Uniform Asymptotic Expansions for Saddle Point Integrals—Application to a Probability Distribution Occurring in Noise Theory

By STEPHEN O. RICE

(Manuscript received June 5, 1968)

The noncentral chi-square distribution occurs in noise interference problems. When the number of degrees of freedom becomes large, the middle portion of the distribution is given by the central limit theorem, and the tails by a classical saddle point expansion. Here recent work by N. Bleistein and F. Ursell on "uniform" asymptotic expansions is combined and extended to obtain an asymptotic series which apparently holds over the entire range of the distribution. General methods for expanding saddle point integrals in uniform asymptotic series are discussed. Recurrence relations are given for the coefficients in two typical cases, (i) when there are two saddle points and (ii) when there is only one saddle point but it lies near a pole or a branch point.

I. INTRODUCTION

This paper deals with the problem of obtaining asymptotic series for the complex integral

$$J = \int_{L'} t^{\lambda-1} g(t) \exp\,[xh(t)]\,dt \tag{1}$$

when x becomes large. Problems of this sort are quite often encountered in applied mathematics, particularly in wave propagation. The material presented here grew out of some recent work by G. H. Robertson[1] on the "Marcum Q-Function." This function, which appears in the study of radar interference, gives the distribution of the random variable (noncentral χ^2)

$$z = (1/x) \sum_{n=1}^{x} y_n^2 \,. \tag{2}$$

Here x is a positive integer and $y_1, y_2, \ldots, y_x$ are independent gaussian random variables with unit variances and mean values which may be different.

Mr. Robertson has devised an algorithm for computing the Q-function which may be used for a wide range of the parameters appearing in the function (that is, in the noncentral χ^2 distribution). In an earlier paper on information theory, by working with an integral of the type in equation (1), I had obtained an asymptotic (for large x) expression for the tails of the distribution.[2] However, comparison with results obtained by Robertson showed that my expression failed badly in the central part of the distribution where the central limit theorem holds.

The need for an asymptotic expansion which holds uniformly over the entire range of the distribution led to a study of the recent work on "uniform" asymptotic expansions of integrals. The first part of this paper is an exposition, plus extensions and generalizations, of some of the procedures which have been used to obtain uniform asymptotic expansions of integrals of the type in equation (1). The theory is then applied to the noncentral χ^2 distribution.

Two procedures are considered. For convenience, we call them the "Bleistein method"[3] and the "Ursell method."[4] Although these names are among the best that suggest themselves, they are not entirely satisfactory because they contain no hint of the earlier work by others, especially Olver, Chester, Friedman, and Ursell.[5, 6] Here we have recast the underlying ideas used by Bleistein and Ursell into forms better suited to our purpose.

Both methods lead to the same asymptotic series. The Bleistein method gives a compact expression for the coefficients in the expansion. However, from the few examples that have been studied, it appears that the labor required to reduce this compact expression to a computable form is at least as great as that required by the Ursell method.

Section III and Appendices A, B, and C are concerned with a preliminary change of variable in the integral J. The case, denoted by "$\lambda = 1$" for brevity, in which the exponent λ is a positive integer, is discussed in Sections IV, V and VI. This material is applied to the problem of two saddle points in Appendix E. The case in which λ is general, denoted briefly by "$\lambda \neq 1$," is discussed in Sections VII, VIII, and IX, and in the examples in Appendices F, G, and H. The results of Section IX are applied in section X to obtain the desired type of expansion for the noncentral χ^2 distribution. Useful results regarding classical saddle point expansions are stated in Appendix D. Some of the results given in Appendix F for the general case of a saddle point near a branch point are applied in Appendix G to obtain an asymptotic series for the Poisson-Charlier polynomial, a polynomial of interest in traffic theory.

II. STATEMENT OF PROBLEM

The general problem is to obtain an asymptotic series for the integral J defined by equation (1) when x becomes large and most of the contribution to J arises from a (rather loosely defined) "critical region" around $t = 0$. The path of integration L' is supposed to start and end at $|t| = \infty$ in "valleys" in the complex t-plane where $\exp[xh(t)] \to 0$ as $|t| \to \infty$. Let the starting and ending valleys be denoted by S and E, respectively. The path L' starts in S, climbs up to and passes through the critical region, and then descends down into E.

The functions $h(t)$ and $g(t)$ are analytic in the critical region; and one or more saddle points, that is, points where $h'(t) = dh(t)/dt$ vanishes, lie in the critical region. We assume $h(0) = 0$ and that x is real and positive. If x were complex, the factor $\exp(i \arg x)$ could be included in $h(t)$.

The path L' may be deformed into a path D consisting of (i) paths of steepest descent which pass through some or all of the saddle points plus possibly (ii) loops around branch cuts and poles. The path D is independent of x. When x is extremely large, all but a negligible part of J arises from contributions of very small portions of D. If $t = 0$ is a singularity, one portion may lie close to $t = 0$. Another portion is centered on the highest (that is, largest $\exp[xh(t)]$) saddle point. If the two highest saddle points are of the same height, a portion is centered on each, and so on. Thus when x is extremely large, the asymptotic series for J may be obtained by the classical or "usual" saddle point method.

However, we may wish to compute J for values of x which, though large, are not large enough to allow J to be evaluated by the classical saddle point method. For such x's the highest saddle points and the singularity (for $\lambda \neq 1$) at $t = 0$ cannot be treated separately, that is, their interaction must be taken into account. If other saddle points of lesser height lie in the critical region, they must also be considered. This is the range of x of interest here. Our problem is to obtain the appropriate expansion of J in descending powers of x. The type of expansion we seek is shown in equation (46) for J.

This type and the type shown in equation (15) have occurred in earlier publications[3–6] and have been called "uniform" asymptotic expansions because they hold uniformly as a saddle point approaches a singularity or another saddle point.

III. CHANGE OF VARIABLE

In Appendix A it is shown that, in the critical region, $h(t)$ behaves much like a polynomial of degree $\mu + 1$ in t. Here μ is the number of saddle points in the critical region. This suggests changing the variable of integration from t to v where

$$F(v) = h(t) \tag{3}$$

and $F(v)$ is a polynomial of degree $\mu + 1$ in v. When $F(v)$ is known, solving (3) for v as a function of t gives $\mu + 1$ branches. The branch chosen for the change of variable is the one for which $dt/dv \approx c$ throughout the critical region, c being a constant. That one and only one of the $\mu + 1$ branches has this property is rendered plausible by the discussion in Appendix A.

Fortunately we do not have to solve equation (3) to obtain the asymptotic series we desire. However, for some steps we do need the values of dt/dv and higher derivatives at the saddle points. These may be obtained by repeated differentiation of (3).

$F(v)$ is not uniquely determined by $h(t)$. The factors which influence its choice are reviewed in Appendix B.

The change of variable from t to v carries the integral (1) for J into

$$J = \int_L v^{\lambda-1} f(v) \exp [xF(v)] \, dv \tag{4}$$

where

$$f(v) = g(t) \; (t/v)^{\lambda-1} t^{(1)}, \qquad t^{(1)} = dt/dv. \tag{5}$$

The path of integration L starts in the v-plane valley corresponding to valley S in the t-plane, passes through the critical region surrounding $v = 0$, then descends into the v-plane valley corresponding to valley E.

IV. THE BLEISTEIN METHOD FOR $\lambda = 1$

For the case $\lambda = 1$, the integral J becomes

$$I = \int_{L'} g(t) \exp [xh(t)]\, dt = \int_L f(v) \exp [xF(v)]\, dv, \tag{6}$$
$$f(v) = g(t) \frac{dt}{dv} = g(t)t^{(1)}.$$

The Bleistein method begins by constructing a polynomial $P_0(v)$ of degree $\mu - 1$ such that $P_0(v_r) = f(v_r)$, $r = 1, 2, \ldots, \mu$ where $v_1, v_2, \ldots, v_\mu$ are the zeros, assumed simple, of $F'(v)$. By Lagrange's interpolation formula,

$$P_0(v) = \sum_{r=1}^{\mu} \frac{f(v_r)F'(v)}{(v - v_r)F''(v_r)} \tag{7}$$

where the primes denote derivatives. The polynomial may be written as

$$\begin{aligned} P_0(v) &= f(v) + \frac{1}{2\pi i} \int_C \frac{f(\zeta)F'(v)\, d\zeta}{(v - \zeta)F'(\zeta)} = \frac{1}{2\pi i} \int_C f(\zeta)\left[\frac{F'(\zeta) - F'(v)}{(\zeta - v)F'(\zeta)}\right] d\zeta \\ &= \frac{1}{2\pi i} \int_C d\zeta\, \frac{f(\zeta)Q(\zeta, v)}{F'(\zeta)} \end{aligned} \tag{8}$$

where $Q(\zeta, v)$ is a polynomial in v of degree $\mu - 1$,

$$Q(\zeta, v) = \frac{F'(\zeta) - F'(v)}{\zeta - v}, \tag{9}$$

and $f(v)$ has been added to remove the contribution of the pole at $\zeta = v$. The path C is taken in the counter-clockwise sense and encloses $\zeta = v$ and the zeros of $F'(\zeta)$ but no singularities of $f(\zeta)$.

The expression for $f(v)$ obtained from (8) gives

$$\begin{aligned} I &= \int_L dv\, f(v) \exp [xF(v)] \\ &= \int_L dv\, P_0(v) \exp [xF(v)] + \int_L dv \exp [xF(v)] \frac{1}{2\pi i} \int_C \frac{d\zeta\, f(\zeta)F'(v)}{(\zeta - v)F'(\zeta)}. \end{aligned} \tag{10}$$

In order to simplify interchanging the order of integration in the double integral, we cut off the tails of L in the usual fashion. The error introduced by truncation is exponentially small compared with the terms that remain. Deforming C so that it encloses the truncated L (in the sense that it encloses the point $\zeta = v$ for all v's on the truncated L), interchanging the order of integration, integrating by parts with respect to v, neglecting the contributions from the integrated portions at the ends of L, and reverting to the original order of integration carries (10) into

$$\int_L dv\, f(v) \exp [xF(v)] = \int_L dv\, P_0(v) \exp [xF(v)] + \frac{1}{x} \int_L dv \exp [xF(v)] \frac{1}{2\pi i} \int_C \frac{d\zeta\, f(\zeta)(-1)}{F'(\zeta)(\zeta - v)^2}. \tag{11}$$

Incidentally, if the contributions from the ends (say at a and b) of the truncated L were not neglected, the right side of (11) would contain the additional term

$$\left[\frac{[f(v) - P_0(v)] \exp [xF(v)]}{xF'(v)}\right]_a^b.$$

The procedure used to establish (11) can be used to show that, for any function $f_n(\zeta)$ analytic inside C, we have

$$\int_L dv\, f_n(v) \exp [xF(v)] = \int_L dv\, P_n(v) \exp [xF(v)] + \frac{1}{x} \int_L dv\, f_{n+1}(v) \exp [xF(v)] \tag{12}$$

where

$$\begin{aligned} f_{n+1}(v) &= \frac{1}{2\pi i} \int_C \frac{d\zeta\, f_n(\zeta)(-1)}{F'(\zeta)(\zeta - v)^2}, \\ P_n(v) &= \frac{1}{2\pi i} \int_C d\zeta\, f_n(\zeta) \left[\frac{Q(\zeta, v)}{F'(\zeta)}\right]. \end{aligned} \tag{13}$$

Setting $f_0(\zeta) = f(\zeta)$ and using (12) repeatedly gives

$$\begin{aligned} I &= \sum_{n=0}^{N} x^{-n} \int_L dv\, P_n(v) \exp [xF(v)] + R_N, \\ R_N &= x^{-N-1} \int_L dv\, f_{N+1}(v) \exp [xF(v)]. \end{aligned} \tag{14}$$

Since $Q(\zeta, v)$ is a polynomial of degree $\mu - 1$ in v, the same is true of $P_n(v)$ and we write

$$\begin{aligned} P_n(v) &= \sum_{l=0}^{\mu-1} p_{nl} v^l, \qquad n = 0, 1, 2, \cdots \\ I &= \sum_{l=0}^{\mu-1} U_l(x) \sum_{n=0}^{N} p_{nl} x^{-n} + R_N \end{aligned} \tag{15}$$

where

$$U_l(x) = \int_L v^l \exp[xF(v)]\, dv, \qquad l = 0, 1, \cdots, \mu - 1. \tag{16}$$

The series (15) is the type of expansion we seek. It would be desirable to have close inequalities for R_N, but none are available at the present time.

Another expression for $P_n(v)$ may be obtained from (13):

$$\begin{aligned} P_n(v) &= \frac{1}{2\pi i}\int_C \frac{d\zeta\, Q(\zeta, v)}{F'(\zeta)} \frac{1}{2\pi i}\int_{C_z} \frac{dz\, f_{n-1}(z)(-1)}{F'(z)(z-\zeta)^2} \\ &= \frac{1}{2\pi i}\int_{C_z} dz\, f_{n-1}(z)\left[\frac{1}{F'(z)}\frac{\partial}{\partial z}\frac{Q(z, v)}{F'(z)}\right] \\ &= \frac{1}{2\pi i}\int_C d\zeta\, f(\zeta)\left[\frac{1}{F'(\zeta)}\frac{\partial}{\partial \zeta}\right]^n \frac{Q(\zeta, v)}{F'(\zeta)}. \end{aligned} \tag{17}$$

In the first line C_z must enclose the point $z = \zeta$ in the z-plane in addition to the zeros of $F'(z)$. Hence initially C_z encloses C. When the order of integration is interchanged, the only singularity of the integrand in the ζ-plane lying outside C is the double pole at $\zeta = z$. Expand C until it consists of a circle of infinite radius at ∞ plus a negative loop around $\zeta = z$. The contribution of the infinite circle vanishes because the integrand is a rational function of ζ of $0(\zeta^{-3})$ at ∞. The contribution of the pole at $\zeta = z$ gives the derivative.

Notice that the coefficients p_{nl} in (15) are independent of the path L in the v-plane.

The procedure used to obtain the integral (17) for $P_n(v)$ may also be used to show that

$$f_{n+1}(v) = \frac{1}{2\pi i}\int_C d\zeta\, f(\zeta)(-1)\left[\frac{1}{F'(\zeta)}\frac{\partial}{\partial \zeta}\right]^n\left[\frac{1}{F'(\zeta)(\zeta - v)^2}\right]. \tag{18}$$

When $\mu = 1$ and $F(v) = v^2$, the polynomial $P_n(v)$ reduces to $(-1)^n f^{(2n)}(0)/(4^n n!)$ and $f_{n+1}(v)$ is equal to $[f_n(v) - vf_n'(v) - f_n(0)]/(2v^2)$.

V. COMPUTATION OF $P_n(v)$, $\lambda = 1$: BLEISTEIN METHOD

We shall regard the functions $U_l(x)$ in the series (15) for I as tabulated or easily computed. For example, when $\mu = 2$ the functions $U_0(x)$ and $U_1(x)$ may be expressed in terms of Airy functions. Then the most difficult step in applying the series is the calculation of the coefficients p_{nl}, $l = 0, 1, \ldots, \mu - 1$, of the polynomial $P_n(v)$. We desire an expression for p_{nl} in terms of the values of the functions $g(t)$, $h(t)$ and their derivatives at the saddle points $t = t_r$, $r = 1, 2, \cdots, \mu$.

Let $t = t(v)$ denote the change of variable from t to v, and let the saddle point $v = v_r$ in the v-plane $[F'(v_r) = 0]$ correspond to t_r in the t-plane: $t_r = t(v_r)$. We shall use the notation

$$h_r^{(n)} = \left[\left(\frac{d}{dt}\right)^n h(t)\right]_{t=t_r}, \qquad g_r^{(n)} = \left[\left(\frac{d}{dt}\right)^n g(t)\right]_{t=t_r}$$
$$t_r^{(n)} = \left[\left(\frac{d}{dv}\right)^n t(v)\right]_{v=v_r}, \qquad r = 1, 2, \cdots, \mu. \tag{19}$$

When convenient, we shall write h_r for $h_r^{(0)} = h(t_r)$ and g_r for $g_r^{(0)} = g(t_r)$.

First consider the expression (7) for $P_0(v)$. As shown in Appendix B, $F(v)$ is a polynomial of degree $\mu + 1$,

$$F(v) = \sum_{j=0}^{\mu+1} A_j v^j \tag{20}$$

whose coefficients A_j may be expressed as functions of the h_r's. When this equation for $F(v)$ is used in (7), the coefficient of v^l in the resulting expression for $P_0(v)$ gives

$$p_{0l} = \sum_{j=l+2}^{\mu+1} jA_j \sum_{r=1}^{\mu} \frac{f(v_r)v_r^{j-2-l}}{F''(v_r)}, \qquad l = 0, 1, \cdots, \mu - 1. \tag{21}$$

Multiplying the right side of (21) by -1 and changing the limits of summarion for j from $l + 2$, $\mu + 1$ to 1, $l + 1$ gives another expression for p_{0l}.

Since $f(v) = g(t)t^{(1)}$, we also need an expression for $t_r^{(1)}$ in terms of $g(t)$ and $h(t)$. Differentiating $F(v) = h(t)$ twice with respect to v and using $h_r^{(1)} = 0$ leads to

$$f(v_r) = g_r t_r^{(1)}, \qquad t_r^{(1)} = [F''(v_r)/h_r^{(2)}]^{\frac{1}{2}}. \tag{22}$$

The sign of the square root is chosen to agree with the constant c in $t \approx cv$, the form assumed by the change of variable throughout the critical region.

Since the A_j's and v_r's may be expressed in terms of the h_r's, equations (21) and (22) show that p_{0l} depends only on the h_r's, g_r's and $h_r^{(2)}$'s.

When n is general, an expression for p_{nl} similar to (21) for p_{0l} may be obtained by expanding the derivative in the integral (17) for $P_n(v)$ in partial factions and then using the Cauchy integral theorem. For

$n = 1$ it is found that

$$p_{1l} = \sum_{j=l+2}^{\mu+1} jA_j \sum_{r=1}^{\mu} \sum_{s=1}^{\mu}{}' \frac{1}{F''(v_s)F''(v_r)} \cdot v_r^{j-2-l}\left[\frac{f(v_r) - f(v_s)}{(v_s - v_r)^2} + \frac{f'(v_r)}{v_s - v_r} + \frac{f''(v_r)}{2}\right] \tag{23}$$

where the prime on $\sum'$ denotes that the term for $s = r$ is omitted. The primes on $f(v)$ and $F(v)$ denote derivatives with respect to v.

The expression obtained for $P_n(v)$ is of the form

$$P_n(v) = \sum_{r=1}^{\mu} \sum_{m=1}^{2n+1} \alpha_{r,m}^{(n)} f^{(m-1)}(v_r)/(m - 1)! \tag{24}$$

where $\alpha_{r,m}^{(n)}$ is a polynomial in v. Recurrence relations for the α's may be obtained with the help of the partial fraction expansion

$$\frac{(\zeta - v_r)^{-m}}{F'(\zeta)} = \sum_{s=1}^{\mu}{}' \frac{(v_s - v_r)^{-m}}{F''(v_s)}\left[\frac{1}{\zeta - v_s} - \sum_{q=0}^{m} \frac{(v_s - v_r)^q}{(\zeta - v_r)^{q+1}}\right]. \tag{25}$$

The relation $\sum' [1/F''(v_s)] = -1/F''(v_r)$ can be used to simplify the coefficient of $(\zeta - v_r)^{-m-1}$.

The mth derivative of $f(v)$ evaluated at v_r,

$$f^{(m)}(v_r) = \sum_{j=0}^{m} \binom{m}{j} t_r^{(j+1)} \left[\left(\frac{d}{dv}\right)^{m-j} g(t)\right]_{t=t_r} \tag{26}$$

contains derivatives of $t(v)$. They may be obtained by extending the method used to get $t_r^{(1)}$. Straightforward differentiation of $F(v) = h(t)$ with respect to v leads to

$$\begin{aligned} t_r^{(2)} &= \frac{1}{3t_r^{(1)}h_r^{(2)}}[F^{(3)} - h^{(3)}t^{(1)3}]_r \\ t_r^{(3)} &= \frac{1}{4t_r^{(1)}h_r^{(2)}}[F^{(4)} - 6h^{(3)}t^{(2)}t^{(1)2} - h^{(4)}t^{(1)4} - 3h^{(2)}t^{(2)2}]_r \end{aligned} \tag{27}$$

where the nth derivative $F^{(n)}$ of $F(v)$ is evaluated at v_r and $h^{(n)}$, $t^{(n)}$ are evaluated at t_r.

The values of $t^{(j+1)}$ for larger j's may be obtained with the help of equation (94), namely

$$F^{(n)}(v) = \sum_{k=1}^{n} h^{(k)}(t)c_{n,k} \tag{28}$$

where $c_{n,1} = t^{(n)}$, $c_{n,n} = t^{(1)n}$ and the remaining c's are given by the recurrence relation (96). Setting $k = 1$ in (96) gives

$$c_{n,2} = nt^{(n-1)}t^{(1)} + \sum_{m=2}^{n-2} \binom{n-1}{m} t^{(n-m)}t^{(m)} \tag{29}$$

for $n \geqq 3$, the last summation being omitted when $n = 3$. The term in (28) for $k = 1$ vanishes when $v = v_r$, $t = t_r$. Substituting for $c_{n,2}$ its value given by (29) and solving for $t^{(n-1)}$ leads to the desired result when $n \geqq 3$:

$$t_r^{(n-1)} = \frac{1}{nt_r^{(1)}h_r^{(2)}} \left[F^{(n)} - \sum_{k=3}^{n} h^{(k)}c_{n,k} - h^{(2)} \sum_{m=2}^{n-2} \binom{n-1}{m} t^{(n-m)}t^{(m)} \right]_r . \tag{30}$$

The value of $F^{(n)}(v)$ is zero for $n > \mu + 1$.

VI. COMPUTATION OF $P_n(v)$, $\lambda = 1$: URSELL METHOD

The Ursell method avoids the evaluation of the derivatives of $f(v)$ which appear in equation (24) for $P_n(v)$. Instead, it makes use of classical saddle point expansions about the individual saddle points in the t and v planes.

Let μ different paths of integration, L_1', L_2', $\ldots$, L_μ' be chosen in (6) such that the chief contributions (as $x \to \infty$) along the paths corresponding to r, namely L_r' in the t-plane and its mate L_r in the v-plane, occur at the saddle points $t = t_r$ and $v = v_r$, respectively. Let the classical asymptotic expansions around t_r and v_r be

$$I_r = \int_{L'_r} g(t) \exp [xh(t)] \, dt \sim \exp [xh(t_r)] \sum_{n=0}^{\infty} \alpha_{rn} x^{-n-\frac{1}{2}} \tag{31}$$

$$[U_l(x)]_r = \int_{L_r} v^l \exp [xF(v)] \, dv \sim \exp [xF(v_r)] \sum_{m=0}^{\infty} \beta_{rlm} x^{-m-\frac{1}{2}} . \tag{32}$$

Using $h(t_r) = F(v_r)$, substituting (31) and (32) in the uniform asymptotic expansion (15) with $N = \infty$, and equating coefficients of $x^{-n-\frac{1}{2}}$ gives

$$\begin{aligned} \alpha_{rn} &= \sum_{l=0}^{\mu-1} \sum_{m=0}^{n} \beta_{rlm} p_{n-m,l} \\ &= \sum_{l=0}^{\mu-1} \beta_{rl0} p_{nl} + \sum_{l=0}^{\mu-1} \sum_{m=1}^{n} \beta_{rlm} p_{n-m,l} \end{aligned} \tag{33}$$

where the second sum in the last line is omitted when $n = 0$. The expression (17) for $P_n(v)$ shows that $P_n(v)$ remains the same, irrespective of the path of integration L, as long as $f(v)$ and $F(v)$, that is, $g(t)$ and $h(t)$, remain the same. Hence, for $n = 0$ and $r = 1, 2, \ldots , \mu$, (33) furnishes μ simultaneous linear equations which may be solved for p_{0l}, $l =$

$0, 1, \cdots, \mu - 1$. Similarly when $n = 1$, (33) determines the p_{1l}'s, and so on. It turns out (see equation 39) that $\beta_{rl0} = v_r^l \beta_{r00}$. This allows us to write the simultaneous equations in the form

$$P_0(v_r) = \alpha_{r0}/\beta_{r00}, \qquad r = 1, 2, \cdots, \mu$$
$$P_n(v_r) = \frac{\alpha_{rn}}{\beta_{r00}} - \sum_{m=1}^{n} \sum_{l=0}^{\mu-1} \left(\frac{\beta_{rlm}}{\beta_{r00}}\right) p_{n-m,l}. \tag{34}$$

Expressions for α_{rn} and β_{rlm} may be obtained from the classical saddle point asymptotic expansion (103) given in Appendix D. Changing n to j in order to agree with the notation of Appendix D gives

$$\alpha_{rj} = (\pi)^{\frac{1}{2}} \left[\frac{-2}{h_r^{(2)}}\right]^{j+\frac{1}{2}} \sum_{n=0}^{2j} \frac{g_r^{(2j-n)}}{(2j-n)!} \sum_{m=0}^{n} b_{mn} (\tfrac{1}{2})_{m+j} \tag{35}$$

where $h_r^{(n)}$, $g_r^{(n)}$ are the derivatives defined in equations (19), and $(x)_0 = 1$, $(x)_n = x(x+1) \ldots (x+n-1)$. The b_{mn}'s are computed from the recurrence relation (100), namely

$$b_{m+1,n+1} = \frac{1}{n+1} \sum_{k=1}^{n-m+1} k a_k b_{m,n-k+1},$$

starting with $b_{00} = 1$ and using

$$a_k = -\frac{2h_r^{(k+2)}}{(k+2)!\, h_r^{(2)}}, \qquad k = 1, 2, \cdots. \tag{36}$$

The value of arg $[-2/h_r^{(2)}]^{\frac{1}{2}}$ is equal to arg $(t - t_r)$ on the portion of L_r' (deformed into a path of steepest descent through t_r) leaving t_r.

Similarly,

$$\beta_{rlj} = (\pi)^{\frac{1}{2}} \left[\frac{-2}{F_r^{(2)}}\right]^{j+\frac{1}{2}} \sum_{n=0}^{2j} \frac{(-1)^{2j-n}(-l)_{2j-n} v_r^{l-2j+n}}{(2j-n)!} \sum_{m=0}^{n} b_{mn} (\tfrac{1}{2})_{m+j} \tag{37}$$

where now the b_{mn}'s are computed from (100) with

$$a_k = \frac{-2F_r^{(k+2)}}{(k+2)!\, F_r^{(2)}}, \qquad F_r^{(n)} = \left[\left(\frac{d}{dv}\right)^n F(v)\right]_{v=v_r}. \tag{38}$$

Setting $j = 0$ in (35) and (37) gives

$$\alpha_{r0} = (\pi)^{\frac{1}{2}} \left[\frac{-2}{h_r^{(2)}}\right]^{\frac{1}{2}} g_r^{(0)}, \qquad \beta_{rl0} = (\pi)^{\frac{1}{2}} \left[\frac{-2}{F_r^{(2)}}\right]^{\frac{1}{2}} v_r^l,$$
$$P_0(v_r) = \frac{\alpha_{r0}}{\beta_{r00}} = \left[\frac{F_r^{(2)}}{h_r^{(2)}}\right]^{\frac{1}{2}} g_r^{(0)} = t_r^{(1)} g_r^{(0)} = f(v_r) \tag{39}$$

where $t_r^{(1)}$ and $f(v_r)$ are the same as in equation (22). The relation

$P_0(v_r) = f(v_r)$ is the starting point for the Lagrange interpolation formula (7) in the Bleistein method.

Setting $n = 1$ in (34) and $j = 1$ in (35) and (37) leads to

$$P_1(v_r) = \frac{\alpha_{r1}}{\beta_{r00}} - \sum_{l=0}^{\mu-1} \frac{\beta_{rl1}}{\beta_{r00}} p_{0l}$$

$$\frac{\alpha_{r1}}{\beta_{r00}} = \left[\frac{-2t_r^{(1)}}{h_r^{(2)}}\right]\left\{\frac{g^{(2)}}{4} - \frac{g^{(1)}h^{(3)}}{4h^{(2)}} + g^{(0)}\left[\frac{-h^{(4)}}{16h^{(2)}} + \frac{5h^{(3)2}}{48h^{(2)2}}\right]\right\}_r \tag{40}$$

$$\frac{\beta_{rl1}}{\beta_{r00}} = \left[\frac{-2}{F_r^{(2)}}\right]\left\{\frac{l(l-1)}{4} v^{l-2} - \frac{lv^{l-1}F^{(3)}}{4F^{(2)}} + v^l\left[\frac{-F^{(4)}}{16F^{(2)}} + \frac{5F^{(3)2}}{48F^{(2)2}}\right]\right\}_r$$

where the subscript r on the braces indicates that the enclosed g's, h's, v's, F's have the subscript r.

VII. THE BLEISTEIN METHOD FOR $\lambda \neq 1$

Here we deal with

$$J = \int_{L'} t^{\lambda-1} g(t) \exp [xh(t)]\, dt = \int_L v^{\lambda-1} f(v) \exp [xF(v)]\, dv$$
$$f(v) = g(t)(t/v)^{\lambda-1} t^{(1)}, \qquad t^{(1)} = dt/dv. \tag{41}$$

The origin is now a singularity, and its vicinity may contribute to J just as the vicinities of the saddle points do. Accordingly, we now require that the polynomial $P_0(v)$ be such that $P_0(0) = f(0)$ in addition to $P_0(v_r) = f(v_r)$, $r = 1, 2, \ldots, \mu$. Assume for the moment that $F'(0) \neq 0$,that is, that the origin is not a saddle point. Starting with Lagrange's interpolation formula and proceeding as in Section IV gives

$$P_0(v) = \frac{f(0)vF'(v)}{vF'(0)} + \sum_{r=1}^{\mu} \frac{f(v_r)vF'(v)}{(v - v_r)v_rF''(v_r)}$$
$$= f(v) + \frac{1}{2\pi i}\int_C \frac{f(\zeta)vF'(v)\, d\zeta}{(v - \zeta)\zeta F'(\zeta)}$$
$$= \frac{1}{2\pi i}\int_C \frac{f(\zeta)Q(\zeta, v)\, d\zeta}{\zeta F'(\zeta)} \tag{42}$$
$$Q(\zeta, v) = \frac{\zeta F'(\zeta) - vF'(v)}{\zeta - v}$$

where C encloses $\zeta = v$, $\zeta = 0$, $\zeta = v_r$, $r = 1, 2, \ldots, \mu$ but no singularities of $f(\zeta)$. Here $P_0(v)$ and $Q(\zeta, v)$ are polynomials of degree μ instead of $\mu - 1$.

When the origin is a saddle point, $P_0(v)$ is still given by the expressions in (42) which contain integrals. In fact, we have $P_0'(0) = f'(0)$ (with primes denoting derivatives) in addition to $P_0(0) = f(0)$.

Much as in Section IV, we obtain

$$\int_L dv\, v^{\lambda-1} f_n(v) \exp[xF(v)]$$

$$= \int_L dv\, v^{\lambda-1} P_n(v) \exp[xF(v)] + \frac{1}{x}\int_L dv\, v^{\lambda-1} f_{n+1}(v) \exp[xF(v)] \tag{43}$$

where $f_0(v) = f(v)$ and

$$\begin{aligned} f_{n+1}(v) &= \frac{1}{2\pi i}\int_C \frac{d\zeta\, f_n(\zeta)(-1)}{\zeta F'(\zeta)} \frac{(\lambda\zeta - \lambda v + v)}{(\zeta - v)^2}, \\ P_n(v) &= \frac{1}{2\pi i}\int_C \frac{d\zeta\, f_n(\zeta) Q(\zeta, v)}{\zeta F'(\zeta)}. \end{aligned} \tag{44}$$

Equations (43) and (44) lead to the desired series for J:

$$\begin{aligned} J &= \sum_{n=0}^{N} x^{-n} \int_L dv\, v^{\lambda-1} P_n(v) \exp[xF(v)] + R_N, \\ R_N &= x^{-N-1} \int_L dv\, v^{\lambda-1} f_{N+1}(v) \exp[xF(v)]\, dv. \end{aligned} \tag{45}$$

When $P_n(v)$ is written out we get

$$\begin{aligned} P_n(v) &= \sum_{l=0}^{\mu} p_{nl} v^l, \qquad n = 0, 1, 2, \cdots \\ J &= \sum_{l=0}^{\mu} V_l(x) \sum_{n=0}^{N} p_{nl} x^{-n} + R_N, \\ V_l(x) &= \int_L v^{l+\lambda-1} \exp[xF(v)]\, dv, \qquad l = 0, 1, 2, \cdots, \mu. \end{aligned} \tag{46}$$

Furthermore, the recurrence relation (44) for $f_n(v)$ leads to

$$\begin{aligned} P_n(v) &= \frac{1}{2\pi i}\int_C \frac{d\zeta\, f_{n-1}(\zeta)}{\zeta F'(\zeta)} \left[\zeta^{\lambda} \frac{\partial}{\partial\zeta} \frac{\zeta^{-\lambda} Q(\zeta, v)}{F'(\zeta)}\right] \\ &= \frac{1}{2\pi i}\int_C d\zeta\, f(\zeta)\zeta^{\lambda-1} \left[\frac{1}{F'(\zeta)}\frac{\partial}{\partial\zeta}\right]^n \frac{Q(\zeta, v)\zeta^{-\lambda}}{F'(\zeta)} \\ f_{n+1}(v) &= \frac{1}{2\pi i}\int_C d\zeta\, f(\zeta)(-1)\zeta^{\lambda-1} \left[\frac{1}{F'(\zeta)}\frac{\partial}{\partial\zeta}\right]^n \frac{(\lambda\zeta - \lambda v + v)\zeta^{-\lambda}}{(\zeta - v)^2 F'(\zeta)}. \end{aligned} \tag{47}$$

When $\lambda = 1$ the formulas of this section do not reduce to those of Section IV since they contain the additional condition $P_n(0) = f(0)$. However, (46) gives the same series for J as (14) does for I because $V_\mu(x)$ can now be expressed as a linear combination of $V_0(x), \ldots, V_{\mu-1}(x)$ [which become $U_0(x), \ldots, U_{\mu-1}(x)$].

The only singularities enclosed by C in the integral (47) for $P_n(v)$ are poles at $\zeta = 0$ and at $\zeta = v_r$, $r = 1, 2, \ldots, \mu$. Evaluating the integral by Cauchy's theorem gives the coefficients in $P_n(v)$ as the sum of derivatives of $f(v)$ at $v = 0$ and at $v = v_r$. The derivatives at the saddle points may be obtained by differentiating

$$\ln f(v) = \ln g(t) + (\lambda - 1) \ln \frac{t}{v} + \ln t^{(1)} \tag{48}$$

with respect to v and using the expressions for $t_r^{(n)}$ developed in Section V. The derivatives $f^{(n)}(0)$ may be computed with the help of the series

$$t/v = t_0^{(1)} + \frac{v}{2!} t_0^{(2)} + \frac{v^2}{3!} t_0^{(3)} + \cdots \tag{49}$$

where $t_0^{(n)}$ denotes the nth derivative of t with respect to v at $v = 0$. The $t_0^{(n)}$'s may be obtained by differentiating $F(v) = h(t)$ repeatedly with respect to v and then setting $v = 0$. If $F'(0) \neq 0$,

$$\begin{aligned} t_0^{(1)} &= \frac{F_0^{(1)}}{h_0^{(1)}} \\ t_0^{(2)} &= \frac{F_0^{(2)} - h_0^{(2)} t_0^{(1)2}}{h_0^{(1)}} \end{aligned} \tag{50}$$

where the subscript 0 refers to $t = 0$ when it is on h and to $v = 0$ when it is on F. Higher order derivatives may be computed by using the results of Appendix C in much the same way as in Section V.

It may be verified that

$$\begin{aligned} f(0) &= g_0^{(0)} t_0^{(1)\lambda} \\ f^{(1)}(0) &= t_0^{(1)\lambda} \left[g_0^{(1)} t_0^{(1)} + \frac{(\lambda + 1)}{2} \frac{g_0^{(0)} t_0^{(2)}}{t_0^{(1)}} \right] \end{aligned} \tag{51}$$

where $g_0^{(n)}$ is the nth derivative of $g(t)$ with respect to t evaluated at $t = 0$.

VIII. THE URSELL METHOD FOR $\lambda \neq 1$

When the origin is not a saddle point, the $\mu + 1$ linear equations to be solved for the coefficients p_{nl}, $l = 0, 1, \ldots, \mu$ in $P_n(v)$ turn out to be

$$P_n(v_r) = \frac{\alpha_{rn}}{\beta_{r00}} - \sum_{m=1}^{n} \sum_{l=0}^{\mu} \left(\frac{\beta_{rlm}}{\beta_{r00}}\right) p_{n-m,l}\ , \qquad r = 1, 2, \cdots, \mu \tag{52}$$

$$p_{n0} = \frac{\alpha_{0n}}{\beta_{000}} - \sum_{q=1}^{n} \sum_{l+m=q} \left(\frac{\beta_{0lm}}{\beta_{000}}\right) p_{n-q,l} \tag{53}$$

where the summations are omitted when $n = 0$. The summation condition $l + m = q$ in (53) is also subject to $0 \leqq l \leqq \mu$, $0 \leqq m \leqq \infty$.

Equations (52) are given by the analysis of Section VI for the case $\lambda = 1$ when $g(t)$ is replaced by $t^{\lambda-1}g(t)$, v^l by $v^{l+\lambda-1}$, I_r by J_r, and $U_l(x)$ by $V_l(x)$. The α's and β's in the rth equation of (52) are the coefficients in the classical saddle point expansions about t_r and v_r:

$$\begin{aligned} J_r &= \int_{L_r'} t^{\lambda-1} g(t) \exp[xh(t)]\, dt \sim \exp[xh(t_r)] \sum_{n=0}^{\infty} \alpha_{rn} x^{-n-\frac{1}{2}} \\ [V_l(x)]_r &= \int_{L_r} v^{l+\lambda-1} \exp[xF(v)]\, dv \sim \exp[xF(v_r)] \sum_{m=0}^{\infty} \beta_{rlm} x^{-m-\frac{1}{2}} \end{aligned} \tag{54}$$

The α_{rj} in (52) (with j for n) is given by equation (35) for α_{rj} with $g_r^{(2j-n)}$ replaced by $\hat{g}_r^{(2j-n)}$ where

$$\begin{aligned} \hat{g}(t) &= t^{\lambda-1} g(t) = \sum_{n=0}^{\infty} \frac{(t-t_r)^n}{n!} \hat{g}_r^{(n)} \\ \hat{g}_r^{(n)} &= t_r^{\lambda} \sum_{k=0}^{n} \binom{n}{k} g_r^{(n-k)} (-1)^k (1-\lambda)_k t_r^{-k-1}. \end{aligned} \tag{55}$$

The β_{rlj} in (52) is given by equation (37) for β_{rlj} with l replaced by $l + \lambda - 1$ on the right side.

Equation (53) arises from a consideration of the region around the singularity at the origin. As described in connection with equation (106) in Appendix D, let L_0' be a loop enclosing the branch cut running out from $t = 0$, and let L_0 be its mate in the v-plane. Then, as $x \to \infty$, the α's and β's in (53) are defined by

$$\begin{aligned} J_0 &= \int_{L_0'} t^{\lambda-1} g(t) \exp[xh(t)]\, dt \sim \sum_{n=0}^{\infty} \alpha_{0n} x^{-n-\lambda} \\ [V_l(x)]_0 &= \int_{L_0} v^{l+\lambda-1} \exp[xF(v)]\, dv \sim \sum_{m=0}^{\infty} \beta_{0lm} x^{-m-l-\lambda}. \end{aligned} \tag{56}$$

Substituting (56) in the uniform asymptotic expansion for J_0 given by (46) and equating coefficients of $x^{-n-\lambda}$ gives (53).

Using the asymptotic series (106) to determine α_{0n} and β_{rlm} leads to

$$\frac{\alpha_{0j}}{\beta_{000}} = t_0^{(1)\lambda} \left(\frac{-1}{h_0^{(1)}}\right)^j \sum_{n=0}^{j} \frac{g_0^{(j-n)}}{(j-n)!} \sum_{m=0}^{n} b_{mn}(\lambda)_{m+j} \tag{57}$$

where the subscript 0 refers to the origin and b_{mn} is computed from (100) with a_k given by

$$a_k = -h_0^{(k+1)}/[(k+1)!\, h_0^{(1)}]. \tag{58}$$

The value of $t_0^{(1)}$ is the value of dt/dv at $v = 0$ determined by the change of variable from t to v. Similarly,

$$\frac{\beta_{0lj}}{\beta_{000}} = \left(\frac{-1}{F_0^{(1)}}\right)^{l+j} \sum_{m=0}^{j} b_{mj}(\lambda)_{m+l+j} \tag{59}$$

where, replacing j by n, b_{mn} is computed from (100) with

$$a_k = -F_0^{(k+1)}/[(k+1)!\, F_0^{(1)}], \qquad k \geqq 1. \tag{60}$$

In Appendix F the theory which has just been developed is applied to the case of one saddle point ($\mu = 1$, $\lambda \neq 1$).

So far in this section it has been assumed that the origin is not a saddle point. Now let the μth saddle point coincide with the origin so that $F_0^{(1)}$ and $h_0^{(1)}$ vanish. The $\mu + 1$ equations determining p_{nl}, $l = 0, 1, \ldots, \mu$ are now

$$P_n(v_r) = \frac{\alpha_{rn}}{\beta_{r00}} - \sum_{m=1}^{n} \sum_{l=0}^{\mu} \left(\frac{\beta_{rlm}}{\beta_{r00}}\right) p_{n-m,l} \qquad r = 1, 2, \cdots, \mu - 1 \tag{61}$$

$$p_{n0} = \frac{\alpha_{0,2n}}{\beta_{000}} - \sum_{q=1}^{n} \sum_{m+l=2q} \left(\frac{\beta_{0lm}}{\beta_{000}}\right) p_{n-q,l} \tag{62}$$

$$p_{n1} = \frac{1}{\beta_{010}} \left[\alpha_{0,2n+1} - \beta_{001} p_{n0} - \sum_{q=1}^{n} \sum_{m+l=2q+1} \beta_{0lm} p_{n-q,l}\right] \tag{63}$$

where the summations are omitted when $n = 0$. The values of l and m occurring in the inner summations in (62) and (63) must also satisfy $0 \leqq l \leqq \mu$ and $0 \leqq m \leqq \infty$.

Equations (61) are the same as (52) except that r runs from 1 to $\mu - 1$ instead of from 1 to μ. The α's and β's in (62) and (63) are the coefficients in the asymptotic expansions

$$\begin{aligned} J_0 &= \int_{L'0} t^{\lambda-1} g(t) \exp[xh(t)]\, dt \sim \sum_{j=0}^{\infty} \alpha_{0j} x^{-(j+\lambda)/2} \\ [V_l(x)]_0 &= \int_{L0} v^{l+\lambda-1} \exp[xF(v)]\, dv \sim \sum_{m=0}^{\infty} \beta_{0lm} x^{-(m+l+\lambda)/2} \end{aligned} \tag{64}$$

where, as discussed in connection with equation (109), the paths L_0', L_0 coincide with the paths of steepest descent through $t = 0$, $v = 0$ except for indentations at those points.

Equation (62) is obtained by substituting (64) in the uniform asymptotic series for J_0 given by (46) and equating coefficients of $x^{-(2n+\lambda)/2}$. Equating coefficients of $x^{-(2n+1+\lambda)/2}$ gives equation (63).

Some results for the case of two saddle points, one of which is at the origin, are stated in Appendix H.

IX. SIMPLE POLE AT THE ORIGIN

When there is a simple pole at $t = 0$ and one saddle point in the critical region, a case discussed briefly by Bleistein,[3] we have

$$J = \int_{L'} t^{-1} \exp [xh(t)]\, dt. \tag{65}$$

In the critical region, L' is assumed to coincide with the linear path running from $\alpha - i\infty$ to $\alpha + i\infty$, $\alpha > 0$.

Let $h(t)$ be real when t is real and in the critical region. Let the saddle point t_1 lie on the real axis. As usual, $h_o = 0$, $h_1^{(1)} = 0$; and we assume $h_1 \leqq 0$, $h_1^{(2)} > 0$. As suggested by example (i) of Appendix 3, we choose

$$F(v) = v^2 - 2v_1 v, \qquad v_1^2 = -h_1$$

where v_1 is real. We write

$$v_1 = \frac{t_1}{|\, t_1 \,|} (-h_1)^{\frac{1}{2}}, \qquad (-h_1)^{\frac{1}{2}} \geqq 0$$

in order to make v_1 and t have the same sign.

Equation (46) shows that the uniform asymptotic expansion for J has the form

$$J \sim V_0(x) \sum_{n=0}^{\infty} p_{n0} x^{-n} + V_1(x) \sum_{n=0}^{\infty} p_{n1} x^{-n} \tag{66}$$

where, with L parallel to, and to the right of, the imaginary v-axis,

$$V_0(x) = \int_L v^{-1} \exp [xF(v)]\, dv = i\pi[1 - \operatorname{erf}(v_1 x^{\frac{1}{2}})]$$

$$V_1(x) = \int_L \exp [xF(v)]\, dv = i(\pi/x)^{\frac{1}{2}} \exp (-xv_1^2)$$

$$\operatorname{erf}(z) = \frac{2}{\pi^{\frac{1}{2}}} \int_0^z \exp (-t^2)\, dt.$$

Putting $\lambda = 0$ in the integral (47) for $P_n(v)$ gives

$$P_n(v) = p_{n0} + p_{n1} v$$

$$= \frac{1}{2\pi i} \int_C \frac{d\zeta\, f(\zeta)}{2^n \zeta} \left(\frac{1}{\zeta - v_1} \frac{\partial}{\partial \zeta} \right)^n \left(\frac{\zeta + v - v_1}{\zeta - v_1} \right) \tag{67}$$

$$f(v) = vt^{(1)}/t, \qquad t^{(1)} = dt/dv$$

where C encloses $\zeta = 0$ and $\zeta = v_1$ but no singularities of $f(\zeta)$. Setting $v = 0$ in (67) gives

$$P_0(0) = p_{00} = f(0) = 1,$$

$$P_n(0) = p_{n0} = 0, \qquad n > 0.$$

Here the series (49) for t/v has been used to show that $f(0) = 1$. Therefore the series for J reduces to

$$J \sim i\pi\{1 - \operatorname{erf}[v_1 x^{\frac{1}{2}}]\} + i(\pi/x)^{\frac{1}{2}} \exp(-xv_1^2) \sum_{n=0}^{\infty} p_{n1} x^{-n}. \tag{68}$$

Setting $v = v_1$ in (67) gives

$$p_{01} = [f(v_1) - 1]/v_1 = \frac{t_1^{(1)}}{t_1} - \frac{1}{v_1} \tag{69}$$

$$p_{n1} = \frac{1}{2\pi i} \int_C \frac{d\zeta\, f(\zeta)}{2^n v_1 \zeta} \left(\frac{1}{\zeta - v_1} \frac{\partial}{\partial \zeta} \right)^n \left(\frac{\zeta}{\zeta - v_1} \right), \qquad n > 0.$$

From (22) and $F''(v) = 2$ it follows that $t_1^{(1)} = [2/h_1^{(2)}]^{\frac{1}{2}}$. The integral for p_{n1} may be evaluated in terms of the $2n$th derivative of $f(v)/v = t^{(1)}/t = (d/dv) \ln t(v)$. Thus, writing $\zeta/(\zeta - v_1)$ as $1 + v_1(\zeta - v_1)^{-1}$ and using

$$\left(\frac{1}{\zeta - v_1} \frac{\partial}{\partial \zeta} \right)^n (\zeta - v_1)^{-1} = (-1)^n 1 \cdot 3 \ldots (2n - 1)(\zeta - v_1)^{-2n-1}, \; n > 0$$

leads to

$$p_{n1} = \frac{(-1)^n (\frac{1}{2})_n}{v_1^{2n+1}} \left\{ \frac{v_1^{2n+1}}{(2n)!} \left[\left(\frac{d}{dv} \right)^{2n} \frac{t^{(1)}}{t} \right]_{v=v_1} - 1 \right\}$$

The first of the p_{n1}'s required in the series (68) for J is given by equation (69) for p_{01}. The remaining ones may be obtained by using the Ursell method equation (52). Since $p_{00} = 1$ and $p_{n0} = 0$ for $n > 0$, equation (52) gives for $n > 0$

$$p_{n1} = \frac{1}{v_1 \beta_{100}} \left[\alpha_{1n} - \beta_{10n} - \sum_{m=1}^{n} \beta_{11m} p_{n-m,1} \right]. \tag{70}$$

From equation (55), $\mathfrak{g}(t) = t^{-1}$ and its nth derivative at t_1 is $\mathfrak{g}_1^{(n)} = (-1)^n n! t_1^{-n-1}$. Replacing g by $\mathfrak{g}$ in (35) leads to

$$\alpha_{1j} = (\pi)^{\frac{1}{2}} \left[\frac{-2}{h_1^{(2)}}\right]^{j+\frac{1}{2}} \sum_{n=0}^{j} (-1)^n t_1^{-2j+n-1} \sum_{m=0}^{n} b_{mn} (\tfrac{1}{2})_{m+j}$$

where $\arg [-2/h_1^{(2)}]^{\frac{1}{2}} = \pi/2$ and the b_{mn}'s are computed from (100) with

$$a_k = -2h_1^{(k+2)}/[(k+2)!\, h_1^{(2)}], \qquad k = 1, 2, \cdots . \tag{71}$$

Equation (54) shows that the β_{1lm}'s are the coefficients in the asymptotic expansion of $[V_l(x)]_1$, that is, in the asymptotic expansion of an integral, which has the same integrand as $V_l(x)$, taken along the path of steepest descent through the saddle point $v = v_1$. Instead of obtaining the β's by the general procedure outlined in Section VIII, we notice that the "asymptotic" series for $V_1(x)$ consists of only one term. Consequently β_{11m} is 0 when $m > 0$ and the summation in equation (70) for p_{n1} disappears. Moreover, the asymptotic series for the error function gives, when $v_1 > 0$,

$$V_0(x) \sim i(\pi)^{\frac{1}{2}} \exp(-xv_1^2) \sum_{m=0}^{\infty} (-1)^m (\tfrac{1}{2})_m (xv_1^2)^{-m-\frac{1}{2}}.$$

When $v_1 > 0$ and $x \to \infty$, $V_0(x)$ is given asymptotically by the contribution from v_1. When $v_1 < 0$, the asymptotic expression for $V_0(x)$ contains the constant term $2\pi i$, but the contribution from a path of steepest descent through v_1 is still given by the same expression as for positive v_1. Hence, irrespective of the sign of v_1,

$$\beta_{10m} = i(\pi)^{\frac{1}{2}} (-1)^m (\tfrac{1}{2})_m v_1^{-2m-1}.$$

These results enable us to write equation (70) (with j for n) as

$$p_{j1} = \frac{(-1)^j (\frac{1}{2})_j}{v_1^{2j+1}} \left\{ \left[\frac{v_1 t_1^{(1)}}{t_1}\right]^{2j+1} \sum_{n=0}^{2j} (-t_1)^n \sum_{m=0}^{n} b_{mn} (j + \tfrac{1}{2})_m - 1 \right\} \tag{72}$$

for $j \geq 0$. Here, to repeat,

$$v_1 = \frac{t_1}{|t_1|} (-h_1)^{\frac{1}{2}}, \quad t_1^{(1)} = \left[\frac{2}{h_1^{(2)}}\right]^{\frac{1}{2}}, \quad \frac{v_1 t_1^{(1)}}{t_1} = \frac{1}{|t_1|} \left[\frac{-2h_1}{h_1^{(2)}}\right]^{\frac{1}{2}},$$
$$(c)_0 = 1, \qquad (c)_n = c(c+1) \cdots (c+n-1) \tag{73}$$

and the b_{mn}'s are computed in succession from equation (100) with a_k given by (71).

The first two p_{n1}'s are

$$\begin{aligned} p_{01} &= \frac{1}{v_1}\left[\frac{v_1 t_1^{(1)}}{t_1} - 1\right] \\ p_{11} &= -\frac{1}{2v_1^3}\left\{\left[\frac{v_1 t_1^{(1)}}{t_1}\right]^3 [1 - \tfrac{3}{2}t_1 a_1 + t_1^2(\tfrac{3}{2}a_2 + \tfrac{15}{8}a_1^2)] - 1\right\} \end{aligned} \tag{74}$$

and the b_{mn}'s for p_{21} may be read from the table in Appendix D.

When $| t_1 |$ is small, the expressions which have been given for p_{n1} are essentially small differences between large numbers. If the calculation is being performed on a digital computer it may be advisable to use double precision. Expanding $h(t)$ about $t = t_1$ and then setting $t = 0$ leads to a series for $-h_1$ which may be used to obtain v_1^{-2i-1} as $[t_1^{(1)}/t_1]^{2i+1}$ times a power series in t_1 . Series of this type can be used to show that

$$\begin{aligned} p_{01} &= \frac{a_1}{2}\, t_1^{(1)} + O(t_1) \\ p_{11} &= -\tfrac{1}{4}t_1^{(1)3}(3a_3 + \tfrac{15}{2}a_1 a_2 + \tfrac{35}{8}a_1^3) + O(t_1) \end{aligned} \tag{75}$$

where a_k is given by (71).

X. THE NONCENTRAL χ^2 DISTRIBUTION

Let x be a positive integer and $y_1, y_2, \ldots, y_x$ be independent gaussian random variables with unit variances and respective mean values $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_x$. Let z be the (noncentral χ^2) random variable

$$z = \frac{1}{x}\sum_{n=1}^{x} y_n^2 \, . \tag{76}$$

It may be shown that the mean value of z is $\bar{z} = 1 + r$ and that its variance is $(2 + 4r)/x$ where

$$r = \frac{1}{x}\sum_{n=1}^{x} \bar{y}_n^2 \, .$$

Furthermore, from Ref. 2 (with a change of variable) the distribution function of z is

$$\text{Prob}\,[0 \leqq z \leqq s] = \frac{1}{2\pi i}\int_{\hat{c}-i\infty}^{\hat{c}+i\infty} t^{-1} \exp\,[xh(t)]\, dt \tag{77}$$

where $\hat{c} > 0$ and

$$h(t) = \tfrac{1}{2}[st - ln(1 + t) + r(1 + t)^{-1} - r]. \tag{78}$$

The integral on the right side of (77) is seen to be equivalent to minus Marcum's Q-function (and also to an expression given by R. A. Fisher) when (77) is written as

$$\text{Prob}\ [0 \leqq z \leqq s] = \frac{x}{2} \int_0^s (z/r)^{(x/4)-\frac{1}{2}} \exp\left[-x(z+r)/2\right] I_{(x/2)-1}[x(rz)^{\frac{1}{2}}]dz.$$

Here I denotes a Bessel function with imaginary argument.

We are interested in computing the distribution of z when x is large. The equation $h^{(1)}(t) = 0$ gives two saddle points. However, as pointed out in Ref. 2, when x is large only the one at

$$t_1 = -1 + \frac{1 + (1 + 4rs)^{\frac{1}{2}}}{2s}$$

need be considered. The value of t_1 is real and > -1. When $s = 1 + r$, t_1 is zero; and when s increases through $1 + r$, t_1 decreases through 0.

From equation (68) the desired asymptotic expansion is

$$\text{Prob}\ [0 \leqq z \leqq s] \sim \tfrac{1}{2}\{1 - \text{erf}\ [v_1 x^{\frac{1}{2}}]\} + \tfrac{1}{2}(\pi x)^{-\frac{1}{2}} \exp(-xv_1^2) \sum_{n=0}^{\infty} p_{n1} x^{-n} \tag{79}$$

where p_{n1} is given by (72). The quantities entering p_{n1} are

$$\begin{aligned} v_1 &= \frac{t_1}{|t_1|}(-h_1)^{\frac{1}{2}}, \qquad h_1 = h(t_1), \qquad t_1^{(1)} = [2/h_1^{(2)}]^{\frac{1}{2}} \\ h^{(n)}(t) &= \tfrac{1}{2}(-1)^n(n-1)!\,[nr(1+t)^{-n-1} + (1+t)^{-n}], \qquad n \geqq 2 \\ h_1^{(2)} &= r(1+t_1)^{-3} + 2^{-1}(1+t_1)^{-2} \\ a_k &= \frac{2(-1)^{k+1}}{(k+2)(1+t_1)^k}\left[\frac{(k+2)r + 1 + t_1}{2r + 1 + t_1}\right], \qquad k \geqq 1. \end{aligned} \tag{80}$$

The values of p_{01} and p_{11} may be obtained from (74) by substitution of the parameter values (80).

When $s - \bar{z} = s - 1 - r$ is small, the central limit theorem in the theory of probability states that

$$\text{Prob}\ [0 \leqq z \leqq s] \sim \frac{1}{2}\left\{1 + \text{erf}\left[\frac{(s - \bar{z})x^{\frac{1}{2}}}{(4 + 8r)^{\frac{1}{2}}}\right]\right\}.$$

This agrees with the approximation given by the error function term in (79) when it is noted that $t_1 \approx -(s - 1 - r)/(1 + 2r)$ and $-h_1 \approx t_1^2 h_1^{(2)}/2$ if $s - 1 - r$ is small.

The ordinary χ^2 distribution is obtained by setting $r = 0$ in the noncentral χ^2 distribution. In this case we have

$$\begin{aligned} t_1 &= \frac{1-s}{s}, \qquad t_1^{(1)} = 2/s, \\ v_1 &= \frac{1-s}{|1-s|}\left(\frac{s-1-\ln s}{2}\right)^{\frac{1}{2}}, \\ a_k &= -\frac{2(-s)^k}{k+2}, \qquad k \geqq 1. \end{aligned} \tag{81}$$

Equations (74) show that the first two coefficients p_{01}, p_{11} in the asymptotic series (79) are now

$$\begin{aligned} p_{01} &= \frac{2}{1-s} - \frac{1}{v_1}, \\ p_{11} &= -\frac{1}{2}\left\{\left(\frac{2}{1-s}\right)^3\left[s + \frac{(1-s)^2}{12}\right] - \frac{1}{v_1^3}\right\}. \end{aligned} \tag{82}$$

When s is close to its average value 1, equation (75) gives

$$\begin{aligned} p_{01} &= 2/3 + O(t_1) \\ p_{11} &= 1/135 + O(t_1). \end{aligned} \tag{83}$$

Setting $x = 2c$ and $r = 0$ gives

$$\begin{aligned} \text{Prob}\,[0 \leqq z \leqq s] &= \frac{1}{\Gamma(c)}\int_0^{cs} u^{c-1}\exp(-u)\,du \\ &= \sum_{n=c}^{\infty}\frac{(cs)^n\exp(-cs)}{n!} \end{aligned} \tag{84}$$

where c is assumed to be an integer (x even) in the last equation. These relations may be combined with the foregoing formulas to obtain asymptotic results for the incomplete gamma function and the Poisson distribution.

There is reason to believe that the asymptotic expansion (79) for Prob $[0 \leqq z \leqq s]$ may hold over the entire range $0 \leqq s \leqq \infty$. For example, consider the ordinary ($r = 0$) χ^2 distribution. In this case the first two terms in (79) give

$$\begin{aligned} \text{Prob}\,[0 \leqq z \leqq s] \sim{} & \tfrac{1}{2}\{1 - \text{erf}\,[v_1 x^{\frac{1}{2}}]\} \\ & + \tfrac{1}{2}[\pi x]^{-\frac{1}{2}}\left[\frac{2}{1-s} - \frac{1}{v_1}\right]\exp(-x v_1^2) \end{aligned} \tag{85}$$

where $x = 2c$ and v_1 is given by (81). Let c be held fixed at some large value and consider further the three cases $s \to 0$, $s \to \infty$, and $s \to 1$. In all three cases it may be verified that the leading terms in Prob $[0 \leqq z \leqq s]$ given by (85) agree with those obtained from the exact equations (84) and the asymptotic properties of the incomplete gamma function. The expressions obtained from (84) are

$$\text{Prob } [0 \leqq z \leqq s] = (2\pi c)^{-\frac{1}{2}} s^c \, [\exp c] \, [1 + O(cs) + O(c^{-1})], \qquad s \to 0$$

$$\text{Prob } [0 \leqq z \leqq s] = 1 - (2\pi c)^{-\frac{1}{2}} s^{c-1} [\exp (c - cs)][1 + O(s^{-1}) + O(c^{-1})], \qquad s \to \infty \tag{86}$$

$$\text{Prob } [0 \leqq z \leqq 1] = \tfrac{1}{2} + \tfrac{1}{3}(2\pi c)^{-\frac{1}{2}} + O(c^{-1}), \qquad s = 1.$$

APPENDIX A

The Behavior of $h(t)$ in the Critical Region

In this appendix we show that, in the critical region, $h(t)$ behaves much like a polynomial of degree $\mu + 1$, and we examine the change of variable from t to v.

First write

$$h(t) = \sum_{j=1}^{\mu+1} \frac{t^j h_0^{(j)}}{j!} + R_{\mu+1} \tag{87}$$

where $h_0^{(j)}$ is the value of $(d/dt)^j h(t)$ at $t = 0$, $h_0^{(\mu+1)}$ is not 0, and $R_{\mu+1}$ is $O(t^{\mu+2})$.

One of the distinguishing features of a polynomial in t of degree $\mu + 1$ is that when t is much larger than r, where $| t | = r$ is the smallest circle which encloses the zeros, the dominant term in the polynomial is the one containing $t^{\mu+1}$. The function $h(t)$ has a corresponding property. Suppose that the saddle points $t_1, t_2, \ldots, t_\mu$ all lie within a distance ϵ of the origin, and for the moment suppose that they may be moved towards the origin so that ϵ may be made as small as we desire. Also suppose that $h_0^{(\mu+1)} = A + O(\epsilon)$ where $A \neq 0$. We shall show that by making ϵ small enough we may find a range $\rho < | t | < \eta$ throughout which

$$h(t) \approx t^{\mu+1} h_0^{\mu+1} / (\mu + 1)! \, . \tag{88}$$

Here η is such that when $|t|$ and ϵ are less than η, the remainder $R_{\mu+1}$ in (87) is negligible in comparison with the $(\mu + 1)$ term $T_{\mu+1} = t^{\mu+1} h_0^{(\mu+1)} / (\mu + 1)!$. Once η is fixed, ρ may be chosen to be arbitrarily small, subject only to $\rho < \eta$.

In order to show that (88) holds when $\rho < | t | < \eta$, notice that by repeated differentiation of the representation

$$h'(t) = (t - t_1)(t - t_2) \ldots (t - t_\mu)\, G(t)$$

it may be shown that $G(0) = h_0^{(\mu+1)}/\mu! + O(\epsilon)$ and that $h_0^{(i)}$ is $O[\epsilon^{\mu+1-j} h_0^{(\mu+1)}]$ for $j = 1, 2, \cdots, \mu$. Hence when $|t| > \epsilon$,

$$\sum_{j=1}^{\mu} \frac{t^j h_0^{(j)}}{j!} = O[t^\mu\, \epsilon\, h_0^{(\mu+1)}] = O\left(\frac{\epsilon}{t} T_{\mu+1}\right).$$

Choosing ρ to be arbitrarily small, subject only to $0 < \rho < \eta$, and then choosing ϵ so that $\epsilon/\rho \ll 1$ establishes (88).

This property of $h(t)$ suggests that some insight into the change of variable from t to v, specified by $F(v) = h(t)$, may be obtained by considering $h(t)$ to be a polynomial $\varphi(t)$ of degree $\mu + 1$. Then a natural choice of $F(v)$ is $F(v) \equiv \varphi(cv + b)$ where c and b are constants. For simplicity we take $c = 1$ and $b = 0$ so that $v_r = t_r$, $r = 1, 2, \ldots, \mu$ where v_r is the rth saddle point on the v-plane. The equation $F(v) = h(t)$ goes into $\varphi(v) = \varphi(t)$ which we write as

$$\begin{aligned}\varphi(v) - \varphi(t) &= \sum_{j=1}^{\mu+1} A_j(v^j - t^j)\\ &= (v - t)[A_1 + A_2(v + t) + A_3(v^2 + vt + t^2) + \cdots]\\ &= 0.\end{aligned}$$

The branch used in the change of variable is

$$v = t \tag{89}$$

for which $dv/dt = 1$ everywhere. The remaining μ branches, which are ignored in the change of variable, may be obtained by solving

$$A_1 + A_2(v + t) + \ldots + A_{\mu+1}(v^\mu + v^{\mu-1}t + \ldots + t^\mu) = 0 \tag{90}$$

for v as a function of t. Writing (90) as

$$G(v, t) = \frac{\varphi(v) - \varphi(t)}{v - t} = 0$$

and expanding $G(v, t)$ about $v = t_r$, $t = t_r$ shows that, near $t = t_r$, one of the remaining branches behaves like $v = t_r - (t - t_r)$. On this branch $v = t_r$ and $dv/dt = -1$ at $t = t_r$. Again, let $v = \hat{v}_s$, $s = 1, 2, \ldots, \mu - 1$, be one of the $\mu - 1$ roots of $G(v, t_r) = 0$ which is not equal to t_r. Expanding $G(v, t)$ about $v = \hat{v}_s$, $t = t_r$ shows that near $t = t_r$ the corresponding branch behaves like

$$v = \hat{v}_s + \frac{(t - t_r)^2 \varphi''(t_r)}{2\varphi'(\hat{v}_s)}$$

and that $dv/dt = 0$ at $t = t_r$.

The examination of special cases suggests that there is a one-to-one correspondence between the μ branches of (90) and the μ saddle points in the sense that dv/dt for a particular branch is equal to -1 at its corresponding saddle point and is zero at the other saddle points. Thus it appears that the branch $v = t$ [or its analogue for general $h(t)$] is the only one suitable for the change of variable throughout the entire critical region.

APPENDIX B

The Choice of $F(v)$

The polynomial $F(v)$ used in changing the variable of integration will be written as

$$F(v) = \sum_{j=0}^{\mu+1} A_j v^j. \tag{91}$$

The positions $t_1, t_2, \ldots, t_\mu$ of the saddle points and the associated values $h_r = h(t_r)$ are supposed known. We require expressions for the A_j's which are either pure numbers or depend only on the h_r's. Although one or more of the zeros $v_1, v_2, \ldots, v_\mu$ of $F'(v) = dF(v)/dv$ may appear in our final expression for $F(v)$, they will always be expressed in terms of the h_r's.

Since $F(v) = h(t)$, we have the 2μ equations

$$\begin{aligned} F(v_r) &= h_r , \\ F'(v_r) &= 0, \qquad r = 1, 2, \cdots, \mu \end{aligned} \tag{92}$$

relating the $2\mu + 2$ unknowns $v_1, v_2, \ldots, v_\mu, A_0, A_1, \ldots, A_{\mu+1}$. Consequently we have at least two arbitrary choices ($A_{\mu+1} = 0$ is forbidden). For the case $\lambda \neq 1$ we shall always require the change of variable to be such that v is 0 when $t = 0$ and thus we take $A_0 = 0$. In some instances the form of $h(t)$ aids in the choice of the A_j's. For example, when $h(t)$ is an even function of t, we can take $F(v)$ to be an even function of v.

In choosing $F(v)$ it is helpful to notice that in the critical region the change of variable takes the form $t \approx cv$, c being a constant. Consequently, from (87),

$$F(v) \approx \sum_{j=1}^{\mu+1} \frac{(cv)^j h_0^{(j)}}{j!} \tag{93}$$

and it follows that the μ zeros, v_r, of $F'(v)$ have nearly the same configuration in the v-plane (except for a possible rotation and magnification given by arg c and $|c|$, respectively) as do the zeros, t_r, of $h'(t)$. For the case $\lambda = 1$ there may also be a small displacement so that $t \approx cv$ can be written more accurately as $t \approx cv + b$, or $dt/dv \approx c$. Furthermore, from (88), we can take c to be one of the roots of

$$A_{\mu+1} = c^{\mu+1} h_0^{(\mu+1)}/(\mu + 1)!\ .$$

The following examples illustrate possible choices of $F(v)$.

(*i*) $\mu = 1$, $\lambda \neq 1$, $t_1 \neq 0$. Initially there are 4 unknowns, v_1, A_0, A_1, A_2 related by 2 equations, and $F(v)$ given by

$$F(v) = A_2 v^2 + A_1 v + A_0.$$

We take $A_0 = 0$ (because $\lambda \neq 1$) and arbitrarily choose $A_2 = 1$ (for convenience). This carries the two equations into

$$v_1^2 + A_1 v_1 = h_1$$

$$2v_1 + A_1 = 0.$$

Consequently

$$F(v) = v^2 - 2v_1 v \qquad v_1^2 = -h_1.$$

This case has been considered by Bleistein.[3]

(*ii*) $\mu = 2$, $\lambda = 1$. Initially there are six unknowns v_1, v_2, A_0, A_1, A_2, A_3 related by four equations, and $F(v)$ given by

$$F(v) = A_3 v^3 + A_2 v^2 + A_1 v + A_0.$$

We take $A_2 = 0$ in order to simplify $F'(v)$. The four equations become

$$F'(v_r) = 3A_3 v_r^2 + A_1 = 0$$

$$3F(v_r) = -A_1 v_r + 3A_1 v_r + 3A_0 = 3h_r\ , \qquad r = 1, 2.$$

It follows that $v_2 = -v_1$, $A_0 = (h_1 + h_2)/2$, $A_1 v_1 = 3(h_1 - h_2)/4$.

For the remaining choice we take A_3 to be equal to $-A_1/3$ and obtain,

$$F(v) = \tfrac{1}{4}(h_2 - h_1)(v^3 - 3v) + \tfrac{1}{2}(h_2 + h_1).$$

Another choice for A_3 is $\frac{1}{3}$ which gives

$$F(v) = \tfrac{1}{3}v^3 - v_1^2 v + \tfrac{1}{2}(h_2 + h_1)$$

$$v_1^3 = \tfrac{3}{4}(h_2 - h_1), \qquad v_2 = -v_1\ .$$

This case has been considered by Chester, Friedman and Ursell.[6]

(*iii*) $\mu = 2$, $\lambda \neq 1$, $t_1 \neq 0$, $t_2 = 0$. Here the unknowns and $F(v)$ are the same as in example (*ii*), but because of $h_2 = h(0) = 0$ it turns out we have three arbitrary choices. Since $\lambda \neq 1$ we take $A_0 = 0$. Then both $F(v_2) = h_2$ and $F'(v_2) = 0$ are satisfied by choosing $v_2 = 0$ and $A_1 = 0$. This leaves $F(v_1) = h_1$ and $F'(v_1) = 0$ to be satisfied by the remaining three unknowns v_1, A_2, A_3. Our third choice is $A_3 = 2$. It leads to

$$F(v) = 2v^3 - 3v_1v^2, \qquad v_1^3 = -h_1 .$$

(*iv*) $\mu = 2$, $\lambda \neq 1$, t_1, t_2, $\neq 0$. This case illustrates the complication encountered for the general case when $\mu \geqq 2$. The value of A_0 must be 0 and we choose $A_3 = 1$. Then

$$F(v) = v^3 + A_2v^2 + A_1v$$

$$F'(v) = 3v^2 + 2A_2v + A_1 .$$

The last equation shows that $A_2 = -3(v_1 + v_2)/2$, $A_1 = 3v_1v_2$. Substituting in $F(v_r) = h_r$ gives

$$v_1^2(-v_1 + 3v_2)/2 = h_1$$

$$v_2^2(-v_2 + 3v_1)/2 = h_2 .$$

Setting $a = h_2/h_1$ and $\rho = v_2/v_1$ leads to

$$\rho^3 - 3\rho^2 + 3\rho a - a = 0$$

which has the three roots

$$\rho_n = 1 + (1 - a)^{\frac{1}{3}}[(1 - a^{\frac{1}{2}})^{\frac{1}{3}}\omega_n + (1 + a^{\frac{1}{2}})^{\frac{1}{3}}\omega_n^*]$$

where $\omega_1 = 1$, $\omega_2 = i^{4/3}$, $\omega_3 = i^{-4/3}$ and the star denotes "conjugate complex." When t_1 and t_2 tend to zero, one of the ρ_n's tends to t_2/t_1, and this is the value of ρ to be used. The value of v_1^3 is equal to $2h_1/(3\rho - 1)$, and we have

$$A_2 = -3v_1(1 + \rho)/2, \qquad A_1 = 3\rho v_1^2.$$

(*v*) $\mu = 3$. The general case of three saddle points may be handled by a procedure similar to that used in example (*iv*). We do not discuss this case beyond mentioning that when we set $v_2 = \rho v_1$, $v_3 = \sigma v_1$ the variable $u = (\rho-1)/(\sigma-1)$ must satisfy the equation

$$u^4 - 2u^3 + 2au - a = 0, \qquad a = (h_1 - h_2)/(h_1 - h_3).$$

(*vi*) $\mu = 3$, $t_3 = 0$, $h(t)$ even. Since $h(t)$ is even we start with

$$F(v) = v^4 + A_2v^2 + A_0$$

and find that

$$F(v) = v^4 - 2v_1^2 v^2, \qquad v_1^4 = -h_1.$$

This case has been treated by Felsen.[7]

APPENDIX C

Derivatives of Composite Functions

A result used in Section V to compute the nth derivative, $t_r^{(n)}$, of $t(v)$ at $v = v_r$ is stated in this appendix. Let the argument u in $h(u)$ be a function $t(v)$ of v. Then

$$\left(\frac{d}{dv}\right)^n h[t(v)] = \sum_{k=1}^{n} h^{(k)} c_{n,k}\,, \qquad n \geqq 1 \tag{94}$$

where $h^{(k)}$ stands for $(d/du)^k h(u)$ evaluated at $u = t(v)$, and the coefficients $c_{n,k}$ are computed from the recurrence relations

$$c_{n,1} = t^{(n)}, \tag{95}$$

$$c_{n+1,k+1} = \sum_{m=k}^{n} \binom{n}{m} t^{(n+1-m)} c_{m,k}\,, \qquad 1 \leqq k \leqq n \tag{96}$$

in which $t^{(n)}$ denotes $(d/dv)^n t(v)$ and $\binom{n}{m}$ the binomial coefficient. Equation (96) may be proved by induction. Differentiating (94) gives

$$c_{n+1,k+1} = t^{(1)} c_{n,k} + \frac{d}{dv} c_{n,k+1}\,, \qquad 1 \leqq k \leqq n - 1. \tag{97}$$

We assume that (96) holds when n is replaced by n-1 and use it to express $c_{n,k+1}$ as a sum. Then one of the terms in the summand for $d\,c_{n,k+1}/dv$ contains $d\,c_{m,k}/dv$. From (97), assuming $k > 1$,

$$\frac{d}{dv} c_{m,k} = c_{m+1,k} - t^{(1)} c_{m,k-1}\,.$$

Equation (96), with $(n-1,\ k-1)$ for $(n,\ k)$, lets us sum the terms containing $t^{(1)} c_{m,k-1}$ with respect to m. Equation (96) follows upon combining binomial coefficients and using $c_{k,k} = t^{(1)} c_{k-1,k-1}$.

The recurrence relations may also be obtained by writing the right side of (94) as a Bell polynomial and using the recurrence relation for these polynomials.[8] Expressions for the $c_{n,k}$'s (up to $n = 8$) as polynomials in the $t^{(n)}$'s may be obtained from Riordan's table of Bell polynomials given on page 49 of Ref. 8.

APPENDIX D

Formulas for Classical Saddle Point Asymptotic Expansions

A result useful in obtaining asymptotic expansions of integrals is

$$\int_0^T \tau^{\rho-1} G(\tau) \exp[xH(\tau)]\, d\tau$$

$$\sim \sum_{j=0}^{\infty} \frac{1}{\nu} \left(\frac{-1}{xH_\nu}\right)^{(\rho+j)/\nu} \sum_{n=0}^{j} G_{j-n} \sum_{m=0}^{n} b_{mn} \Gamma\left(m + \frac{\rho + j}{\nu}\right) \tag{98}$$

where $x \to \infty$, $Re\ \rho > 0$, ν is a positive integer, and

$$G(\tau) = \sum_{n=0}^{\infty} \tau^n G_n , \qquad H(\tau) = \sum_{n=\nu}^{\infty} \tau^n H_n . \tag{99}$$

The b_{mn}'s depend only on $H(\tau)$ and are computed in succession, starting with $b_{00} = 1$ and $b_{0n} = 0$ for $n \geq 1$, from

$$b_{m+1,n+1} = \frac{1}{n+1} \sum_{k=1}^{n-m+1} k a_k b_{m,n-k+1} . \tag{100}$$

Here $a_k = -H_{\nu+k}/H_\nu$, $k = 1, 2, \ldots$.

Special values of b_{mn} are given in Table I.

The asymptotic expansion (98) is based upon the gamma function integral

$$\int_0^\infty u^{z-1} \exp(-u^\nu)\, du = \frac{1}{\nu} \Gamma\left(\frac{z}{\nu}\right) \tag{101}$$

and the expansion

$$\exp\left[y \sum_1^\infty a_n \xi^n\right] = \sum_{n=0}^{\infty} \xi^n \sum_{m=0}^{n} b_{mn} y^m . \tag{102}$$

The recurrence relation (100) may be obtained by differentiating (102) with respect to ξ, replacing the exponential by its series, and then equating coefficients of $\xi^n y^{m+1}$.

TABLE I — SPECIAL VALUES OF b_{mn}

n	b_{0n}	b_{1n}	b_{2n}	b_{3n}	b_{4n}	b_{nn}
0	1					
1	0	a_1				
2	0	a_2	$a_1^2/2$			
3	0	a_3	$a_1 a_2$	$a_1^3/6$		
4	0	a_4	$a_1 a_3 + 2^{-1} a_2^2$	$2^{-1} a_1^2 a_2$	$a_1^4/24$	
$n > 1$	0	a_n	—	—	—	$a_1^n/n!$

For an integral, say I_r in equation (31), having a simple saddle point at $t = t_r$ and a path of integration L_r' which runs up to, through, and then down from t_r, we can use (98) with $\nu = 2$, $\rho = 1$, $\tau = t - t_r$, $G(\tau) = g(t_r + \tau)$, and $H(\tau) = h(t_r + \tau) - h(t_r)$. The contribution of the saddle point is obtained by deleting the terms in (98) for which j is odd, doubling the terms for j even, and taking arg $(-1/xH_2)^{\frac{1}{2}}$ = arg $[-2/xh_r^{(2)}]^{\frac{1}{2}}$ to be equal to arg $(t - t_r)$ on the part of L_r' just leaving t_r. The result is

$$\int_{L_r'} g(t) \exp [xh(t)]\, dt$$

$$\sim \exp [xh_r] \sum_{j=0}^{\infty} \left[\frac{-2}{xh_r^{(2)}}\right]^{j+\frac{1}{2}} \sum_{n=0}^{2j} \frac{g_r^{(2j-n)}}{(2j-n)!} \sum_{m=0}^{n} b_{mn}\Gamma(m + j + \tfrac{1}{2}) \quad (103)$$

where the derivatives of $g(t)$, $h(t)$ are defined in equation (19) and b_{mn} is computed with

$$a_k = -\frac{2h_r^{(k+2)}}{(k+2)!\, h_r^{(2)}}. \quad (104)$$

The gamma function $\Gamma(m + j + \frac{1}{2})$ may be written as $\sqrt{\pi}\, (\frac{1}{2})_{m+j}$.

For the integral J_0 given by equation (56) most of the contribution comes from the region near the branch point at $t = 0$. When $t = 0$ is not a saddle point, there is only one path of steepest descent {for exp $[xh(t)]$} leaving $t = 0$. This path may be taken to be the branch cut in the t-plane and the path of integration L_0' for J_0 may be taken to be a positive loop enclosing the cut. Then the asymptotic series for J_0 may be obtained from (98) by setting $\nu = 1$, $\rho = \lambda$, $\tau = t$, $G(\tau) = g(t)$, $H(\tau) = h(t)$ and using in place of (101) the integral

$$\int_{+\infty}^{(0+)} u^{z-1} \exp(-u)\, du = [1 - \exp(-i2\pi z)]\Gamma(z) = \frac{2\pi i \exp(-i\pi z)}{\Gamma(1-z)} \quad (105)$$

Here arg u is 0 on the part of the path of integration leaving $t = 0$. The positive real u-axis in (105) is a branch cut. The path of integration starts at $u = +\infty$ on the top side of the cut, comes in along the cut, encircles $u = 0$ in the positive direction, then runs out to $u = +\infty$ along the bottom side of the cut.

It is found that

$$J_0 = \int_{L_0'} t^{\lambda-1} g(t) \exp [xh(t)]\, dt$$
$$\sim \sum_{j=0}^{\infty} \left[\frac{-1}{xh_0^{(1)}}\right]^{\lambda+j} \sum_{n=0}^{j} \frac{g_0^{(j-n)}}{(j-n)!} \cdot \sum_{m=0}^{n} b_{mn}[1 - \exp(-2\pi i\lambda)]\Gamma(m + \lambda + j) \tag{106}$$

where arg $[-1/xh_0^{(1)}]$ is equal to arg t on the part of L_0' leaving $t = 0$, and b_{mn} is computed with

$$a_k = -\frac{h_0^{(k+1)}}{(k+1)!\, h_0^{(1)}}. \tag{107}$$

The last relation in (105) may be used to handle the case in which λ is 0 or a negative integer.

When $t = 0$ is a simple saddle point as well as a branch point, the path L_0' can be taken to coincide with the path of steepest descent through $t = 0$ except for an indentation at $t = 0$. The indentation is chosen so that a man walking in the positive direction along L_0' would have the point $t = 0$ on his left. We put $\nu = 2$, $\rho = \lambda$, $G(\tau) = g(t)$, $H(\tau) = h(t)$ in (98) and use in place of (101) the integral

$$\int_K u^{z-1} \exp(-u^2)\, du = \tfrac{1}{2}[1 - \exp(-i\pi z)]\Gamma\left(\frac{z}{2}\right) = \frac{i\pi \exp(-i\pi z/2)}{\Gamma\left(1 - \frac{z}{2}\right)} \tag{108}$$

Here K runs from $u = -\infty$ to $u = +\infty$ with a downward indentation at $u = 0$, and arg u is 0 on the part of K leaving $u = 0$. Instead of (106) we now have

$$\int_{L_0'} t^{\lambda-1} g(t) \exp [xh(t)]\, dt \sim \sum_{j=0}^{\infty} \left(\frac{-2}{xh_0^{(2)}}\right)^{(\lambda+j)/2} \sum_{n=0}^{j} \frac{g_0^{(j-n)}}{(j-n)!} \cdot \sum_{m=0}^{n} b_{mn} \tfrac{1}{2}\{1 - \exp[-i\pi(\lambda + j)]\}\Gamma\left(m + \frac{\lambda + j}{2}\right) \tag{109}$$

where arg $[-2/xh_0^{(2)}]^{\frac{1}{2}}$ is equal to arg t on the part of L_0' leaving $t = 0$ and b_{mn} is computed with

$$a_k = -\frac{2h_0^{(k+2)}}{(k+2)!\, h_0^{(2)}}. \tag{110}$$

APPENDIX E

Two Saddle Points

In order to illustrate some of the results of Sections IV, V, and VI, we consider the case of two saddle points, $\mu = 2$. This case has been discussed by Chester, Friedman, and Ursell.[4, 6] From (15) the desired expansion is of the form

$$\int_{L'} g(t) \exp [xh(t)] \, dt \sim U_0(x) \sum_{n=0}^{\infty} p_{n0} x^{-n} + U_1(x) \sum_{n=0}^{\infty} p_{n1} x^{-n}, \tag{111}$$
$$P_n(v) = p_{n0} + p_{n1} v$$

where, from example (*ii*) of Appendix B and equation (16),

$$U_l(x) = \int_L v^l \exp [xF(v)] \, dv, \qquad l = 0, 1$$
$$F(v) = \tfrac{1}{3}v^3 - v_1^2 v + A_0 , \tag{112}$$
$$A_0 = \tfrac{1}{2}(h_2 + h_1), \qquad v_1^3 = \tfrac{3}{4}(h_2 - h_1), \qquad v_2 = -v_1 .$$

Arg v_1 is determined by the correspondence of v_1 with t_1 which comes with the change of variable from t to v.

Let L' and the change of variable from t to v be such that L runs in from $v = \infty \exp(-i\pi/3)$ to the critical region near $v = 0$ and then out to $\infty \exp(i\pi/3)$ (it may be necessary to reverse the direction of L'). Then

$$U_0(x) = 2\pi i x^{-\frac{1}{3}} Ai(x^{\frac{2}{3}} v_1^2) \exp (xA_0) ,$$
$$U_1(x) = -2\pi i x^{-\frac{2}{3}} Ai'(x^{\frac{2}{3}} v_1^2) \exp (xA_0) \tag{113}$$

where $Ai(z)$ is the Airy function and $Ai'(z)$ its derivative with respect to z.

From $F'(v) = v^2 - v_1^2$ and equation (9) it follows that $Q(\zeta, v)$ is $\zeta + v$. Consequently equation (17) for $P_n(v)$ gives

$$P_n(v) = \frac{1}{2\pi i} \int_C d\zeta \, f(\zeta) \left[\frac{1}{\zeta^2 - v_1^2} \frac{\partial}{\partial \zeta} \right]^n \frac{\zeta + v}{\zeta^2 - v_1^2}. \tag{114}$$

Here $f(v) = g(t)t^{(1)}$ in which $t^{(1)} = dt/dv$ is obtained by differentiating $F(v) = h(t)$ with respect to v. The path of integration C encloses $\zeta = \pm v_1$ but no singularities of $f(\zeta)$.

From $v_2 = - v_1$ and

$$p_{00} + v_r p_{01} = P_0(v_r) = f(v_r), \qquad r = 1, 2$$

we get

$$p_{00} = \frac{f(v_1) + f(-v_1)}{2}, \qquad p_{01} = \frac{f(v_1) - f(-v_1)}{2v_1} \tag{115}$$

which may also be obtained from equation (21) for p_{0l}.

Putting $v = v_1$, $n = 1$ in (114) and expanding the integrand in partial fractions leads to

$$P_1(v_1) = \frac{1}{8v_1^3}\,[f(-v_1) - f(v_1) + 2v_1 f^{(1)}(v_1) - 2v_1^2 f^{(2)}(v_1)]. \tag{116}$$

Expressions for p_{10} and p_{11} which agree with equation (23) for p_{1l} may be obtained from (116) by changing the sign of v_1 to get $P_1(-v_1)$ and using $n = 1$ in

$$\begin{aligned} p_{n0} &= \frac{P_n(v_1) + P_n(-v_1)}{2} \\ p_{n1} &= \frac{P_n(v_1) - P_n(-v_1)}{2v_1}. \end{aligned} \tag{117}$$

If we were to continue with the Bleistein method we would have to evaluate $f^{(n)}(v_r)$ by using equations (26) through (30). Instead we turn to the problem of obtaining $P_n(v_r)$ by the Ursell method. For $\mu = 2$, equations (34) become

$$\begin{aligned} P_0(v_r) &= \alpha_{r0}/\beta_{r00}\,, \qquad r = 1, 2 \\ P_n(v_r) &= \frac{\alpha_{rn}}{\beta_{r00}} - \sum_{m=1}^{n} \frac{\beta_{r0m}p_{n-m,0} + \beta_{r1m}p_{n-m,1}}{\beta_{r00}} \end{aligned} \tag{118}$$

where α_{rj} and β_{rlj} are given by equations (35) and (37).

Since $g(t)$ and $h(t)$ in the original integral (111) are quite general, we use equation (35) for α_{rj} as it stands. However, equation (37) for β_{rlj} simplifies considerably. This is to be expected since it gives essentially the coefficients in the asymptotic expansions of $Ai(z)$ and $Ai'(z)$. From $F_r^{(2)} = 2v_r$, $F_r^{(3)} = 2$, and $F_r^{(n)} = 0$ for $n > 3$ we have $a_1 = -1/(3v_r)$, and $a_k = 0$ for $k > 1$. It turns out that b_{mn} is 0 for $m \neq n$ and $b_{nn} = a_1^n/n!$. When l is set equal to 0 in (37), all terms vanish except the one for $n = 2j$, $m = 2j$. When $l = 1$, all terms vanish except those for $n = 2j$, $m = 2j$ and $n = 2j - 1$, $m = 2j - 1$. It is found that

$$\begin{aligned} \beta_{r00} &= \left(\frac{-\pi}{v_r}\right)^{\frac{1}{2}}, \qquad \beta_{r0j} = \beta_{r00}\left(\frac{-1}{v_r}\right)^{3j} \frac{(\frac{1}{2})_{3j}}{9^j(2j)!} \\ \beta_{r1j} &= v_r\left(\frac{1 + 6j}{1 - 6j}\right)\beta_{r0j}\,. \end{aligned} \tag{119}$$

Changing $P_n(v_r)$ into $P_k(v_r)$ in order to avoid confusion with the n used in Appendix D carries (118) into

$$P_k(v_r) = t_r^{(1)}\left[\frac{-2}{h_r^{(2)}}\right]^k \sum_{n=0}^{2k} \frac{g_r^{(2k-n)}}{(2k-n)!} \sum_{m=0}^{n} b_{mn}(\tfrac{1}{2})_{m+k}$$
$$- \sum_{m=1}^{k} \frac{9^{-m}}{(2m)!}\left(\frac{-1}{v_r}\right)^{3m}(\tfrac{1}{2})_{3m}\left[p_{k-m,0} + v_r\left(\frac{1+6m}{1-6m}\right)p_{k-m,1}\right]. \quad (120)$$

Here $r = 1, 2$; $v_2 = -v_1$; and b_{mn} is computed from (100) with a_k given by equation (36). The value of $t_r^{(1)}$ is given by

$$t_r^{(1)} = \left(\frac{dt}{dv}\right)_{v=v_r} = \left\{\left[\frac{-2}{h_r^{(2)}}\right]^{\frac{1}{2}} \Big/ \left(\frac{-1}{v_r}\right)^{\frac{1}{2}}\right\} \quad (121)$$

where arg $t_r^{(1)}$ is calculated either from (i) the form $t \approx cv$ assumed by the change of variable in the critical region or from (ii) arg $[-2/h_r^{(2)}]^{\frac{1}{2}}$ and arg $(-1/v_r)^{\frac{1}{2}}$ being equal to arg $(t - t_r)$ and arg $(v - v_r)$, respectively, on the portions of the paths of steepest descent leaving t_r and v_r. The last summation in (120) is omitted when $k = 0$. The expression for $P_1(v_r)$ may be written with the help of equations (40).

Equation (120) was checked by using it to obtain the first few terms in the known[10] uniform asymptotic expansion for the Bessel function $H_x^{(1)}(xz)$ with $0 < z < 1$. Here $h(t) = z \sinh t - t$, the saddle points are at $\pm t_1$ ($t_1 > 0$) on the real axis, and the path of integration runs from $t = -\infty$ to $t = \infty + i\pi$. If the direction of the path of integration is reversed [so that (111) gives $-H_x^{(1)}(xz)$], the paths L' and L can be brought into correspondence by a rotation of 120°. In the approximate form $t \approx cv$ of the change of variable, arg $c = 2\pi/3$; and v_1 corresponding to t_1 is $v_1 = |-3h_1/2|^{\frac{1}{3}} \exp(-i2\pi/3)$. Furthermore, $f(v)$ turns out to be an even function, (114) gives $P_n(-v) = (-1)^n P_n(v)$, and $p_{2n,1}$, $p_{2n+1,0}$ are zero for $n = 0, 1, 2, \cdots$.

When t_1 and t_2 approach each other, $h_1^{(2)}$, $h_2^{(2)}$, v_1 and v_2 tend to zero. In this case the asymptotic behavior of the integral (111) may be determined with the help of the equation obtained by setting $\nu = 3$ in equation (98). However, if one is interested in the behavior of the coefficients p_{nl}, the following relations are useful. Putting $v_1 = 0$ in the integral (114) for $P_n(v)$ shows that in the limit

$$p_{00} = f(0), \qquad p_{01} = f'(0),$$
$$p_{10} = -f^{(3)}(0)/3!, \qquad p_{11} = -2f^{(4)}(0)/4! \quad (122)$$

and so on. The derivatives $t^{(n)}$ appearing in the derivatives of $f(v) = g(t)t^{(1)}$ are now obtained by differentiating $3^{-1}v^3 = h(t)$ repeatedly with respect to v. The leading coefficients are found to be

$$\begin{aligned} p_{00} &= g_1^{(0)} t_1^{(1)} & t_1^{(1)} &= [2/h_1^{(3)}]^{\frac{1}{3}} \\ p_{01} &= g_1^{(1)} t_1^{(1)2} + g_1^{(0)} t_1^{(2)} & t_1^{(2)} &= -h_1^{(4)} t_1^{(1)5}/12. \end{aligned} \quad (123)$$

APPENDIX F

Saddle Point Near Branch Point

Here we apply the theory of Sections VII and VIII to a case discussed by Bleistein,[3] namely, $\lambda \neq 1$ and $\mu = 1$. The paths L', L and the functions $h(t)$, $F(v)$ are assumed to be the same as in Section IX where the singularity at the origin was a simple pole instead of a branch point. In the critical region L' is parallel to, and to the right of, the imaginary t-axis. Only the case in which t_1 and v_1 are real and of the same sign will be considered. When t_1 and v_1 are positive the cut associated with the branch point is assumed to start out from the origin along the positive real axis, and then quickly bend downward to run out to $-i\infty$. When t_1 and v_1 are negative, the cut starts out along the negative axis and then bends downward to $-i\infty$.

Equation (46) and $F(v) = v^2 - 2v_1 v$ lead to

$$J \sim V_0(x) \sum_{n=0}^{\infty} p_{n0} x^{-n} + V_1(x) \sum_{n=0}^{\infty} p_{n1} x^{-n} \quad (124)$$

where, with $c > 0$ and the path of integration lying to the right of the cut,

$$V_0(x) = \int_{c-i\infty}^{c+i\infty} v^{\lambda-1} \exp [xF(v)]\, dv = i\pi x^{-\lambda/2} \sum_{n=0}^{\infty} \frac{(-2v_1 x^{\frac{1}{2}})^n}{n!\, \Gamma\left(1 - \dfrac{\lambda+n}{2}\right)}.$$

Replacing λ by $\lambda + 1$ gives $V_1(x)$. $V_0(x)$ and $V_1(x)$ are parabolic cylinder functions (Bleistein,[3] and pair No. 740.2 in Campbell and Foster Table[9]).

$$\begin{aligned} P_n(v) &= p_{n0} + p_{n1} v \\ &= \frac{1}{2\pi i} \int_C \frac{d\zeta\, f(\zeta) \zeta^{\lambda-1}}{2^n} \left[\frac{1}{\zeta - v_1} \frac{\partial}{\partial \zeta}\right]^n \frac{(\zeta + v - v_1)\zeta^{-\lambda}}{\zeta - v_1} \end{aligned} \quad (125)$$

where C encloses $\zeta = 0$ and $\zeta = v_1$. Setting $n = 0$ and using $f(v) = g(t)\ (t/v)^{\lambda-1} t^{(1)}$ leads to

$$\begin{aligned} P_0(0) &= p_{00} = f(0) = g_0^{(0)} t_0^{(1)\lambda}, & t_0^{(1)} &= -2v_1/h_0^{(1)} \\ P_0(v_1) &= f(v_1), & t_1^{(1)} &= [2/h_1^{(2)}]^{\frac{1}{2}}. \end{aligned}$$

Setting $n = 1$ gives

$$P_1(0) = \frac{\lambda}{2v_1^2}[f(0) - f(v_1) + v_1 f^{(1)}(0)]$$

$$P_1(v_1) = \frac{1-\lambda}{2v_1^2}[f(0) - f(v_1) + v_1 f^{(1)}(v_1)] - \frac{f^{(2)}(v_1)}{4}.$$

The values of p_{n0} and p_{n1} may be obtained by following the steps outlined in the first part of Section VIII. Equation (52) and (53) become

$$P_n(v_1) = \frac{1}{\beta_{100}}\left[\alpha_{1n} - \sum_{m=1}^{n}(\beta_{10m}p_{n-m,0} + \beta_{11m}p_{n-m,1})\right]$$
$$p_{n0} = \frac{1}{\beta_{000}}\left[\alpha_{0n} - \sum_{q=1}^{n}(\beta_{00q}p_{n-q,0} + \beta_{0,1,q-1}p_{n-q,1})\right]. \tag{126}$$

The path L_1 in (54) is parallel to the imaginary axis and passes through the saddle point at $v = v_1$. The path L_0 in (56) runs up along the right side of the cut, encircles the origin in a positive direction, and then runs down to $v = -i\infty$ along the left side of the cut.

To get β_{1lj} we replace l on the right side of (37) by $l + \lambda - 1$ and notice that a_k given by (38) is 0 for the values of k used in computing b_{mn}. Consequently, the only nonvanishing b_{mn} is $b_{oo} = 1$ and (37) gives

$$\beta_{100} = i(\pi)^{\frac{1}{2}}v_1^{\lambda-1}, \qquad \frac{\beta_{1lj}}{\beta_{100}} = \frac{(-4)^{-j}}{j!}(-l-\lambda+1)_{2j}v_1^{l-2j}.$$

The expression for α_{1n} obtained by replacing $g_1^{(2j-n)}$ in (35) by $\hat{g}_1^{(2j-n)}$, where $\hat{g}(t) = t^{\lambda-1}g(t)$, contains $[-2/h_1^{(2)}]^{\frac{1}{2}}$ which may be written as $it_1^{(1)}$.

In equation (57) for α_{0j}/β_{000} we replace b_{mn} by $\hat{b}_{mn}$ to indicate that $\hat{b}_{mn}$ is computed with a_k given by (58) instead of the a_k used in computing α_{1n}/β_{100}. A still different a_k, given by (60), is used to compute β_{0lj}/β_{000}. From equation (60) all of the a_k's used to compute β_{0lj}/β_{000} are zero except $a_1 = 1/(2v_1)$. Therefore b_{mn} is zero unless $m = n$, and it follows from (59) that

$$\beta_{0lj}/\beta_{000} = (\lambda)_{l+2j}(2v_1)^{-l-2j}/j!.$$

When all of these results are used in the expression (126) for p_{no} we get, with j for n,

$$p_{j0} = (2v_1)^{-j}t_0^{(1)(\lambda+j)}\sum_{n=0}^{j}\frac{g_0^{(j-n)}}{(j-n)!}\sum_{m=0}^{n}\hat{b}_{m,n}(\lambda)_{m+j}$$
$$-\sum_{s=1}^{j}\frac{(\lambda)_{2s-1}}{s!\,(2v_1)^{2s}}[(\lambda+2s-1)p_{j-s,0} + 2v_1 s p_{j-s,1}] \tag{127}$$

where $h(0) = h_0 = 0$, $h_1 < 0$, $h_1^{(2)} > 0$, $v_1 = t_1 \mid t_1 \mid^{-1}(-h_1)^{\frac{1}{2}}$, $t_0^{(1)} = -2v_1/h_0^{(1)}$, and $\hat{b}_{mn}$ is computed from (100) with a_k replaced by $\hat{a}_k$ where

$$\hat{a}_k = -h_0^{(k+1)}/[(k+1)!\, h_0^{(1)}], \qquad k \geqq 1.$$

When $j = 0$ the summation with respect to s is omitted. Similarly, $P_n(v_1)$ gives

$$p_{j1} = (-1)^j v_1^{-\lambda} t_1^{(1)(2j+1)} \sum_{n=0}^{2j} \frac{\hat{g}_1^{(2j-n)}}{(2j-n)!} \sum_{m=0}^{n} b_{mn}(\tfrac{1}{2})_{m+j} - \frac{p_{j0}}{v_1}$$
$$- \sum_{s=1}^{j} \frac{(-1)^s(1-\lambda)_{2s-1}}{s!\,(2v_1)^{2s}} \left[\frac{-\lambda+2s}{v_1} p_{j-s,0} - \lambda p_{j-s,1}\right] \tag{128}$$

where $t_1^{(1)} = [2/h_1^{(2)}]^{\frac{1}{2}}$, $\hat{g}(t) = t^{\lambda-1}g(t)$, and b_{mn} is computed from (100) with

$$a_k = -2h_1^{(k+2)}/[(k+2)!\, h_1^{(2)}].$$

It is interesting to notice that when $\lambda = 0$, (127) and (128) give values of p_{j0} and p_{j1} which agree with those obtained in Section IX.

When the saddle point approaches the branch point, t_1, h_1, $h_0^{(1)}$, and v_1 tend to zero. In this case the integral J may be evaluated with the help of equation (109). The behavior of the coefficient p_{nl} may be studied by putting $v_1 = 0$ in the integral (125) for $P_n(v)$. It is found that

$$p_{00} = f(0), \qquad p_{01} = f^{(1)}(0),$$
$$p_{10} = -\lambda f^{(2)}(0)/2, \qquad p_{11} = -(\lambda+1)f^{(3)}(0)/12 \tag{129}$$

and so on. The derivatives $t^{(n)}$ appearing in the derivatives of $f(v) = g(t)\ (t/v)^{\lambda-1}t^{(1)}$ are now obtained by differentiating $v^2 = h(t)$ repeatedly with respect to v.

For $n = 1$ and 2,

$$t_0^{(1)} = [2/h_0^{(2)}]^{\frac{1}{2}}, \qquad t_0^{(2)} = -h_0^{(3)} t_0^{(1)4}/6. \tag{130}$$

Substituting these values in equations (51) for $f(0)$ and $f^{(1)}(0)$ and using (129) gives the limiting expressions for p_{00} and p_{01}.

APPENDIX G

Poisson-Charlier Polynomial

In this appendix equations (127) and (128) for p_{j0} and p_{j1} are used to obtain an asymptotic series for the Poisson-Charlier polynomial $c_n(y, a)$ when y is $O(1)$ and both n and a are large and positive.

Multiplying the generating function

$$(1 - u)^y \exp (au) = \sum_{m=0}^{\infty} \frac{a^m u^m}{m!} c_m(y, a) \tag{131}$$

by u^{-n-1} and integrating u around a small circle enclosing $u = 0$ gives a contour integral for $c_n(y, a)$. Instead of $c_n(y, a)$, we find it more convenient to deal with the polynomial (in y)

$$d_n(y, a) = \frac{a^n \exp(-a)}{n!} c_n(y, a). \tag{132}$$

Setting $x = n + 1$ and making the change of variable $t = 1 - u$ in the integral for $c_n(y, a)$ leads to

$$\begin{aligned} d_n(y, a) &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{t^y \exp(-at)}{(1-t)^x} \, dt, \qquad x > y + 1 \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} t^y \exp[xh(t)] \, dt \\ &= \frac{J}{2\pi i} = \hat{J}, \\ h(t) &= -rt - \ln(1 - t) \end{aligned} \tag{133}$$

where $r = a/x$. The ratio r is positive.

We wish to use equations (127) and (128) to compute p_{n0}, p_{n1} in the expansion obtained by dividing (124) by $2\pi i$, namely

$$d_n(y, a) = \hat{J} \sim \hat{V}_0(x) \sum_{n=0}^{\infty} p_{n0} x^{-n} + \hat{V}_1(x) \sum_{n=0}^{\infty} p_{n1} x^{-n}. \tag{134}$$

Here $\hat{V}_0(x) = V_0(x)/2\pi i$ with $\lambda = y + 1$. The function $\hat{V}_1(x)$ is obtained from $\hat{V}_0(x)$ by increasing y by 1.

We have

$$\begin{aligned} \hat{V}_0(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} v^y \exp[x(v^2 - 2v_1 v)] \, dv, \qquad c > 0 \\ &= x^{-(y+1)/2} G[y, v_1 x^{\frac{1}{2}}] \end{aligned} \tag{135}$$

where arg v is 0 at $v = c$ and $G(y, z)$ is the parabolic cylinder function

$$\begin{aligned} G(y, z) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^y \exp(u^2 - 2zu) \, du \\ &= 2^{-1} \sum_{n=0}^{\infty} \frac{(-2z)^n}{n! \, \Gamma\left(\frac{1-y-n}{2}\right)}. \end{aligned} \tag{136}$$

The function G is related to the function U discussed and tabulated in chapter 19 of Ref. 10 by the equation

$$G(y, z) = \frac{2^{-y/2}}{2(\pi)^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2}z^2\right) U[-y - \tfrac{1}{2}, z2^{\frac{1}{2}}].$$

The saddle point $t = t_1$ is obtained by setting the derivative $h^{(1)}(t) = -r + (1 - t)^{-1}$ to zero; and the saddle point $v = v_1$ is given by the relation $v_1^2 = -h_1$ together with the condition that v_1 and t_1 be of the same sign:

$$t_1 = 1 - \frac{1}{r}, \qquad v_1 = \frac{t_1}{|t_1|}(r - 1 - \ln r)^{\frac{1}{2}}. \tag{137}$$

The values of the derivatives $t^{(1)} = dt/dv$ at $v = 0$ and $v = v_1$ are

$$t_0^{(1)} = -2v_1/h_0^{(1)} = \frac{2v_1}{r - 1}$$

$$t_1^{(1)} = [2/h_1^{(2)}]^{\frac{1}{2}} = \frac{2^{\frac{1}{2}}}{r}.$$

For $k > 1$ the kth derivative $h^{(k)}(t)$ is $(k - 1)!(1 - t)^{-k}$ and the coefficients used to compute $\hat{b}_{m,n}$, $b_{m,n}$ from (100) are

$$\hat{a}_k = -k!/[(k + 1)!\,(1 - r)] = 1/[(k + 1)(r - 1)]$$

and

$$a_k = -2(k + 1)!\,r^{k+2}/[(k + 2)!\,r^2] = -2r^k/(k + 2),$$

respectively.

Comparison of the integral (133) for $d_n(y, a)$ with the integral (1) for J shows that $g(t) = 1$ and $\lambda = y + 1$. Consequently $\mathfrak{g}(t) = t^{\lambda-1}g(t)$ becomes $\mathfrak{g}(t) = t^y$. For $k > 0$ the derivatives are $g^{(k)} = 0$ and $\mathfrak{g}^{(k)} = y(y - 1) \cdots (y - k + 1)t^{y-k}$.

Setting $j = 0$ in (127) and (128), and using the results just obtained gives

$$p_{00} = \left(\frac{2v_1}{r - 1}\right)^{y+1}, \qquad p_{01} = \left(\frac{t_1}{v_1}\right)^y \frac{(2)^{\frac{1}{2}}}{rv_1} - \frac{p_{00}}{v_1} \tag{138}$$

for the leading coefficients in the asymptotic expansion for $d_n(y, a)$. Here t_1 and v_1 are given by (137). The next two coefficients, obtained by setting $j = 1$, reduce to

$$\begin{aligned} p_{10} &= (y + 1)(y + 2)p_{00}\left[\frac{1}{2(r - 1)^2} - \frac{1}{4v_1^2}\right] - \frac{(y + 1)p_{01}}{2v_1} \\ p_{11} &= -\left(\frac{t_1}{v_1}\right)^{y+1}\left[\frac{(2)^{\frac{1}{2}}}{r - 1}\right]^3\left[\frac{y(y - 1)}{4} - \frac{yt_1r}{2} + \frac{t_1^2r^2}{24}\right]. \end{aligned} \tag{139}$$

Some idea of the behavior of the series (134) for $d_n(y, a)$ may be gained from Table II. Equations (127) and (128) were programmed for calculation on a high speed digital computer. The table lists results for the typical case $x = 30$, $a = 25$, and $y = -5$. Here $\text{Term}_{2n} = \hat{V}_0(x)p_{n0}x^{-n}$, $\text{Term}_{2n+1} = \hat{V}_1(x)p_{n1}x^{-n}$, and $S_m = t_0 + t_1 + \cdots + t_m$. The "exact" value, 381.02, was calculated by using the recurrence relation for the Poisson-Charlier polynomials.

No study was made to decide whether the relatively large value of Term_7 results from accumulated round-off error (an accuracy of 1 part in 10^7 was used) or whether the asymptotic series actually starts its divergence around $m = 5$ or 6.

When r is near unity, v_1 and t_1 are small and the individual terms in the expressions (127) and (128) for p_{j0}, p_{j1}, become large. In this case considerable cancellation occurs, and a high degree of precision in the calculations is required to obtain accurate values of p_{j0} and p_{j1}.

An asymptotic series (nonuniform) which is useful when $r - 1$ is small may be obtained by a variation of the classical method which is sometimes used in cases of this sort. Instead of using an expansion about both $t = 0$ and $t = t_1$, which is done (in effect) in obtaining the uniform asymptotic expansion, an expansion is made only about $t = 0$. Thus, the exponent $xh(t)$ may be written as

$$xh(t) = [-x(r - 1)t + xt^2/2] + (xt^2)(t/3 + t^2/4 + \cdots).$$

Changing the variable of integration from t to $u = t(x/2)^{1/2}$ and assuming that $r - 1$ is so small that $z = (r-1)(x/2)^{1/2}$ is $O(1)$ gives

$$\begin{aligned} \exp[xh(t)] &= \exp[-2zu + u^2] \exp[u^2(2t/3 + 2t^2/4 + \cdots)] \\ &= \exp[-2zu + u^2] \sum_{n=0}^{\infty} (2/x)^{n/2} \sum_{m=0}^{n} b_{mn}u^{2m+n}. \end{aligned} \tag{140}$$

The last series is obtained from equation (102) with $\xi = t = u(2/x)^{1/2}$,

TABLE II — PARTIAL SUMS FOR $d_{29}(-5, 25)$

m	Term_m	S_m	m	Term_m	S_m
0	276.62	276.62	4	0.015	381.03
1	82.82	359.44	5	−0.008	381.02
2	20.19	379.63	6	0.008	381.03
3	1.39	381.01	7	−0.144	380.88

Exact $d_{29}(-5, 25) = 381.02$

$y = u^2$, and $a_n = 2/(n+2)$. The coefficient b_{mn} is computed from the a_n's and **(100)**.

Substituting (140) in the integral (133) for $d_n(y, a)$ leads to

$$d_n(y, a) \sim \sum_{n=0}^{\infty} (2/x)^{(n+y+1)/2} \sum_{m=0}^{n} b_{mn} K_{2m+n} \tag{141}$$

where

$$K_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{n+y} \exp (u^2 - 2zu\, du)$$
$$= G(n + y, z) = G[n + y, (r - 1)(x/2)^{\frac{1}{2}}].$$

The recurrence relation

$$K_{n+1} = zK_n - \frac{n + y}{2} K_{n-1} \tag{142}$$

permits K_{2m+n} in (141) to be expressed as a linear function of K_0 and K_1. Equation (142) is obtained by integrating the derivative

$$\frac{d}{du} u^{n+y} \exp (u^2 - 2zu) = [(n + y)u^{-1} + 2u - 2z]u^{n+y} \exp (u^2 - 2zu)$$

As $r \to 1$ the leading term, $(2/x)^{(y+1)/2} K_0$, in (141) tends to the leading term $\hat{V}_0(x)p_{00}$ in the uniform asymptotic series (134). Although (141) is much simpler than (134), it does not hold for nearly as wide a range of values of $r - 1$.

APPENDIX H

Saddle Point at Origin

Here we are concerned with the leading term when $\lambda \neq 1$ and there are two saddle points, one at $t = 0$ and the other at $t = t_1$. We assume that t_1 is real and positive, and that $h(t)$ is real on the real axis. Furthermore, we assume $h_1 < 0$, $h_0^{(2)} < 0$, $h_1^{(2)} > 0$ so that the saddle point at t_1 is lower than the one at 0, and the paths of steepest descent at 0 and t_1 are parallel to the real and imaginary axes, respectively. A cut extends from 0 to $-\infty$ along the negative real axis.

The paths of integration L' and L are taken to run in from $\infty \exp(-i\pi/3)$, cross the positive real axis in the critical region, and the run out to $\infty \exp(i\pi/3)$. Example (*iii*) of Appendix B leads us to choose

$$F(v) = 2v^3 - 3v_1 v^2, \qquad v_1^3 = -h_1 \tag{143}$$

with arg $v_1 = 0$. From equation (46), the uniform asymptotic expansion is of the form

$$J \sim V_0(x) \sum_{n=0}^{\infty} p_{n0} x^{-n} + V_1(x) \sum_{n=0}^{\infty} p_{n1} x^{-n} + V_2(x) \sum_{n=0}^{\infty} p_{n2} x^{-n} \tag{144}$$

where

$$V_l(x) = \int_L v^{l+\lambda-1} \exp [xF(v)]\, dv, \qquad l = 0, 1, 2.$$

Expanding $\exp(-3xv_1v^2)$ and integrating termwise with the help of

$$\int_L u^\rho \exp (u^3)\, du = 2\pi i \Big/ \left[3\Gamma\left(\frac{2-\rho}{3}\right) \right]$$

gives

$$V_0(x) = \frac{2\pi i}{3} (2x)^{-\lambda/3} \sum_{n=0}^{\infty} \frac{(-3v_1)^n (x/4)^{n/3}}{n!\, \Gamma\left(1 - \dfrac{\lambda + 2n}{3}\right)} \tag{145}$$

from which $V_l(x)$ may be obtained by replacing λ by $\lambda + l$. When $\lambda = 1$, $V_0(x)$ reduces to the product of an Airy function and an exponential.

Setting $n = 0$ in the integral (47) for $P_n(v)$ gives

$$\begin{aligned} P_0(v) &= p_{00} + p_{01} v + p_{02} v^2 \\ &= \frac{1}{2\pi i} \int_C f(\zeta) \frac{[\zeta^2 + \zeta(v - v_1) + (v^2 - v_1 v)]}{\zeta^2(\zeta - v_1)}\, d\zeta \\ &= f(0) + f'(0)v + [f(v_1) - f(0) - v_1 f'(0)] v^2 v_1^{-2}. \end{aligned} \tag{146}$$

The values of $t_0^{(1)}$, $t_0^{(2)}$, $t_1^{(1)}$ appearing in $f(0)$, $f'(0)$, $f(v_1)$ are

$$t_0^{(1)} = \left[\frac{-6v_1}{h_0^{(2)}} \right]^{\frac{1}{2}}, \qquad t_0^{(2)} = \frac{12 - h_0^{(3)} t_0^{(1)3}}{3 h_0^{(2)} t_0^{(1)}}$$
$$t_1^{(1)} = \left[\frac{6v_1}{h_1^{(2)}} \right]^{\frac{1}{2}}. \tag{147}$$

The derivatives $t_0^{(1)}$, $t_1^{(1)}$ are positive and nearly equal when the saddle points are close together.

When $n = 0$, the Ursell equations (61), (62), and (63) become

$$\begin{aligned} P_0(v_1) &= \alpha_{10}/\beta_{100} \\ p_{00} &= \alpha_{00}/\beta_{000} \\ p_{01} &= \frac{\alpha_{01}}{\beta_{010}} - \frac{\beta_{001}}{\beta_{010}} p_{00}\,. \end{aligned} \tag{148}$$

The values of α_{10}, β_{100} otbained from the leading terms in the asymptotic series (54) defining the α_{rn}'s and β_{rlm}'s give

$$P_0(v_1) = g_1^{(0)}(t_1/v_1)^{\lambda-1}t_1^{(1)}. \tag{149}$$

Similarly, comparing the asymptotic series (64) defining the α_{0j}'s and β_{0lm}'s with the series (109) leads to expressions which give

$$\begin{aligned} p_{00} &= g_0^{(0)} t_0^{(1)\lambda}, \\ p_{01} &= \left\{ g_0^{(1)} - (\lambda + 1) g_0^{(0)} \left[\frac{1}{3v_1 t_0^{(1)}} + \frac{h_0^{(3)}}{6h_0^{(2)}} \right] \right\} t_0^{(1)(\lambda+1)}. \end{aligned} \tag{150}$$

The remaining coefficient, p_{02}, in $P_0(v)$ may now be obtained by combining (149) and (150). The coefficients p_{0l} give the leading part of the desired expansion (144) for J.

ACKNOWLEDGMENTS

I wish to express my appreciation for the computations made by Mr. G. H. Robertson to check my first approximations against his algorithm. I am also indebted to him for values which enabled me to check the expansion for the noncentral χ^2 distribution given in Section X. In addition, it gives me pleasure to acknowledge helpful comments made by L. A. Shepp and several other colleagues.

REFERENCES

1. Robertson, G. H., "Computation of the Noncentral Chi-Square Distribution," scheduled to be published in B.S.T.J., *48*, No. 1 (January 1969).
2. Rice, S. O., "Communication in the Presence of Noise—Probability of Error for Two Encoding Schemes," B.S.T.J., *29*, No. 1 (January 1950), pp. 60–93.
3. Bleistein, N., "Uniform Asymptotic Expansions of Integrals with Stationary Point Near Algebraic Singularity," Comm. Pure and Applied Math. *19*, No. 4 (1966), pp. 353–370.
4. Ursell, F., "Integrals with a Large Parameter. The Continuation of Uniformly Asymptotic Expansions," Proc. Cambridge Phil. Soc., *61* (1965), pp. 113–128.
5. Olver, F. W. J., "The Asymptotic Expansions of Bessel Functions of Large Order," Phil. Trans. Royal Soc., Series A, *247* (1954), pp. 328–368.
6. Chester, C., Friedman, B., and Ursell, F., "An Extension of the Method of Steepest Descents," Proc. Cambridge Phil. Soc., *53* (1957), pp. 599–611.
7. Felsen, L. B., "Radiation from a Uniaxially Anisotropic Plasma Half-Space," IEEE Trans. Antennas and Propagation, *AP-11* (1963), pp. 469–484.
8. Riordan, J., *An Introduction to Combinatorial Analysis,* New York: John Wiley and Sons, 1958.
9. Campbell, G. A., and Foster, R. M., *Fourier Integrals for Practical Applications,* New York: Van Nostrand, 1948.
10. Abramowitz, M., and Stegun, I. A., *Handbook of Mathematical Functions,* Washington, D. C.: National Bureau of Standards, 1964, equation (9.3.37).