Gain Control for Diversity Receivers

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Previous work on optimum gain control is extended to an important class of diversity receivers used for digital data transmission through fading media and for radar. As in the single diversity case the optimum gain (which yields minimum average cost of receiver saturation) is extremely insensitive to relative costs of saturation at the upper and lower dynamic range bounds. The sensitivity to relative cost decreases as the order of diversity increases.

Optimum gain and performance characteristics are given from which dynamic range requirements for diversity receivers can be deduced.

I. INTRODUCTION

A good part of detection theory literature deals with the determination of statistically optimum or near optimum receiver structures. However, in any practical implementation of these receivers the signal processing must be performed by components of finite dynamic range. To effectively use the amplitude range of a signal processing chain it is common to scale the received signal by adjusting the receiver gain. Optimum gain settings for minimum average cost of excluding (from a receiver's finite dynamic range) the envelope of a narrowband signal plus gaussian noise were presented last year.[1] Here similar results are presented for an important class of diversity receivers used for comunications through fading media and for radar. For the single diversity ($M = 1$) case, these results reduce to those given previously.

II. PREVIOUS RESULTS

Consider the problem of determining the normalized attenuation, a to optimally scale a positive homogeneous functional, ξ, of the received signal so that the average cost, l, of excluding ξ from the receiver's dynamic range is minimized. It follows from Ref. 1 that the average exclusion cost is given by

$$l = \int_0^a p_\omega(\xi)\, d\xi + \nu \int_{ad}^\infty p_\omega(\xi)\, d\xi \tag{1}$$

in which d denotes the dynamic range of the receiver (such that $D(\text{dB}) = 20 \log_{10} d$, $d > 1$), ν is the ratio of cost of saturation at the upper dynamic range bound to the cost of saturation at the lower, ω is a vector parameter determined by signal noise and channel conditions, and p_ω is the probability density function of ξ. When the optimizing value of a is a stationary point of l it can be found as a real positive solution to

$$p_\omega(a) = \nu\, dp_\omega(ad). \tag{2}$$

If a_s is the optimum a then the minimum average exclusion cost is

$$l = P_\omega(a_s) + \nu[1 - P_\omega(a_s d)] \tag{3}$$

in which P_ω is the cumulative distribution corresponding to p_ω. For $\nu = 1$, l becomes the exclusion probability. Ref. 1 considered this problem in detail for the case in which ξ represents the envelope of a narrowband signal plus gaussian noise received through a Rician fading medium. The results are based upon the solution of (2) for the case in which p_ω is the Rician[2] probability density function defined by

$$p_\gamma(\xi) = \xi \exp[-(\xi^2 + \gamma^2)/2] I_0(\gamma\xi) \qquad \gamma, \xi \geqq 0 \tag{4}$$

where γ is a suitably defined signal-to-noise ratio. ($\gamma_{\text{dB}} = 20 \log_{10}\gamma$).

III. GAIN SETTINGS FOR DIVERSITY RECEIVERS

In various diversity receivers formation of the test statistic leads to the generalized Rician probability density function given by

$$p_\omega(R) = R(R(M)^{\frac{1}{2}}/\gamma)^{M-1} \exp[-(R^2 + \gamma^2/M)/2] I_{M-1}[\gamma R/(M)^{\frac{1}{2}}] \tag{5}$$

$$R, \gamma, \geqq 0$$

$$M = 1, 2, \cdots$$

where I_K denotes the modified Bessel function of the first kind and order K and ω is the vector (γ, M). Such is the case for example in square-law combining M-fold diversity receivers for noncoherent frequency shift keyed signaling through Rician or Rayleigh (if $\gamma = 0$) fading channels, in radar receivers using post detection square law integration of M pulses,† and in partially coherent diversity reception of N-ary orthogonal

† In these cases the functional ξ is the test statistic. More generally however, the functional used for determining the optimum gain need not be actually formed in the receiver.

signals transmitted through M independent slow Rician fading channels.[3,4]

The probability density function (5) has interesting properties. It can be shown that (5) is the probability density function of the square root of the sum of squares of M independent normalized Rician variates, each variate having a probability density function of the form (4) with γ replaced by $\gamma/(M)^{1/2}$. For $M = 1$, (5) of course reduces to the Rician probability density function (4), and if in addition $\gamma = 0$ it becomes the Rayleigh probability density function. The density (5) can be viewed as the probability density function of the square root of a noncentral chi-square variate with $2M$ degrees of freedom and noncentrality parameter γ^2. With $\gamma = 0$ it becomes the density of the square root of a chi-square variate with $2M$ degrees of freedom.

In many practical cases γ^2 is proportional to the ratio of the total specular energy received via the M diversity branches to the sum of the scatter and noise energy received via any diversity branch (assuming that this latter sum is the same for any diversity branch). Thus γ^2/M can be thought of as the power signal-to-noise ratio per diversity branch or per pulse in the case of time diversity if, as is commonly assumed, the diversity branches are statistically independent but have identical parameters.

Since (5) arises in various applications, let us consider the canonical problem. Specifically the solutions to (2) will be obtained in which p_ω is given by (5). Letting $a = A(2)^{\frac{1}{2}}$ and $\alpha = \gamma/(M)^{\frac{1}{2}}$ leads to the following transcendental equation for A:

$$A^2 = A_c^2 + [(M + 1)/2]\, A_0^2 + [1/(d^2 - 1)] \cdot \{\ln I_{M-1}[\alpha A\, d(2)^{\frac{1}{2}}] - \ln I_{M-1}[\alpha A(2)^{\frac{1}{2}}]\} \qquad (6)$$

in which

$$A_0^2 \triangleq \frac{2 \ln d}{d^2 - 1} \qquad (7)$$

determines the optimum required attenuation for the Rayleigh case with unity cost ratio ($\nu = 1$), and

$$A_c^2 \triangleq \frac{\ln \nu}{d^2 - 1}. \qquad (8)$$

For the single diversity case ($M = 1$), (6) reduces to the transcendental equation encountered previously.[1] One is thus led to seek an iterative solution along the same lines. Before obtaining the required iteration equations consider some properties of (6). The equation can be written in the form

$$A^2 = A_c^2 + MA_0^2 + [1/(d^2 - 1)]\{\ln [(\alpha A\, d(2)^{\frac{1}{2}})^{M-1} I_{M-1}(\alpha A\, d(2)^{\frac{1}{2}})]$$

$$- (M - 1) \ln d^2 - \ln [(\alpha A(2)^{\frac{1}{2}})^{M-1} I_{M-1}(\alpha A(2)^{\frac{1}{2}})]\}. \qquad (9)$$

Combining the logarithmic terms on the right side of (9) and using the fact that $I_K(Z) \to (Z^K)/2^K K!$ as $Z \to 0$ it is seen that the quantity in braces goes to zero as $\gamma \to 0$. Hence the optimum attenuation for the chi-square case ($\gamma = 0$) is determined explicitly by $A = A_{cs}$ where

$$A_{cs}^2 \triangleq A_c^2 + MA_0^2 . \qquad (10)$$

In the same manner it is seen that if $A_{cs}^2 = 0$, then $A = 0$ is a solution to (9) for any γ and d. It is easy to show that the right hand side of (9) is an even function of A having a minimum of A_{cs}^2 at $A = 0$. The left hand side of (9) is of course a standard parabola centered at the origin. These curves (*i*) do not intersect if $A_{cs}^2 < 0$, (*ii*) intersect only at $A = 0$ if $A_{cs}^2 = 0$, and (*iii*) intersect at positive (and negative) values of A if $A_{cs}^2 > 0$. Thus meaningful values of A which minimize l are stationary if and only if $A_{cs}^2 > 0$. From (7), (8), and (10) this requires $\nu d^{2M} > 1$, which for $M = 1$ reduces to the constraint encountered previously.[1]

The solution to (6) or (9) can be obtained using the extrapolated iteration scheme described in Ref. 1. The iteration formulas require the derivative of the right side of (9) which can be found using the identity

$$d/d\zeta\, [\zeta^n I_n(\zeta)] = \zeta^n I_{n-1}(\zeta) \qquad n = \cdots -2, -1, 0, 1, 2, \cdots . \qquad (11)$$

The result is

$$f'(A^2) = \frac{\gamma(2)^{\frac{1}{2}}}{2(d^2 - 1)A} \left\{ d\, \frac{I_{M-2}[A\, d\alpha(2)^{\frac{1}{2}}]}{I_{M-1}[A\, d\alpha(2)^{\frac{1}{2}}]} - \frac{I_{M-2}[A\alpha(2)^{\frac{1}{2}}]}{I_{M-1}[A\alpha(2)^{\frac{1}{2}}]} \right\} \qquad (12)$$

in which f denotes the right hand side of (9) and the prime denotes differentiation with respect to the argument.

For computational purposes it is convenient to define the functions

$$\Psi_K(\zeta) \triangleq (\exp -\zeta)\, I_K(\zeta) \qquad (13)$$

which are uniformly bounded on the semi-infinite interval $[0, \infty]$. For any argument ζ these functions can be readily generated by numerical techniques using the recurrence relations and asymptotic expansions for the modified Bessel functions.

Using (12) and (13) the following iteration formulas to solve (6) are obtained.

$$
\begin{aligned}
A_{i+1}^2 &= F_i - \beta_i(A_i^2 - F_i) \qquad \beta_i \neq -1 \\
F_i &= A_c^2 + [(M+1)/2]\, A_0^2 \\
&\quad + \frac{A_i\alpha(2)^{\frac{1}{2}}}{d+1} + [1/(d^2-1)] \left[\ln \frac{\Psi_{M-1}(A_i\, d\alpha(2)^{\frac{1}{2}})}{\Psi_{M-1}(A_i\alpha(2)^{\frac{1}{2}})} \right] \\
G_i &= \frac{\alpha(2)^{\frac{1}{2}}}{2(d^2-1)A_i} \left\{ d\, \frac{\Psi_{M-2}[A_i\, d\alpha(2)^{\frac{1}{2}}]}{\Psi_{M-1}[A_i\, d\alpha(2)^{\frac{1}{2}}]} - \frac{\Psi_{M-2}[A_i\alpha(2)^{\frac{1}{2}}]}{\Psi_{M-1}[A_i\alpha(2)^{\frac{1}{2}}]} \right\} \\
\beta_i &= G_i/(1 - G_i) \qquad G_i \neq 1
\end{aligned}
\tag{14}
$$

The iteration is begun with $i = 1$, γ small, and $A_1^2 = A_{cs}^2$ and stopped when $|(A_{i+1} - A_i)/A_i|$ is less than the allowable error. By this method the optimum required normalized attenuation was found for various values of ν, γ, d, and M.

Inasmuch as the optimum attenuation satisfies the nonlinear equation (6) it will be helpful in interpreting results to find some useful approximations. Accordingly one notes that for $\gamma^2/M \gg 1$ and $d \gg 1$ the second term in the brackets on the right side of (6) is negligible compared with the first. Then taking $I_M(x) \approx \exp x$ leads to a quadratic equation in A whose solution

$$A = \frac{\gamma}{(2M)^{\frac{1}{2}}\, d} + \left[\frac{\gamma^2}{2M\, d^2} + A_c^2 + \frac{(M+1)}{2} A_0^2 \right]^{\frac{1}{2}} \tag{15}$$

approximates the required attenuation over the range specified.

For $\gamma^2/M \ll 1$, on the other hand, one may take $I_M(x) \approx (x/2)^M/M!$ in (6). Some further approximations and manipulation lead to the result

$$A \approx A_{cs}[1 + (\gamma^2/2M)]^{\frac{1}{2}} \qquad \gamma^2/M \ll 1 \tag{16}$$

which is exact for $\gamma = 0$.

When the solution to (6) is found for given parameters the minimum average exclusion cost can be determined. For the probability density function (5), (3) becomes

$$l = 1 - Q_M[\alpha, A(2)^{\frac{1}{2}}] + \nu Q_M[\alpha, Ad(2)^{\frac{1}{2}}] \tag{17}$$

where

$$Q_M(x, y) = \int_y^\infty \zeta(\zeta/x)^{M-1} \exp[-(\zeta^2 + x^2)/2] I_{M-1}(x\zeta)\, d\zeta \tag{18}$$

is the generalized Marcum Q-function.[3]

IV. EFFECT OF COST RATIO ON REQUIRED ATTENUATION

Since one cannot easily decide how much better or worse it is for the receiver to saturate at its upper limit than at its lower limit, the ratio, ν, is generally difficult to assess accurately. It is interesting (even fortunate!) that here as in the single diversity case,[1] the solutions obtained using (14) show that the optimum receiver attenuations for a wide range of cost ratios do not differ appreciably from those for the minimum exclusion probability case ($\nu = 1$).

For given ν, γ, d and M one may define the sensitivity, S_c, of the optimum attenuation to the cost ratio by the difference in required attenuation between the given case and the corresponding minimum exclusion probability case. Specifically, the sensitivity to cost ratio is

$$S_c(\nu, \gamma, d, M) \triangleq 20 \log_{10} A(\nu, \gamma, d, M) - 20 \log_{10} A(\nu = 1, \gamma, d, M) \tag{19}$$

in which the functional dependence is shown explicitly. In (19) a positive value of S_c indicates an increase in required attenuation compared with the minimum exclusion probability case. It can be shown that (*i*) the sign of (19) depends only on ν (positive if $\nu > 1$, negative if $\nu < 1$), and, (*ii*) $| S_c(\nu, \gamma, d, M) | \leqq | S_c(\nu, 0, d, M) |$. Thus, one can define a maximum sensitivity

$$S_c^* \triangleq S_c(\nu, 0, d, M) \equiv 10 \log_{10} A_{cs}^2 - 10 \log_{10} MA_0^2 \tag{20}$$

where the second equality follows from (19) and (10). Using (7) and (8), (20) can be written

$$S_c^* = 10 \log_{10} [1 + (\nu_{\text{dB}}/2MD)] \tag{21}$$

in which

$$\nu_{\text{dB}} \triangleq 20 \log_{10} \nu \tag{22}$$

is the cost ratio expressed in dB. S_c^* is an easily calculated bound which gives, with the correct algebraic sign, the maximum change (in dB) of the optimum required attenuation from the optimum for the minimum exclusion probability case. Figure 1 is a plot of S_c^*. It follows from (21)

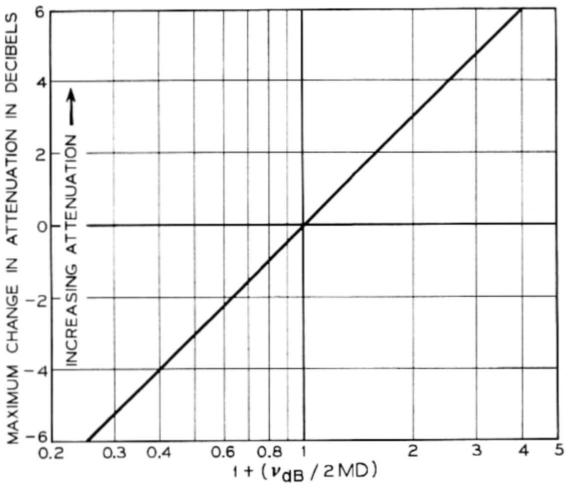

Fig. 1 — Maximum change in required receiver attenuation caused by nonunity cost ratio.

that for cost ratios in the range $-MD \leqq \nu_{\text{dB}} \leqq 2MD$, the maximum change in required receiver attenuation is less than about 3 dB. Equivalently, with γ, ν, and d fixed, the maximum sensitivity, S_c^* decreases as the order of diversity, M, increases.

Since the optimum attenuation is extremely insensitive to cost ratio for typical parameters, the minimum exclusion probability case ($\nu = 1$) is of special import among all average cost criteria of the form (1). The *numerical* results presented in this paper, therefore, include only the case $\nu = 1$, although the formulas derived apply more generally and can be used to generate numerical results in an entirely similar manner.

V. EFFECT OF DIVERSITY ON REQUIRED ATTENUATION

It is interesting to consider how the order of diversity affects the optimum required attenuation. Accordingly, in a manner analogous to (19) one can define the difference in required receiver attenuation resulting from diversity by

$$S_m(\nu, \gamma, d, M) = 20 \log_{10} A(\nu, \gamma, d, M) - 20 \log_{10} A(\nu, \gamma, d, M = 1). \qquad (23)$$

Let

$$\Delta_n(\nu, \gamma, d, M) \triangleq 20 \log_{10} A(\nu, \gamma, d, M) \tag{24}$$

be the required normalized attenuation in dB. Then (23) can be written

$$S_m(\nu, \gamma, d, M) = \Delta_n(\nu, \gamma, d, M) - \Delta_n(\nu, \gamma, d, 1). \tag{25}$$

Values of $S_m(1, \gamma, d, M)$ were obtained for various γ, d, and M using (14) and (23). These are shown in Fig. 2 where all quantities except M are in dB. The optimum normalized attenuation $\Delta_n(1, \gamma, d, M)$ required for the minimum exclusion probability case can be found using (25). Specifically one finds $S_m(1, \gamma, d, M)$ from Fig. 2, and adds to it the quantity $\Delta_n(1, \gamma, d, 1)$ from Fig. 6 in Ref. 1. Notice that in Ref. 1 only the single diversity case ($M = 1$) was considered so that the functional dependence of Δ_n on M was suppressed in the notation. That is, $\Delta_n(\nu, \gamma, d, 1)$ here is identical to $\Delta_n(\nu, \gamma, d)$ in Ref. 1.

From Fig. 2 it can be seen that if γ is sufficiently small (or large), S_m is positive (or negative) so that more (or less) attenuation is required if multiple diversity is used than would be required if the same specular energy were concentrated in a single diversity branch or pulse. Also for sufficiently small (or large) γ the required attenuation increases (or decreases) as the order of diversity, M, increases. There is of course a transition region which bridges the above cases and in which, for γ fixed, the differences S_m cross one another depending on the particular values of M, and D (and, in the general case, ν). The curves for $\gamma_{dB} = 15$, for example, exhibit this behavior.

Using (10) and (16) it can be shown that for $\gamma \to 0$

$$S_m(\nu, \gamma, d, M) \approx 10 \log_{10} M + 10 \log_{10} \left[\frac{1 + (\nu_{dB}/2MD)}{1 + (\nu_{dB}/2D)}\right] + 10 \log_{10} \left[\frac{1 + (\gamma^2/2M)}{1 + (\gamma^2/2)}\right] \tag{26}$$

which is exact for $\gamma = 0$. For the minimum exclusion probability case and $\gamma = 0$ (26) yields $S_m(1, 0, d, M) = 10 \log_{10} M$ which is independent of d. Similarly it can be shown using (15) that

$$\lim_{\gamma \to \infty} S_m(\nu, \gamma, d, M) = -10 \log_{10} M \tag{27}$$

which is independent of ν and d. The differences S_m for $\gamma_{dB} = \pm\infty$ therefore appear as horizontal lines in Fig. 2. One also observes that over the range of parameters shown, the limit (27) is approached within

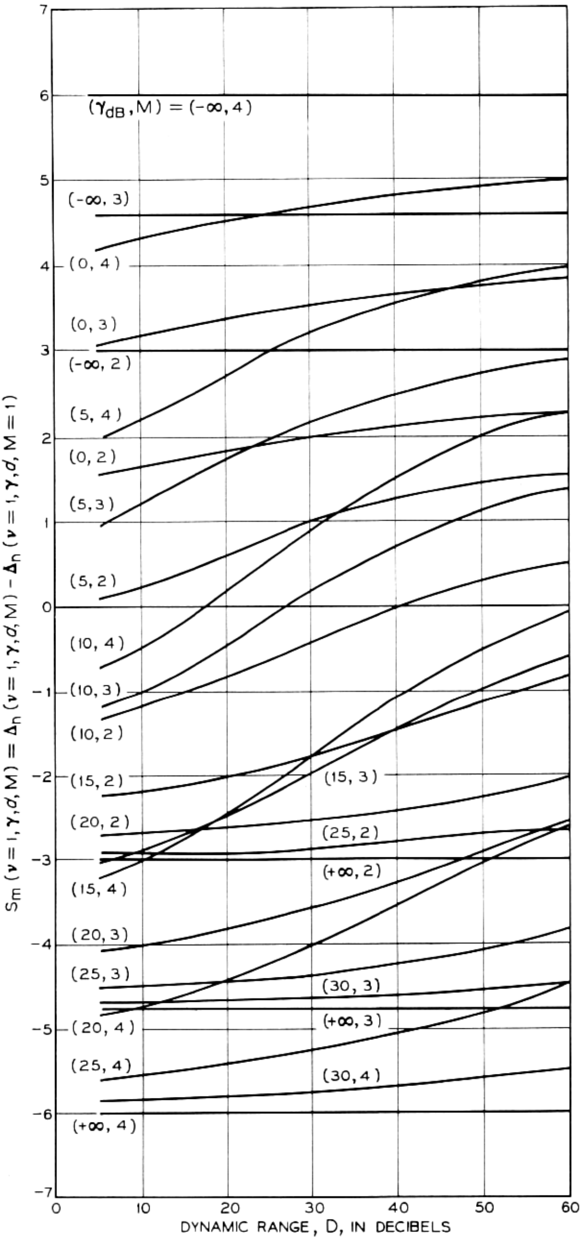

Fig. 2 — Differences in required optimum attenuation resulting from diversity. $\nu = 1$.

0.5 dB for $\gamma_{dB} = 30$. Noting that for $\gamma^\circ < \gamma < \gamma^*$, S_m is bounded by

$$S_m(\nu, \gamma^*, d, M) < S_m(\nu, \gamma, d, M) < S_m(\nu, \gamma^\circ, d, M) \tag{28}$$

it follows from (26) through (28) that for the minimum exclusion probability case all the differences $S_m(1, \gamma, d, M)$ lie between two horizontal lines in Fig. 2 determined only by the order of diversity. $| S_m(1, \gamma, d, M) | \leqq 10 \log_{10} M$. The difference between the required optimum attenuation for dual diversity and that required for single diversity with the same total received specular energy is less than about 3 dB.

VI. EXCLUSION COSTS FOR DIVERSITY RECEIVERS

The optimum normalized attenuations obtained using the iteration equations (14) were used to obtain the minimum average exclusion costs (17) for the case $\nu = 1$. These are shown in Fig. 3(a) and for smaller values of D in Fig. 3(b). The generalized Q function (18) was evaluated by computer, using relations derived from those given by Sagon.[5]

It can be seen that for small values of γ, the smallest dynamic range D required to obtain a given exclusion probability decreases rapidly as the order of diversity is increased; the most substantial decrease is obtained in going from single to dual diversity. This trend is lessened as the available signal-to-noise ratio γ increases. As a matter of fact if γ is sufficiently large (for example, $\gamma = 20$ dB) the dynamic range required to achieve a given exclusion probability increases as M increases. However at the large values of γ where this latter effect is apparent, even modest values of D yield extremely small exclusion probabilities. Moreover on the types of channels where diversity receivers are useful one would generally encounter small values of γ.

Consider a diversity receiver operating in a small signal-to-noise ratio and let the dynamic range of the components used be such that the probability of excluding the signal at any point in the receiver is the same throughout. Then it follows from Fig. 3 and the foregoing discussion that the dynamic range required of the components used in the post-combining portions of the receiver may be considerably smaller than that required of those components used in the individual diversity branches.

VII. SUMMARY AND CONCLUSIONS

An important class of diversity receivers used for communications through fading media and for radar is considered. The required gain

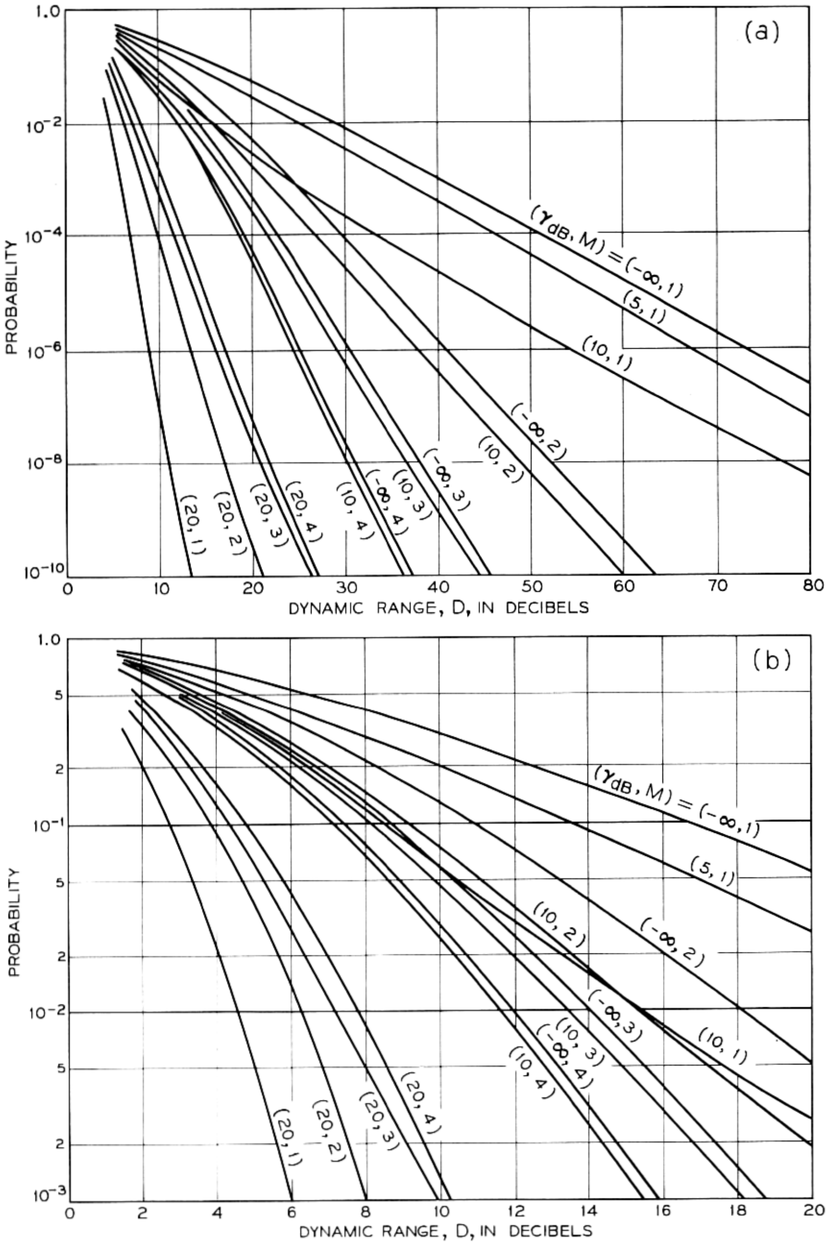

Fig. 3 — Minimum exclusion probabilities for diversity receivers.

is determined which minimizes the average cost of excluding from a finite dynamic range the signal appearing in the post-combining portions of the receiver. For the single diversity case ($M = 1$) the results reduce to those given previously.

It is shown that the required receiver gain is extremely insensitive to the relative costs of saturation at the upper and lower dynamic range bounds, differing at most by about 3 dB from the optimum for the (equal cost) minimum exclusion probability case for relative costs in the range $-MD \leqq \nu_{dB} \leqq 2MD$. One also finds that the sensitivity to relative cost therefore decreases as the order of diversity increases.

The difference between the required optimum receiver gain for various orders of diversity M, and that required for a single diversity receiver having the same total received specular energy is considered. Exact differences are given for the minimum exclusion probability case, and it is shown that these are less than $10 \log_{10} M$ dB independent of other parameters. Bounds on the difference are also given for non-unity cost ratio.

Performance characteristics derived show minimum exclusion probabilities obtainable as a function of dynamic range for various signal-to-noise ratios and orders of diversity. For a small signal-to-noise ratio the dynamic range required of the components used in the post-combining portions of the receiver can be considerably smaller than that required of those components in the individual diversity branches in order to achieve uniform exclusion probability throughout.

Notice that in some applications the normalization assumed in writing (4) and (5) may depend upon M and γ. This fact must be accounted for if one is calculating the *actual* required attenuation from the required *normalized* attenuation discussed in Sections IV and V. The optimum exclusion costs however, depend on the normalized attenuation and not on the normalizing factor. The results of Section VI therefore apply directly.

VIII. ACKNOWLEDGEMENT

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