta_0^2) \pm 2S_{13}l_0\beta_0], \tag{45}$$

where l_1 corresponds to the "+" sign and m_1 to the "−" sign, and in equations (44) and (45) β_1 is given in equation (43). Notice that from (43) and (44), $f_1 = g_1 = 0$ for the TE case.

The expressions for the coefficients are

$$A_1^{(0)} = -\beta_0(\beta_0 S_{12} + ip_0 S_{23}) \cos (kf_0 w)/[2K_0 p_0(p_1 - q_1)], \tag{46}$$

$$A_2^{(0)} = -q_1 \cos (kf_0 w)/(p_1 - q_1), \tag{47}$$

$$A_3^{(0)} = i(p_0/\beta_0)A_1^{(0)}, \tag{48}$$

$$B_1^{(0)} = -A_1^{(0)}, \tag{49}$$

$$B_2^{(0)} = p_1 \cos (kf_0 w)/(p_1 - q_1), \tag{50}$$

$$B_3^{(0)} = i(p_0/\beta_0)B_1^{(0)}, \tag{51}$$

$$F_\alpha^{(0)} = G_\alpha^{(0)} = 0, \qquad \alpha = 1, 3 \tag{52}$$

$$F_2^{(0)} = G_2^{(0)} = \tfrac{1}{2} \tag{53}$$

$$L_\alpha^{(0)} = M_\alpha^{(0)} = 0, \qquad \alpha = 1, 2, 3 \tag{54}$$

$$\begin{matrix} F_1^{(1)} \\ G_1^{(1)} \end{matrix} = \tfrac{1}{2}c\,[\mp S_{23}\beta_0 f_0 - S_{12}(K_3 - f_0^2)]\,, \tag{55}$$

$$F_2^{(1)} = G_2^{(1)}, \tag{56}$$

$$\begin{matrix} F_3^{(1)} \\ G_3^{(1)} \end{matrix} = \tfrac{1}{2}c\,[\mp S_{12}\beta_0 f_0 - S_{13}(K_1 - \beta_0^2)]\,, \tag{57}$$

$$\begin{matrix} L_3^{(1)} \\ M_3^{(1)} \end{matrix} = \tfrac{1}{4}l_0 \cos (kwf_0)\{aS_{23}[2\beta_0^2(K_1 - K_0)c - 1] \pm \frac{b\beta_0}{p_0} S_{12}[2p_0^2(K_0 - K_3)c - 1]\}\,, \tag{58}$$

$$L_1^{(1)} = (\beta_0 K_3/l_0 K_1)L_3^{(1)}, \tag{59}$$

$$M_1^{(1)} = -(\beta_0 K_3/l_0 K_1)M_3^{(1)}, \tag{60}$$

$$L_2^{(1)} = M_2^{(1)} = 0, \tag{61}$$

where

$$a = [l_0K_0 \cos(kwl_0) + p_0K_3 \sin(kwl_0)]^{-1}, \tag{62}$$

$$b = [l_0K_0 \sin(kwl_0) - p_0K_3 \cos(kwl_0)]^{-1}, \tag{63}$$

$$c = (K_1K_3 - K_3\beta_0^2 - K_1f_0^2)^{-1}, \tag{64}$$

and p_1 and q_1 are the values appropriate to the TE modes.

Finally, we write down the three combinations

$$A_1^{(1)} + B_1^{(1)} = -i(\beta_0/p_0)[A_3^{(1)} + B_3^{(1)}] - \cos(kf_0w)[(p_0^2 - K_0)S_{12} + i\beta_0p_0S_{23}]/(2K_0p_0^2), \tag{65}$$

$$A_2^{(1)} + B_2^{(1)} = \cos(kf_0w)[F_2^{(1)} + G_2^{(1)}], \tag{66}$$

$$A_3^{(1)} + B_3^{(1)} = [F_3^{(1)} + G_3^{(1)}] \cos(kf_0w) + i[F_3^{(1)} - G_3^{(1)}] \sin(kf_0w) + [L_3^{(1)} + M_3^{(1)}] \cos(kl_0w) + i[L_3^{(1)} - M_3^{(1)}] \sin(kl_0w). \tag{67}$$

The coefficients $F_2^{(1)} = G_2^{(1)}$, and hence $A_2^{(1)} + B_2^{(1)}$, are arbitrary and correspond to an overall multiplicative constant. They can be set equal to zero with no loss in generality. We discuss this point further in Section III. Moreover, the individual coefficients $A_1^{(1)}$, $A_3^{(1)}$, $B_1^{(1)}$, and $B_3^{(1)}$ cannot be determined at this stage. However, the terms we have are sufficient to determine each component of the field up through order one in η.

For the perturbed TM modes p_0, l_0, and β_0 are the positive solutions of the system of equations (33), (36), and (37), and f_0 is given in terms of β_0 as the positive root in equation (34). The parameters q_0 and p_0 are still related by equation (40), m_0 and l_0 by equation (42), and g_0 and f_0 by equation (41). The remaining parameters are

$$\beta_1 = [1/(2\beta_0)][K_0(K_1l_0^2 + K_3p_0^2) + \zeta]^{-1} \cdot \{[S_{11}\beta_0^2/K_1][K_0K_3p_0^2 + K_1^2l_0^2 + \zeta] + [K_1S_{33}l_0^2/K_3^2][K_3(K_3 - K_0)p_0^2 + \zeta]\}, \tag{68}$$

where

$$\zeta = (kp_0w)(K_3^2p_0^2 + K_0^2l_0^2). \tag{69}$$

With the aid of (68) and (69) for β_1, p_1 and q_1 are determined by equation (38), f_1 and g_1 by equation (44), and l_1 and m_1 by equation (45).

The expressions for the coefficients are

$$A_1^{(0)} = -K_1 \cos(kwl_0)(2\beta_0\beta_1 - 2p_0p_1 - S_{22})/[2p_0K_0(p_1 - q_1)], \quad (70)$$

$$A_2^{(0)} = -K_1 \cos(kwl_0)(\beta_0S_{12} + ip_0S_{23})/[2p_0\beta_0K_0(p_1 - q_1)], \quad (71)$$

$$B_1^{(0)} = K_1 \cos(kwl_0)(2\beta_0\beta_1 - 2p_0q_1 - S_{22})/[2p_0K_0(p_1 - q_1)], \quad (72)$$

$$B_2^{(0)} = -A_2^{(0)}, \quad (73)$$

where β_1 is given by equations (68) and (69), p_1 and q_1 are the values appropriate to the TM modes, and $A_3^{(0)}$ is related to $A_1^{(0)}$ by equation (48) and $B_3^{(0)}$ to $B_1^{(0)}$ by equation (51). Furthermore,

$$F_\alpha^{(0)} = G_\alpha^{(0)} = 0, \qquad \alpha = 1, 2, 3 \quad (74)$$

$$L_1^{(0)} = M_1^{(0)} = \tfrac{1}{2}, \quad (75)$$

$$L_2^{(0)} = M_2^{(0)} = 0, \quad (76)$$

$$L_3^{(0)} = -M_3^{(0)} = (l_0K_1)/(2\beta_0K_3), \quad (77)$$

$$F_1^{(1)} = F_3^{(1)} = G_1^{(1)} = G_3^{(1)} = 0, \quad (78)$$

$$\begin{matrix} F_2^{(1)} \\ G_2^{(1)} \end{matrix} = \cos(kwl_0)\{\beta_0aS_{12}[K_1 + 2cp_0^2(K_0 - K_3)]$$

$$\pm p_0bS_{23}[K_1 + 2c\beta_0^2(K_1 - K_0)]\}/(4p_0\beta_0K_0), \quad (79)$$

$$\begin{matrix} L_2^{(1)} \\ M_2^{(1)} \end{matrix} = -\tfrac{1}{2}c[S_{12} \pm l_0K_1S_{23}/(\beta_0K_3)] \quad (80)$$

$$A_2^{(1)} + B_2^{(1)} = [F_2^{(1)} + G_2^{(1)}]\cos(kwf_0) + i[F_2^{(1)} - G_2^{(1)}]\sin(kwf_0)$$
$$+ [L_2^{(1)} + M_2^{(1)}]\cos(kwl_0) + i[L_2^{(1)} - M_2^{(1)}]\sin(kwl_0), \quad (81)$$

where

$$a = [p_0 \cos(kwf_0) - f_0 \sin(kwf_0)]^{-1}, \quad (82)$$

$$b = [f_0 \cos(kwf_0) + p_0 \sin(kwf_0)]^{-1}, \quad (83)$$

$$c = (K_2 - l_0^2 - \beta_0^2)^{-1}. \quad (84)$$

Just as in the perturbed TE case, the coefficients cannot all be determined uniquely. We can with no loss of generality set

$$L_3^{(1)} = -M_3^{(1)} = 0. \quad (85)$$

Once this choice is made, we have

$$L_1^{(1)} = [S_{13} - l_0\beta_1 - l_1\beta_0 + (S_{33} - 2l_0l_1)(l_0K_1/\beta_0K_3)]/(2\beta_0l_0), \quad (86)$$

$$M_1^{(1)} = [-S_{13} - l_0\beta_1 - m_1\beta_0 + (S_{33} - 2l_0m_1)(l_0K_1/\beta_0K_3)]/(2\beta_0l_0), \quad (87)$$

$$A_3^{(1)} + B_3^{(1)} = \{-(S_{13}p_0kw)/K_0 + i[\tfrac{1}{2}kw(K_3\beta_0^2S_{11}/K_1 + K_1l_0^2S_{33}/K_3 - 2K_3\beta_0\beta_1)]/(K_3\beta_0)\} \cos(kwl_0) \quad (88)$$

$$A_1^{(1)} + B_1^{(1)} = -i(\beta_0/p_0)[A_3^{(1)} + B_3^{(1)}] + (K_1 \cos(kwl_0)/K_0\beta_0p_0) \cdot[-K_0\beta_1/p_0 - \beta_0S_{11}(p_0^2 - K_0)/(2p_0K_0) + p_0\beta_0S_{33}/(2K_0) - ip_0^2S_{13}/K_0]. \quad (89)$$

A knowledge of these terms is sufficient to determine each component of the field up through order one in η.

IV. DETAILS OF THE CALCULATIONS

In standard fashion **H** can be eliminated from equations (5) and (6) by taking the curl of equation (5) and by making use of the assumed form of the solutions, equations (7) and (8). There results the system of equations

$$i\beta \frac{de_3}{d\xi} - \beta^2 e_1 + \sum_{\alpha=1}^{3} K_{1\alpha}e_\alpha = 0, \quad (90)$$

$$\frac{d^2e_2}{d\xi^2} - \beta^2 e_2 + \sum_{\alpha=1}^{3} K_{2\alpha}e_\alpha = 0, \quad (91)$$

$$\frac{d^2e_3}{d\xi^2} + i\beta \frac{de_1}{d\xi} + \sum_{\alpha=1}^{3} K_{3\alpha}e_\alpha = 0, \quad (92)$$

where we have introduced the new independent variable

$$\xi = kx. \quad (93)$$

The standard boundary conditions[21] on **E** and **H** yield the conditions that e_2, e_3, $de_2/d\xi$, and $de_3/d\xi + i\beta e_1$ must be continuous at $\xi = \sigma = kw$.

The general plan of the calculation is first to consider the equations obtained by substituting into equations (90) through (92) the expressions for e_α in the various regions given by equations (9) through (11). From these equations, one can determine up through first order in η all but one of the parameters and some of the coefficients as functions of the parameter β_1. Upon substituting these values into the boundary condition equations, a set of equations is obtained from which β_1 and some of the remaining coefficients can be determined.

Since the components of the electromagnetic field satisfy a linear, homogeneous system of equations, it follows that if $e_\alpha(x)$, $\alpha = 1, 2, 3$ is a solution set, then so is $(1 + a_1\eta + a_2\eta^2 + \ldots)e_\alpha(x)$, $\alpha = 1, 2, 3$, where the constants $a_1, a_2, \ldots$ are arbitrary. For example, if the coefficients $A_\alpha, B_\alpha, \ldots$ given by expansions of the form (12) represents a solution, then the coefficients given by expansions of the form

$$A_\alpha = A_\alpha^{(0)} + \eta[a_1 A_\alpha^{(0)} + A_\alpha^{(1)}] + \cdots, \tag{94}$$

with the same a_1 used in each expansion, represent another solution. Thus unless the corresponding zeroth order coefficient is zero, the first order coefficient cannot be uniquely determined. We do, however, have the arbitrary constant a_1 at our disposal. The multiplicative constant $(1 + a_1\eta + \ldots)$ can only be determined from a knowledge of the excitation of the mode.

If the assumed expressions for e_α in $\xi \geqq \sigma$ given by equation (9) are substituted into equations (90) through (92) we get the set of homogeneous, linear equations in A_α, $\alpha = 1, 2, 3$,

$$(K_0 + \eta S_{11} - \beta^2)A_1 + \eta S_{12}A_2 + (\eta S_{13} - i\beta p)A_3 = 0, \tag{95}$$

$$\eta S_{12}A_1 + (K_0 + \eta S_{22} + p^2 - \beta^2)A_2 + \eta S_{23}A_3 = 0, \tag{96}$$

$$(\eta S_{13} - i\beta p)A_1 + \eta S_{23}A_2 + (K_0 + \eta S_{33} + p^2)A_3 = 0, \tag{97}$$

plus a similar set of equations with A_α replaced by B_α and p replaced by q. The condition that these equations have a nontrivial solution, the vanishing of the determinant of coefficients, yields a relation between β and p of the form

$$D(p, \beta) = 0, \tag{98}$$

where $D(p, \beta)$ is a quartic polynomial in p and β. The second set of equations involving the B_α and q yields the same determinantal equation with p replaced by q,

$$D(q, \beta) = 0. \tag{99}$$

That is, q is a second root of the quartic. If p, q, and β are expanded in powers of η as in equation (13), equations (98) and (99) can be expanded in powers of η and the coefficients of the various powers of

η can be equated to zero. The vanishing of the lowest order term yields equation (33), which is satisfied by both p_0 and q_0, thus also yielding equation (40). The vanishing of the first order coefficients shows that p_1 and q_1 are the two roots of a quadratic which are given by equation (38). These results are independent of the TE or TM character of the mode.

Equations (95) through (97) can now be expanded in powers of η by substituting in the expansions of p, β, and A_a. The three similar equations involving q, β, and B_a can be expanded in powers of η in the same way. Because p_0 and β_0 satisfy equation (33), equation (96) vanishes to zeroth order in η, while equations (95) and (97) yield

$$(K_0 - \beta_0^2)A_1^{(0)} - i\beta_0 p_0 A_3^{(0)} = 0, \tag{100}$$

$$-i\beta_0 p_0 A_1^{(0)} + (K_0 + p_0^2)A_3^{(0)} = 0. \tag{101}$$

The determinant of this pair of homogeneous equations vanishes because equation (33) is satisfied, so a nontrivial solution exists. The quantities $B_1^{(0)}$ and $B_3^{(0)}$ satisfy the same equations. Using equation (33), it follows from equation (101) that $A_1^{(0)}$ and $A_3^{(0)}$ are related by equation (48), and $B_1^{(0)}$ and $B_3^{(0)}$ are related by equation (51).

To first order in η, equations (95) through (97) are

$$\begin{aligned}(K_0 - \beta_0^2)&A_1^{(1)} - i\beta_0 p_0 A_3^{(1)} \\ &= -[(S_{11} - 2\beta_0\beta_1)A_1^{(0)} + S_{12}A_2^{(0)} + (S_{13} - ip_0\beta_1 - ip_1\beta_0)A_3^{(0)}],\end{aligned} \tag{102}$$

$$S_{12}A_1^{(0)} + (2p_0p_1 - 2\beta_0\beta_1 + S_{22})A_2^{(0)} + S_{23}A_3^{(0)} = 0, \tag{103}$$

$$\begin{aligned}-i\beta_0 p_0 &A_1^{(1)} + (K_0 + p_0^2)A_3^{(1)} \\ &= -[(S_{13} - ip_0\beta_1 - ip_1\beta_0)A_1^{(0)} + S_{23}A_2^{(0)} + (S_{33} + 2p_0p_1)A_3^{(0)}].\end{aligned} \tag{104}$$

With the replacement of A_a by B_a and p by q in equations (102) through (104) we obtain the first order equations satisfied by the B_a. At this stage we must differentiate between the perturbed TE and TM modes. For the perturbed TE modes we must have

$$A_1^{(0)} + B_1^{(0)} = -i(\beta_0/p_0)[A_3^{(0)} + B_3^{(0)}] = 0, \tag{105}$$

$$A_2^{(0)} + B_2^{(0)} = \cos(kf_0w), \tag{106}$$

while for the perturbed TM modes

$$A_1^{(0)} + B_1^{(0)} = -i(\beta_0/p_0)[A_3^{(0)} + B_3^{(0)}] = (K_1/K_0)\cos(kl_0w), \tag{107}$$

$$A_2^{(0)} + B_2^{(0)} = 0. \tag{108}$$

If we now add to equation (103) the equivalent equation in B_α, and make use of the fact that $A_\alpha^{(0)} + B_\alpha^{(0)}$, $\alpha = 1, 2, 3$ are prescribed for the perturbed TE modes, in equations (105) and (106), we get a new equation involving only $A_2^{(0)}$ and $B_2^{(0)}$. This equation together with equation (106) can be solved for $A_2^{(0)}$ and $B_2^{(0)}$ to yield (47) and (50). Once $A_2^{(0)}$ is known, $A_1^{(0)}$ and $A_3^{(0)}$ can be determined from equations (103) and (48) yielding (46). We get $B_1^{(0)}$ and $B_3^{(0)}$ from equation (105). In the same fashion we determine $A_\alpha^{(0)}$ and $B_\alpha^{(0)}$, $\alpha = 1, 2, 3$ for the perturbed TM modes.

Equations (102) and (104) [and the two equivalent equations in $B_1^{(1)}$ and $B_3^{(1)}$] are two inhomogeneous equations whose determinant vanishes. Thus the left side of (102) is a multiple of the left side of (104), and the equations are compatible only if the right side of (102) is the same multiple of the right side of (104). This can be shown to be the case, and so (102) and (104) provide just one relationship between $A_1^{(1)}$ and $A_3^{(1)}$. There is a corresponding relationship between $B_1^{(1)}$ and $B_3^{(1)}$.

By replacing A_α, B_α, p, q by C_α, D_α, $-r$, $-s$, respectively, in the equations so far obtained, the formulas for the region $\xi \leqq -\sigma$ are obtained. Here $-r$ and $-s$ are the remaining two roots of the quartic $D(p, \beta) = 0$.

Next, if the assumed expressions for e_α in $|\xi| < \sigma$ given by equation (11) are substituted into equations (90) through (92) we get four sets of three homogeneous, linear equations in F_α, G_α, L_α, and M_α, respectively, which hold for both the perturbed TE and TM modes. These equations are obtained from equations (95) through (97) by replacing A_α and p by F_α and $-if$, G_α and ig, L_α and $-il$, and M_α and im, respectively.

The determinental equation for each of these four sets of homogeneous equations can again be expanded in powers of η, and the coefficient of each power of η separately equated to zero. The vanishing of the zeroth order coefficients yields equations (34), (41), (36), and (42) relating f_0, g_0, l_0, and m_0 to β_0. The vanishing of the first order coefficients yields equations (44) and (45) relating f_1, g_1, l_1, and m_1 to β_1.

Each of the four sets of homogeneous equations can be expanded in powers of η, just as for the equations describing the region $\xi \geqq \sigma$. To proceed further, we must again differentiate between the perturbed TE and TM modes. For the perturbed TE modes, equations (52) through (54) must be satisfied, while for the perturbed TM modes, equations (74) through (77) must be satisfied. These values satisfy the lowest order equations identically.

For both the perturbed TE and TM modes, the first order equations can be written as

$$(K_1 - \beta_0^2)F_1^{(1)} - \beta_0 f_0 F_3^{(1)} = -S_{12}F_2^{(0)}, \tag{109}$$

$$-\beta_0 f_0 F_1^{(1)} + (K_3 - f_0^2)F_3^{(0)} = -S_{23}F_2^{(0)}, \tag{110}$$

$$(K_2 - f_0^2 - \beta_0^2)F_2^{(1)} = -S_{12}F_1^{(0)} - S_{23}F_3^{(0)} - (S_{22} - 2f_0f_1 - 2\beta_0\beta_1)F_2^{(0)}. \tag{111}$$

For both the perturbed TE and TM modes, equation (111) vanishes and yields no information, while equations (109) and (110) have a nonzero determinant and so can be solved for $F_1^{(1)}$ and $F_3^{(1)}$ yielding the solutions given in (55), (57), and (78). If we replace F_α and f by G_α and $-g$, we get equations which can be solved for $G_1^{(1)}$ and $G_3^{(1)}$ yielding solutions given in (55), (57), and (78). The equations obtained when F_α and f are replaced by L_α and l, and M_α and $-m$, respectively, have a different character. The two equations in $L_2^{(1)}$ and $M_2^{(1)}$ corresponding to (111) do not vanish identically and can be solved for $L_2^{(1)}$ and $M_2^{(1)}$. The solutions are given in (61) and (80). The equations in $L_1^{(1)}$ and $L_3^{(1)}$, and $M_1^{(1)}$ and $M_3^{(1)}$, have a vanishing determinant. In the perturbed TE case, the equations are homogeneous and yield (59) and (60). In the perturbed TM case, the equations are nonhomogeneous but compatible, and yield the relations

$$-\beta_0 l_0 L_1^{(1)} + (K_3\beta_0^2/K_1)L_3^{(1)} = -\tfrac{1}{2}(S_{13} - l_0\beta_1 - l_1\beta_0) - \tfrac{1}{2}(S_{33} - 2l_0l_1)(l_0K_1/\beta_0K_3), \tag{112}$$

$$\beta_0 l_0 M_1^{(1)} + (K_3\beta_0^2/K_1)M_3^{(1)} = -\tfrac{1}{2}(S_{13} + l_0\beta_1 + m_1\beta_0) + \tfrac{1}{2}(S_{33} - 2l_0m_1)(l_0K_1/\beta_0K_3). \tag{113}$$

We finally turn to the boundary conditions at $\xi = \pm\sigma$, of which there are eight, four at each boundary. They can be grouped as follows

$$A_2 + B_2 = F_2e^{if\sigma} + G_2e^{-ig\sigma} + L_2e^{il\sigma} + M_2e^{-im\sigma}, \tag{114}$$

$$C_2 + D_2 = F_2e^{-if\sigma} + G_2e^{ig\sigma} + L_2e^{-il\sigma} + M_2e^{im\sigma}, \tag{115}$$

$$-pA_2 - qB_2 = ifF_2e^{if\sigma} - igG_2e^{-ig\sigma} + ilL_2e^{il\sigma} - imM_2e^{-im\sigma}, \tag{116}$$

$$rC_2 + sD_2 = ifF_2e^{-if\sigma} - igG_2e^{ig\sigma} + ilL_2e^{-il\sigma} - imM_2e^{im\sigma}, \tag{117}$$

$$A_3 + B_3 = F_3e^{if\sigma} + G_3e^{-ig\sigma} + L_3e^{il\sigma} + M_3e^{-im\sigma}, \tag{118}$$

$$C_3 + D_3 = F_3e^{-if\sigma} + G_3e^{ig\sigma} + L_3e^{-il\sigma} + M_3e^{im\sigma}, \tag{119}$$

$$-pA_3 - qB_3 + i\beta(A_1 + B_1) = i(fF_3 + \beta F_1)e^{if\sigma} + i(-gG_3 + \beta G_1)e^{-ig\sigma} + i(lL_3 + \beta L_1)e^{il\sigma} + i(-mM_3 + \beta M_1)e^{-im\sigma}, \quad (120)$$

$$rC_3 + sD_3 + i\beta(C_1 + D_1) = i(fF_3 + \beta F_1)e^{-if\sigma} + i(-gG_3 + \beta G_1)e^{ig\sigma} + i(lL_3 + \beta L_1)e^{-il\sigma} + i(-mM_3 + \beta M_1)e^{im\sigma}. \quad (121)$$

These equations split naturally into two groups, one group involving only the subscript 2 and the other group involving only the subscripts 1 and 3. These equations can be expanded in powers of η. The zeroth order equations are satisfied as long as (35) holds in the perturbed TE case and (37) holds in the perturbed TM case.

For the perturbed TE modes, the first order expansion of equations (114) through (117) yields four nonhomogeneous equations in $A_2^{(1)} + B_2^{(1)}$, $C_2^{(1)} + D_2^{(1)}$, $F_2^{(1)}$ and $G_2^{(1)}$. The inhomogeneous terms on the right side of these equations contain the parameter β_1. The determinant of the equations vanishes, and then the condition that they be compatible provides an equation from which β_1, given in (43), is determined. Once β_1 is determined, these equations yield (56) and (66). We can now choose the arbitrary parameter a_1—indicated in (94)—so that $F_2^{(1)} = 0$. Then from (56) and (66) $G_2^{(1)} = A_2^{(1)} + B_2^{(1)} = 0$. In addition, since β_1 is real, it can now be shown that $r_1 = p_1^*$, $s_1 = q_1^*$, $C_\alpha^{(0)} = A_\alpha^{(0)*}$, $D_\alpha^{(0)} = B_\alpha^{(0)*}$, and $C_\alpha^{(1)} + D_\alpha^{(1)} = [A_\alpha^{(1)} + B_\alpha^{(1)}]^*$, $\alpha = 1, 2, 3$, which justifies equation (39). Finally, the first order expansion of equations (118) through (121) can be combined with equations (59) and (60), equation (102), and the corresponding three equations in $B_1^{(1)}$ and $B_3^{(1)}$, $C_1^{(1)}$ and $C_3^{(1)}$, and $D_1^{(1)}$ and $D_3^{(1)}$ to form a set of equations from which $A_\alpha^{(1)} + B_\alpha^{(1)} = [C_\alpha^{(1)} + D_\alpha^{(1)}]^*$, and $M_\alpha^{(1)}$, $\alpha = 1, 3$, can be determined. These are listed in (58), (65), and (67).

For the perturbed TM modes the procedure is virtually the same, except that it is now the first order expansion of equations (118) through (121) which has a vanishing determinant. The condition that these be compatible then yields the expressions (68) and (69) for β_1. This set of equations also yields the result that

$$L_3^{(1)} + M_3^{(1)} = 0. \quad (122)$$

We can now pick the arbitrary parameter a_1 so that $L_3^{(1)} = 0$, which combined with (122) yields (85). Equations (112) and (113) now yield (86) and (87). The first order term of equation (118) then yields (88), and this result, combined with the equation obtained by adding equation (102) to the corresponding equation in B yields (89). Finally, equations (114) through (117) yield expressions (79) through (84) for $F_2^{(1)}$, $G_2^{(1)}$, $L_2^{(1)}$, $M_2^{(1)}$, $A_2^{(1)} + B_2^{(1)}$.

ACKNOWLEDGMENTS

The authors gratefully acknowledge several helpful conversations with D. F. Nelson and F. K. Reinhart, and the assistance of Miss A. Marucchi in programming the numerical calculations.

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