Synthesis of Rational Transfer Function Approximations Using a Tapped Distributed RC Line With Feedback

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This paper describes a simple procedure for synthesizing an active distributed RC network which, by using dominant poles and zeros, realizes a very accurate approximation of an arbitrary stable rational transfer function. The network uses a single uniformly distributed RC line with taps spaced along its length. A linear combination of tap voltages is added to the input signal to form the driving voltage for the RC line; the output signal is also a linear combination of the tap voltages.

The network offers a number of significant advantages. Since it realizes a nearly rational transfer function, the approximation problem can be conveniently solved using readily available results on rational function approximation. Also, the network uses only one uniform RC line, the transfer function can be changed simply by changing resistor values, and the frequency can be scaled by minor connection changes. Thus one standard network with minor modification is useful for a wide variety of applications.

This paper develops the design procedure and derives the various sensitivity functions of importance. Two example designs are carried out: an approximation to a second-order low-pass transfer function and an approximation to a second-order band-pass transfer function with a Q of 100. The sensitivities for the examples are very reasonable and the measurements made on laboratory models indicate excellent agreement with theoretical predictions.

I. INTRODUCTION

The progress being made in miniaturizing electronic circuits has stimulated a continuing interest in the synthesis of networks using distributed RC components. Numerous techniques are available for synthesizing transfer functions using distributed RC components in conjunction with various active network elements.[1] Generally, these synthesis procedures are applicable only if the transfer function has a very special form. This form is not a rational function of the complex frequency variable s but involves hyperbolic functions of s. If the problem posed to the network designer were merely to realize given transfer functions of this special form using distributed RC networks, there would be no difficulty.

However, the problem is generally not this but rather to realize a network which achieves certain system specifications such as band-limiting or pulse shaping. Thus, a realizable transfer function must be developed which approximates the specifications (that is, the approximation problem must be solved) before a network can be synthesized. Because the transfer functions realizable by distributed RC networks have a somewhat complicated form, the approximation part of the network designer's work is more difficult when using distributed RC networks. This fact has led to a continuing effort to develop distributed RC networks which realize rational transfer functions. Since rational functions are easier to manipulate, and many applicable results are readily available in the literature, the approximation problem is made much easier. This paper develops a simple procedure and network for realizing an accurate approximation to a rational transfer function using an active network incorporating a distributed RC line.

Available techniques for synthesizing rational transfer functions using distributed RC networks are documented by Heizer, Barker, Woo and Hove, and Fu and Fu.[2–6] Each of these techniques uses the fact, first demonstrated by Heizer, that some of the immittance parameters of a distributed RC line can be made rational functions of s by cutting the conducting layer of the RC line in a particular manner.

These synthesis techniques have some definite disadvantages. They require two RC lines with cuts in the conducting layer which depend upon the transfer function being realized; this is undesirable from a manufacturing point of view and makes tuning difficult. Also, the synthesis procedure involves a test to determine that the curve cut in the conductor satisfies certain restrictions, that is, it does not "attempt" to create a negative capacitance in the line. If it does, a new try at the design is required. Fu and Fu eliminate this problem at the expense of a significant increase in circuit complexity.[6]

Recently techniques have become available for approximating rational transfer functions by using the dominant poles and zeros of distributed networks. A few representative approaches are those of

Kerwin, Bello and Gausi, and Wyndrum. Kerwin's approach, in general, requires the use of lumped components.[7] Bello and Gausi consider only low-pass transfer functions and use different configurations to realize an arbitrary transfer function.[8] Wyndrum's technique also deals only with low-pass transfer functions.[9]

The synthesis technique described here offers advantages over the other available techniques since only one uniformly distributed RC line is used and it is capable of realizing an accurate approximation to an arbitrary transfer function. In addition, the design procedure is very simple.

II. TRANSFER FUNCTION OF UNIFORM RC LINE WITH FEEDBACK

Chen and Levine[10, 11] have suggested that filters could be built using a uniform RC line driven by an input voltage source and having the output formed as a linear combination of the voltages appearing along the line as in Fig. 1. This procedure is useful in some cases but is not general enough because it synthesizes transfer functions by using zeros of transmission. What is needed in addition to zeros are poles; poles can be realized by using feedback as in Fig. 2.

The network of Fig. 2 consists of a uniform RC line with taps spaced along its length. The tap voltages are appropriately scaled by the infinite input impedance coefficients a_i and added to the input signal to form the driving voltage for the line. The output voltage is the sum of the tap voltages appropriately scaled by the infinite input impedance coefficients b_i. The RC line is the three-layer structure shown in Fig. 3, where it will be assumed that there is no voltage variation in the y direction.

To determine the voltage transfer function of the network of Fig.

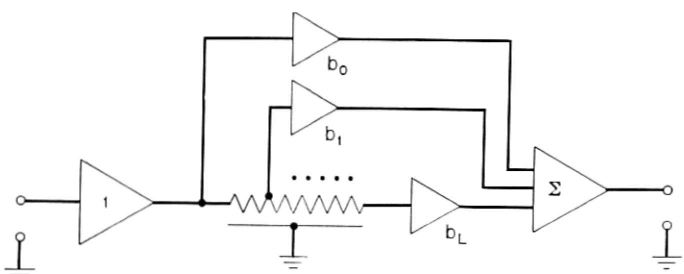

Fig. 1 — Tapped RC line.

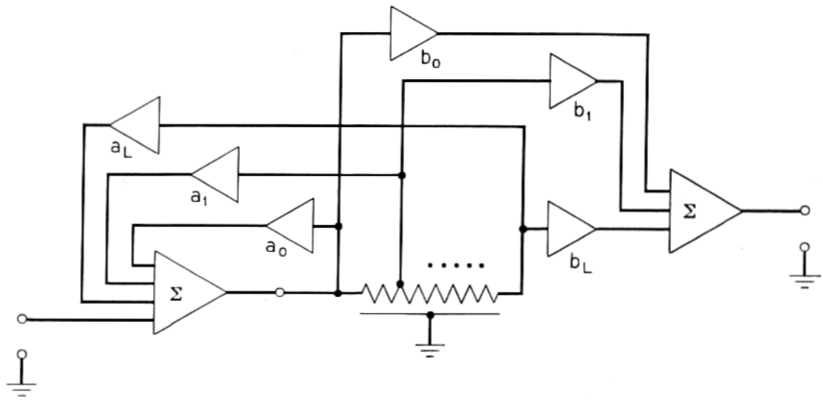

Fig. 2 — Tapped RC line with feedback.

2, we first determine the voltage gain $G_i(s)$ from the input of the line to a point x_i meters from the input. The result is[1]

$$G_i(s) = \frac{\cosh (l - x_i)(rcs)^{\frac{1}{2}}}{\cosh l(rcs)^{\frac{1}{2}}},$$

where l is the total length of the line in meters, and r and c are the resistance and capacitance per meter. If the distances l and x_i are constrained to be integral multiples* of some fixed length d_0, that is, $l = Ld_0$ and $x_i = id_0$, and we let $\tau = rcd_0^2$, $G_i(s)$ becomes

$$G_i(s) = \frac{\cosh (L - i)(\tau s)^{\frac{1}{2}}}{\cosh L(\tau s)^{\frac{1}{2}}}. \tag{1}$$

Using (1) the voltage transfer function of the network of Fig. 2 becomes

$$G(s) = K \frac{\sum_{i=0}^{L} b_i \cosh (L - i)(\tau s)^{\frac{1}{2}}}{\sum_{i=0}^{L} c_i \cosh (L - i)(\tau s)^{\frac{1}{2}}}, \tag{2}$$

where $c_0 = 1$ and $c_i = -a_i$ for $i \neq 0$† . The real constant K is such that $b_i = 1$ for the smallest i for which $b_i \neq 0$. By making the substitution $p = \exp (\tau s)^{\frac{1}{2}}$ and factoring the resulting polynomials in p, it can be

* For any set of x_i and l, a small enough d_0 can be found that error in this assumption is negligible.

† a_0 has been set to zero which can be done without any loss of generality.

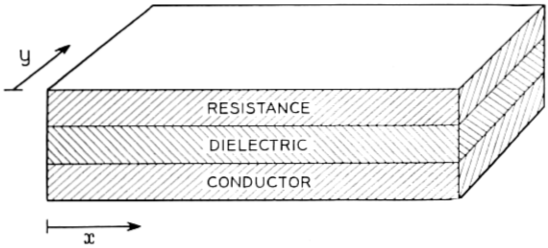

Fig. 3 — Uniform RC line.

shown that (2) can be factored into the form

$$G(s) = K \frac{2^{R-L} \prod_{i=1}^{R} [\cosh (\tau s)^{\frac{1}{2}} - Z_i]}{\prod_{i=1}^{L} [\cosh (\tau s)^{\frac{1}{2}} - P_i]} \tag{3}$$

where $1 \leqq R \leqq L$ [unless the numerator in (2) is unity in which case the numerator of (3) is 2^{1-L}] and the quantities P_i and Z_i are real or occur in complex conjugate pairs.

Before considering the question of stability, we will determine the locations of the poles and zeros of $G(s)$. Notice that, in spite of the fact that $(s)^{\frac{1}{2}}$ is involved, $G(s)$ is single valued. To determine the pole (zero) locations, we set the denominator (numerator) factors in (3) equal to zero and solve for s. For a typical denominator factor $(\cosh (\tau s)^{\frac{1}{2}} - P_i)$ we calculate the s-plane pole positions to be

$$\tau s_i = \ln^2 | p_i | - (\arg p_i + 2n\pi)^2 + 2j \ln | p_i | (\arg p_i + 2n\pi) \tag{4}$$

where $n = 0, \pm 1, \pm 2, \ldots$ and $p_i = P_i + (P_i^2 - 1)^{\frac{1}{2}}$. The term p_i comes from the solution of a quadratic equation which has two roots. However, these roots are always reciprocals of one another and, as can be seen from the form of (4), these two values of p_i give the same s_i. Hence, only one of them need be used. A simple check shows that each of the poles resulting from the single term $(\cosh (\tau s)^{\frac{1}{2}} - P_i)$ as given by (4) is simple.*

When P_i is real, the s_i given by (4) are on the negative real axis for $| P_i | \leqq 1$ and occur in complex conjugate pairs for $| P_i | > 1$. When P_i is complex, (3) involves a term $[\cosh (\tau s)^{\frac{1}{2}} - P_i^*]$ which gives poles that are the complex conjugates of those of (4).

It is easy to see from (4) that the infinite set of poles generated by one

* For $P = \pm 1$ double roots occur but not for $P = +1$ with $n = 0$.

denominator factor lie on a parabola given by

$$\sigma = \frac{\ln^2 |p_i|}{\tau} - \frac{\omega^2 \tau}{4 \ln^2 |p_i|}. \tag{5}$$

Figure 4 shows the location of these poles in the normalized, $\tau = 1$, s-plane. The poles due to the term $[\cosh (\tau s)^{\frac{1}{2}} - P_i]$ are indicated by single circles and those due to the term $[\cosh (\tau s)^{\frac{1}{2}} - P_i^*]$ by double circles. Similar comments hold in the case of numerator factors in (3).

Knowing the locations of the poles of $G(s)$ permits the question of stability to be answered easily. For simplicity we assume that in (2) $b_0 = 0$. If this is not the case, $G(s)$ can be separated into the sum of a constant plus a $\hat{G}(s)$ which is of the form of (2) where $\hat{G}(s)$ has the same denominator as $G(s)$ but different numerator and $b_0 = 0$. The constant gain is stable. With $b_0 = 0$, $G(s)$ is stable, that is, its impulse response remains bounded for large values of time, if all the poles lie in the left

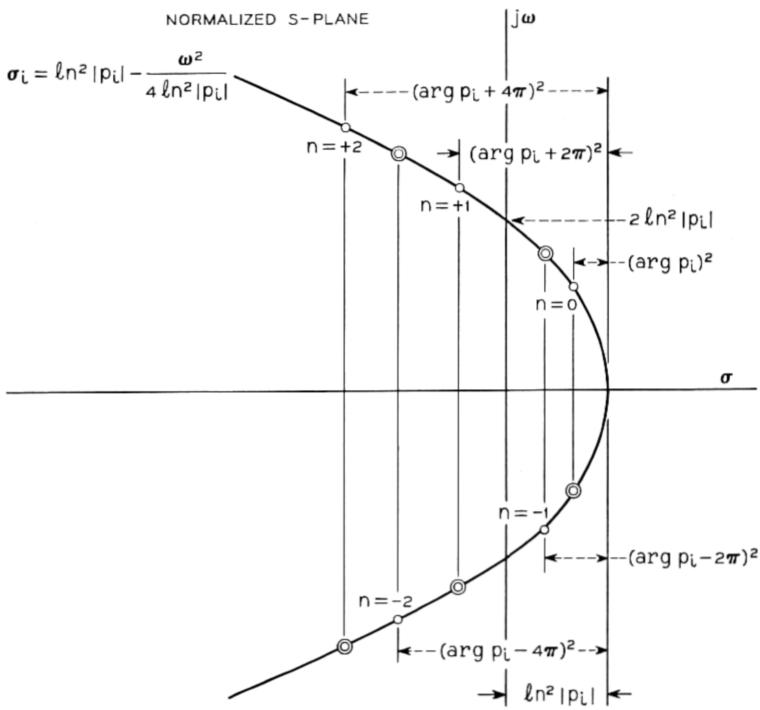

Fig. 4 — S-plane roots resulting from a pair of complex conjugate factors in $G(s)$.

half of the s-plane and those on the $j\omega$ axis are simple. This result can be proved by finding the inverse transform of $G(s)$ by using the Cauchy residue calculus.[12]

Notice that $G(s)$ is a meromorphic function and $G(s) \to 0$ as $s \to \infty$. The Laplace inversion relation is written as

$$g(t) = \lim_{n\to\infty} \frac{1}{2\pi j} \int_{\alpha - jy_n}^{\alpha + jy_n} G(s)e^{st}\, ds.$$

This integral is evaluated by closing the contour in the left half plane so that it does not pass through any poles of $G(s)$ and encloses a finite number of poles. The value of the closed contour integral is determined by the residues of the poles enclosed. As $n \to \infty$ $y_n \to \infty$ and the contours in the left half plane become larger without bound. Using Jordan's lemma[13] the integral over the left half plane contour approaches zero and $g(t)$ is determined. For large values of time the behavior of $g(t)$ is dominated by that pole with the most positive real part. The stability requirement follows directly from this.

III. TRANSFER FUNCTION SYNTHESIS

A glance at Fig. 4 shows that, if the $n = 0$ pole is close to the $j\omega$ axis, the response of the network will approximate that of this single pole alone for values of ω near the pole. An examination of (4) shows that this dominance can always be made to occur by an appropriate selection of τ. From (4) the pole positions in the s-plane are proportional to τ^{-1}. Therefore, by decreasing τ the poles become more widely spaced and hence those near the $j\omega$ axis become more dominant. Since p_i can be adjusted so as to cause the $n = 0$ pole to be arbitrarily close to the s-plane origin, a decrease in τ can be offset, for the $n = 0$ pole, by changing p_i. Therefore, the $n = 0$ pole can be made dominant. Hence, a rational transfer function can be approximated by the system considered here by making its dominant poles and zeros match those of the desired rational function. To calculate the feedback and feed forward coefficients of (2) we calculate the P_i and Z_i of (3) by using the desired pole or zero for s_i in

$$\left.\begin{matrix} P_i \\ Z_i \end{matrix}\right\} = \cosh\,(\tau s_i)^{\frac{1}{2}} \tag{6}$$

and multiply the factors in (3).

The scale factor τ controls the dominance of the $n = 0$ poles and zeros; the dominance improves as τ is reduced. A lower limit on practical values of τ occurs because the network sensitivity generally deteriorates with reduced values of τ. An upper limit on τ occurs because the $n = 0$ poles are restricted to the shaded area of Fig. 5. If τ is large enough so that a desired dominant pole s_i lies outside the shaded region, the network will realize this pole for a nonzero value of n. It is clear from Fig. 4 that the network will then have an $n = 0$ pole with a more positive real part than that of s_i. This pole can destroy the desired dominance or cause instability if it lies in the right half s-plane. The region permitted for $n = 0$ poles in Fig. 5 is determined from (4) by setting $n = 0$, substituting a value for ω_i and solving for the most negative value of σ_i. The resulting restriction is

$$0 \geqq \sigma_i \geqq \frac{\omega_i^2 \tau}{4\pi^2} - \frac{\pi^2}{\tau}. \tag{7}$$

Except in rather unusual situations τ will be much smaller than the maximum implied by (7).

To synthesize an approximation of a given rational transfer function, the following simple steps are performed.

(*i*) τ is selected so that all the poles of the transfer function lie in the region shown in Fig. 5, and the resulting $n = 0$ poles and zeros realized by the RC line are dominant.

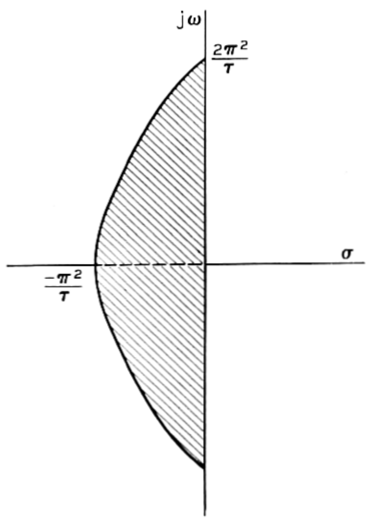

Fig. 5 — Permitted $n = 0$ pole positions in s-plane.

(*ii*) The desired transfer function poles and zeros are used in (6) to determine the P_i and Z_i which, when substituted in (3) and multiplied out, yield the feedback and feed forward coefficients for the network.

(*iii*) The exact response of the network is calculated using (2) or (3) to verify that a good approximation has been achieved.

As pointed out in the examples in Section VI, a wide range of values of τ gives a very accurate approximation. Thus, a little experience will then make step *iii* unnecessary. The selection of τ also affects the sensitivity of the network; hence sensitivity considerations may determine the best value of τ.

IV. SENSITIVITY

One of the most important aspects of any active network synthesis technique is its sensitivity to various parameter variations. In addition, sensitivity results are necessary to show how a physical network may be tuned to achieve an accurate realization of the requirements. Of the several different sensitivity functions that could be derived, we have chosen to consider the relative changes of the poles and zeros with a variety of parameters. These seem to give good physical insight into the behavior of the circuit and result in reasonably concise expressions. The sensitivity functions derived are the relative changes of the poles resulting from relative changes in feedback coefficients, τ, tap positions, and tap loading. Similar results hold for the zero sensitivity functions. The details of the derivations are contained in the Appendix.

If λ_j is the pole in question and the sensitivity of that pole to some parameter X is defined as

$$S_X^{\lambda_j} = \frac{\partial \lambda_j}{\partial X} \frac{X}{\lambda_j},$$

and $P_q = \cosh(\tau\lambda_q)^{1/2}$ where λ_q is the qth pole, then we have the following:*

(*i*) Pole sensitivity to feedback coefficients:

$$S_{a_i}^{\lambda_j} = \frac{a_i \cosh (L - i)(\tau\lambda_j)^{\frac{1}{2}}}{2^{L-2}(\tau\lambda_j)^{\frac{1}{2}} \sinh (\tau\lambda_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ \neq j}}^{L} (P_j - P_k)}. \tag{8}$$

* λ_j is assumed to be a simple pole.

Simple relations for determining the numerator of (8) are given in Appendix equations (20) and (21).

(*ii*) Pole sensitivity to RC product, τ:

$$S_{\tau}^{\lambda_j} = -1. \tag{9}$$

(*iii*) Pole sensitivity to improper tap spacing*

$$S_{l_i}^{\lambda_j} = -\frac{i a_i \sinh (L - i)(\tau\lambda_j)^{\frac{1}{2}}}{2^{L-2} \sinh (\tau\lambda_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ \neq j}}^{L} (P_j - P_k)}, \qquad i \neq L \tag{10a}$$

$$S_{l_L}^{\lambda_j} = -\frac{L \sum_{k=0}^{L-1} c_k \sinh (L - k)(\tau\lambda_j)^{\frac{1}{2}}}{2^{L-2} \sinh (\tau\lambda_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ \neq j}}^{L} (P_j - P_k)}. \tag{10b}$$

(*iv*) Pole sensitivity to tap loading:

$$S_{g_i}^{\lambda_j} = -\frac{g_i R \cosh (L - i)(\tau\lambda_j)^{\frac{1}{2}} \sum_{k=0}^{i-1} c_k \sinh (i - k)(\tau\lambda_j)^{\frac{1}{2}}}{2^{L-2}\tau\lambda_j \sinh (\tau\lambda_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ \neq j}}^{L} (P_j - P_k)} \tag{11}$$

where g_i is the conductance loading the ith tap and $R = d_0 r$.

V. SECOND ORDER DESIGN EQUATIONS

For the case where $L = 2$, that is, where the RC line is realizing an approximation to a second order transfer function $H(s)$, the design and sensitivity relations given above take on the very simple forms below (λ and ρ, which are complex, are the pole and zero positions in the upper left half s plane):

$$P = \cosh (\tau\lambda)^{\frac{1}{2}}, \qquad Z = \cosh (\tau\rho)^{\frac{1}{2}}$$

$$a_0 = 0,\ a_1 = 4\ \mathrm{Re}(P),\ a_2 = -\ (1 + 2 \mid P \mid^2)$$

$$b_0 = b_1 = 0,\ b_2 = 1 \text{ for } H(s) \text{ with no finite zeros}$$

$b_0 = 1$, $b_1 = -4\ \mathrm{Re}(Z)$, $b_2 = 1 + 2 \mid Z \mid^2$ for $H(s)$ with finite complex zeros

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* $l_i d_0 = x_i$ where x_i is the distance from the input of the RC line to the ith tap. Nominally $l_i = i$.

$b_0 = 0,\ b_1 = 1,\ b_2 = -1$ for $H(s)$ with one zero at zero

$$S_\lambda^\tau = -1$$

$$S_{a_1}^\lambda = \frac{2P\ \mathrm{Re}\,(P)}{j(\tau\lambda)^{\frac{1}{2}} \sinh\,(\tau\lambda)^{\frac{1}{2}}\ \mathrm{Im}\,(P)}, \qquad S_{a_2}^\lambda = \frac{-(1 + 2\,|\,P\,|^2)}{2j(\tau\lambda)^{\frac{1}{2}} \sinh\,(\tau\lambda)^{\frac{1}{2}}\ \mathrm{Im}\,(P)}$$

$$S_{l_1}^\lambda = \frac{-2\ \mathrm{Re}\,(P)}{j\ \mathrm{Im}\,(P)}, \qquad S_{l_2}^\lambda = \frac{2P^*}{j\ \mathrm{Im}\,(P)}$$

$$S_{g_1}^\lambda = \frac{-g_1RP}{2j\tau\lambda\ \mathrm{Im}\,(P)}, \qquad S_{g_2}^\lambda = \frac{g_2RP^*}{j\tau\lambda\ \mathrm{Im}\,(P)}$$

$$S_{b_0}^\rho = -\frac{2Z^2 - 1}{2j(\tau\rho)^{\frac{1}{2}} \sinh\,(\tau\rho)^{\frac{1}{2}}\ \mathrm{Im}\,(Z)} \text{ for } H(s) \text{ with finite complex zeros.}$$

For the case where $H(s)$ has two complex zeros, the zero sensitivities are the same as for the poles with ρ and Z replacing λ and P, except for $S_{b_0}^\rho$ which is given. For the case where $H(s)$ has a zero at zero and at infinity, the sensitivity of the zero at zero is infinite (due to the normalization by $1/\rho$), but unnormalized,

$$\frac{\partial\rho}{\partial b_1} = \frac{\partial\rho}{\partial b_2} = -\frac{2}{\tau} \quad \text{and} \quad \frac{\partial\rho}{\partial g_2} = -\frac{2R}{\tau}.$$

Other sensitivities not given are zero.

VI. EXAMPLES

Two examples of approximations to second order rational transfer functions will be worked out and compared with experimental results achieved with a thin film line. The two functions to be approximated are, normalized in frequency,

$$G_1(s) = \frac{(s/4)^2 + 1}{s^2 + (2)^{\frac{1}{2}}s + 1} \tag{12}$$

$$G_2(s) = \frac{0.01\ s}{s^2 + 0.01\ s + 1}. \tag{13}$$

The first is a noncritical low-pass function with a pair of zeros on the $j\omega$ axis and the second is a band-pass function with a Q of 100.

For the low-pass function (12), sensitivity is not a problem because the poles are very low Q. Therefore, τ can be selected to satisfy (7) and to insure dominance of the poles. Letting $\tau = 1$ we have the following results:

$$\lambda = (-1 + j)/(2)^{\frac{1}{2}}, \qquad P = 0.646 + 0.313j,$$

$$\rho = +j4, \qquad Z = 0.342 + j1.9$$

$$a_0 = 0, \quad a_1 = 2.59, \quad a_2 = -2.032$$

$$b_0 = 1, \quad b_1 = -1.36, \quad b_2 = 8.5, \quad K = 0.0544$$

$$S^{\lambda}_{a_1} = 3.33\angle 154°, \qquad S^{\lambda}_{a_2} = 3.65\angle -52°$$

$$S^{\lambda}_{l_1} = 4.12\angle 90°, \qquad S^{\lambda}_{l_2} = 4.6\angle -116°$$

$$S^{\lambda}_{g_1} = 1.15 g_1 R\angle -19°, \qquad S^{\lambda}_{g_2} = 2.3 g_2 R\angle 109°$$

$$S^{\rho}_{b_0} = 0.508\angle 125°, \qquad S^{\rho}_{b_1} = 0.16\angle -137°, \qquad S^{\rho}_{b_2} = 0.512\angle -37°$$

$$S^{\rho}_{l_1} = 0.359\angle 90°, \qquad S^{\rho}_{l_2} = 2.03\angle -170°$$

$$S^{\rho}_{g_1} = 0.126 g_1 R\angle 80°, \qquad S^{\rho}_{g_2} = 0.254 g_2 R\angle 100°$$

Figure 6 shows a block diagram of the experimental circuit, the theoretical response, and the measured results. Notice that the theoretical response realized by the RC line and that of the rational function cannot be distinguished on the scale used for this figure, since they differ by 1 percent at most.

In the case of $G_2(s)$ which has a pole with a $Q = 100$, dominance is achieved for a wide range of values of τ for which (7) holds, and the selection of τ is influenced primarily by sensitivity considerations. The parameter τ affects the sensitivity in a rather complicated way as can be seen from the various sensitivity relations. An examination of the pole sensitivity to coefficient variations has shown that S^{λ}_{a2} has a rather broad minimum in the range $2 \leqq \tau \leqq 14$ and that S^{λ}_{a1} goes to zero in this range when $a_1 = 0$. Therefore, without an exhaustive study to determine an optimum value of τ, we select that value which gives $a_1 = 0$, that is, $\tau = 4.94$. With this value of τ the following result:

$$\lambda = -0.005 + j \qquad P = j2.34$$

$$a_0 = a_1 = 0 \qquad a_2 = -11.95$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = -1 \quad K = 0.051$$

$$S^{\lambda}_{a_2} = 0.452\angle -45° \qquad S^{\lambda}_{l_2} = 2\angle 180°$$

$$S^{\lambda}_{g_1} = 0.1015 g_1 R\angle 90° \qquad S^{\lambda}_{g_2} = 0.203 g_2 R\angle 90°$$

$$\frac{\partial \rho}{\partial b_1} = \frac{\partial \rho}{\partial b_2} = 0.405\angle 180° \qquad \frac{\partial \rho}{\partial g_2} = 0.405R\angle 180° \text{ for the zero at zero.}$$

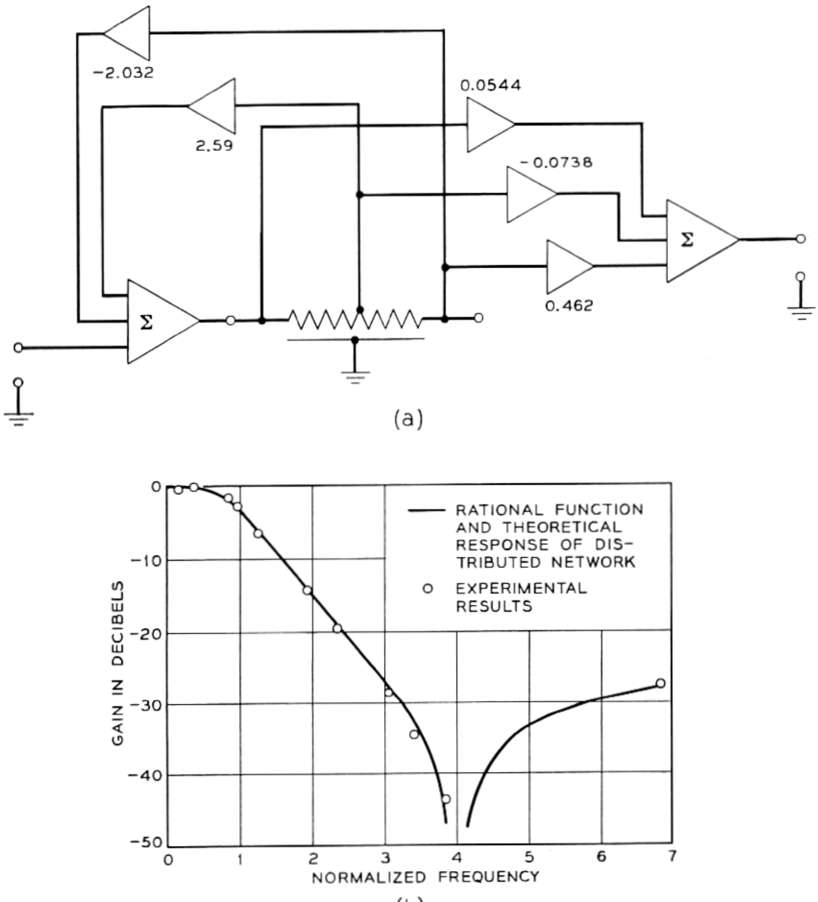
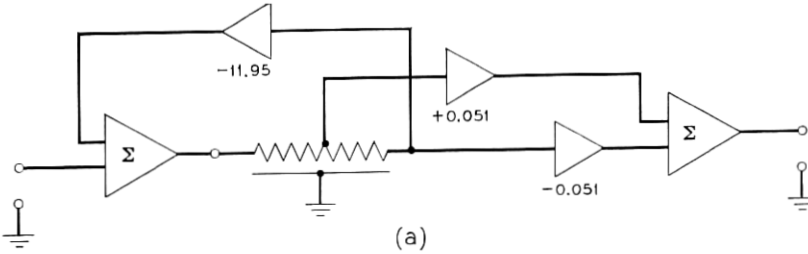


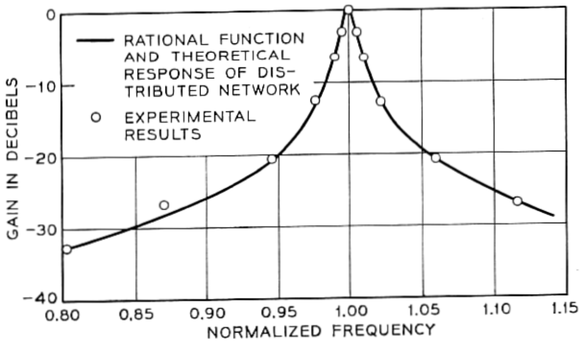
Fig. 6 — (a) RC line with feedback approximating $G_1(s)$. (b) Gain vs frequency for $G_1(s)$.

The block diagram of the experimental circuit, the theoretical response, and measured results are shown in Fig. 7. The difference between the theoretical gain of the RC line and that of the rational function is not noticeable since it is approximately 0.1 percent over the frequency range shown in the figure.

The sensitivity of this network is quite acceptable. $S^{\lambda}_{g_2}$ can be controlled by the proper selection of impedance levels; l_2 , a_2 and τ can be stabilized so that the values of $S^{\lambda}_{l_2}$, $S^{\lambda}_{a_2}$ and S^{λ}_{τ} are satisfactory. l_2 should not change after manufacture; a_2 can be made to depend only on the



(a)



(b)

Fig. 7 — (a) RC line with feedback approximating $G_2(s)$. (b) Gain vs frequency for $G_2(s)$.

ratio of two resistors which track with temperature and τ can be stabilized by selecting the temperature coefficients of the resistive and capacitive materials of the line to be negatives of one another.*

Several final notes concerning the network are in order. By isolating the taps on the line with emitter followers when necessary, it is possible to reduce to two the number of operational amplifiers in the network used for combining and scaling, one for the feedback voltages and another for the feed-forward voltages. When several of these networks are cascaded, one of these two can be eliminated by using an operational amplifier from the succeeeding network. One RC line can be constructed with a large number of taps. Then by selecting the appropriate set of taps, the line can be used for a variety of purposes and at different frequencies.

* Tantalum resistors on a substrate can be made to track within ± 5 ppm/°C and RC products can be made to track within ± 30 ppm/°C.

Although only second-order examples were worked out and built, it is not unreasonable to expect that advances in building thin film RC lines and resistors using tantalum may eventually yield the stability of the various parameters required to make higher-order realizations possible.

VII. CONCLUSIONS

A network has beeen described which uses a single uniform RC line with feedback to approximate an arbitrary rational transfer function. The design procedure is simple as is the physical network. Theoretical calculations indicate that the transfer function realized by the RC line is an accurate approximation of the desired rational transfer function and measurements made on experimental circuits agree well with the theory.

VIII. ACKNOWLEDGMENT

The author thanks W. W. Armstrong for his helpful discussions.

APPENDIX

Derivation of Sensitivity Expressions

This appendix derives the sensitivity expressions given by (8) through (11). The sensitivity of quantity λ to parameter α is defined as

$$S_\alpha^\lambda = \frac{\partial\lambda}{\partial\alpha}\frac{\alpha}{\lambda}. \tag{14}$$

If s_j is a network pole and $D(s)$ is the denominator of (2), $D(s_j) = 0$. The equation $D(s_j) = 0$ defines s_j as an implicit function of the parameters in $D(s)$. By differentiating the equation $D(s_j) = 0$ with respect to a parameter α, we can determine the quantity $\partial s_j/\partial\alpha$. This result will hold for general values of the various parameters in $D(s)$. For the particular case when all the parameters in $D(s)$ have their nominal values, s_j will in fact be one of the desired network poles, that is, $s_j = \lambda_j$. Furthermore, the factorization used in going from (2) to (3) can then be used to simplify the expression for $\partial\lambda_j/\partial\alpha$. The sensitivity of λ_j to α is then determined by using (14). A similar procedure using the numerator of (2) gives the sensitivity functions of the zeros.

A. 1 Sensitivity of Poles to Feedback Coefficients

From (2) the denominator of $G(s)$ is

$$D(s) = \sum_{k=0}^{L} c_k \cosh (L - k)(\tau s)^{\frac{1}{2}}. \tag{15}$$

If s_j is a root of $D(s_j) = 0$, we have

$$\frac{\partial}{\partial c_i} D(s_i) = 0 = \left[\sum_{k=0}^{L} c_k(L - k) \frac{\sinh (L - k)(\tau s_i)^{\frac{1}{2}}}{2(\tau s_i)^{\frac{1}{2}}} \tau \right] \frac{\partial s_i}{\partial c_i} + \cosh (L - i)(\tau s_i)^{\frac{1}{2}}.$$

Setting all the parameters to their nominal values gives $s_i = \lambda_i$. As shown in (18), the term in brackets is nonzero if λ is a simple pole and $\tau\lambda_i \neq -n^2\pi^2$ where n is a nonzero integer. Therefore, solving the above equation gives

$$\frac{\partial \lambda_i}{\partial c_i} = -\frac{\cosh (L - i)(\tau \lambda_i)^{\frac{1}{2}}}{\sum_{k=0}^{L} c_k(L - k) \frac{\sinh (L - k)(\tau \lambda_i)^{\frac{1}{2}}}{(\tau \lambda_i)^{\frac{1}{2}}} \tau}. \tag{16}$$

As was done in (3), (15) can be factored, when all parameters have their nominal values and $P_k = \cosh (\tau\lambda_k)^{1/2}$, into

$$D(s) = \sum_{k=0}^{L} c_k \cosh (L - k)(\tau s)^{\frac{1}{2}} = 2^{L-1} \prod_{k=1}^{L} [\cosh (\tau s)^{\frac{1}{2}} - P_k]. \tag{17}$$

Differentiating this equation with respect to s gives

$$\sum_{k=0}^{L} c_k \frac{(L - k) \sinh (L - k)(\tau s)^{\frac{1}{2}}}{2(\tau s)^{\frac{1}{2}}} \tau = 2^{L-1} \sum_{m=1}^{L} \tau \frac{\sinh (\tau s)^{\frac{1}{2}}}{2(\tau s)^{\frac{1}{2}}} \prod_{\substack{k=1 \\ \neq m}}^{L} [\cosh (\tau s)^{\frac{1}{2}} - P_k],$$

and letting $s = \lambda_j$, we have

$$\sum_{k=0}^{L} c_k \frac{(L - k) \sinh (L - k)(\tau \lambda_i)^{\frac{1}{2}}}{2(\tau \lambda_i)^{\frac{1}{2}}} \tau = 2^{L-2} \tau \frac{\sinh (\tau \lambda_i)^{\frac{1}{2}}}{(\tau \lambda_i)^{\frac{1}{2}}} \prod_{\substack{k=1 \\ \neq i}}^{L} (P_i - P_k). \tag{18}$$

(18) is nonzero provided λ_i is a simple pole and $\tau\lambda_i \neq -n^2\pi^2$ where n is nonzero integer.

Using this and $\partial c_i/\partial a_i = -1$, $i \neq 0$, with (16) gives

$$S_{a_i}^{\lambda_i} = \frac{a_i \cosh (L - i)(\tau \lambda_i)^{\frac{1}{2}}}{2^{L-2}(\tau \lambda_i)^{\frac{1}{2}} \sinh (\tau \lambda_i)^{\frac{1}{2}} \prod_{\substack{k=1 \\ \neq i}}^{L} (P_i - P_k)}. \tag{19}$$

The numerator can be simplified by expressing a_i in terms of the P_k by using (17) and by factoring $\cosh(L - i)(\tau\lambda_j)^{1/2}$. The results are

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 2 \sum_{i=1}^{L} P_i \\
a_2 &= -L - 2 \sum_{i=1}^{L} P_i\Big[\sum_{\substack{k=1\\ \neq i}}^{L} P_k\Big] \qquad (20) \\
a_3 &= 2(L-1) \sum_{i=1}^{L} P_i + \frac{4}{3} \sum_{i=1}^{L} P_i\Big\{\sum_{\substack{j=1\\ \neq i}}^{L} P_j\Big[\sum_{\substack{k=1\\ \neq i,j}}^{L} P_k\Big]\Big\}^* \\
&\vdots
\end{aligned}
$$

and

$$\cosh (L - i)(\tau\lambda_j)^{\frac{1}{2}} = 2^{(L-i-1)} \prod_{k=1}^{(L-i)} \Big\{P_j - \cos\Big[\frac{2k-1}{2(L-i)}\,\pi\Big]\Big\}, \qquad i \neq L. \qquad (21)$$

A. 2 Sensitivity of Poles to Tap Position

The tap positions are directly proportional to the integers k in (15). If k in (15) is replaced by l_k which is no longer constrained to be an integer and s_j is a root of the resulting $D(s)$, we have

$$D(s_j) = \sum_{k=0}^{L} c_k \cosh (l_L - l_k)(\tau s_j)^{\frac{1}{2}} = 0.$$

As in the previous section, differentiating with respect to l_k, solving for $\partial s_j/\partial l_k$, using the nominal values $l_k = k$ so that $s_j = \lambda_j$, and using (18), we have

$$S_{l_i}^{\lambda_j} = -\frac{i a_i \sinh (L - i)(\tau\lambda_j)^{\frac{1}{2}}}{2^{L-2} \sinh (\tau\lambda_j)^{\frac{1}{2}} \prod_{\substack{k=1\\ \neq j}}^{L} (P_j - P_k)} \qquad i \neq L \qquad (22)$$

and

$$S_{l_L}^{\lambda_j} = -\frac{L \sum_{k=0}^{L} c_k \sinh (L - k)(\tau\lambda_j)^{\frac{1}{2}}}{2^{L-2} \sinh (\tau\lambda_j)^{\frac{1}{2}} \prod_{\substack{k=1\\ \neq j}}^{L} (P_j - P_k)}. \qquad (22)$$

---

* Divide a_i by 2 if $i = L$.

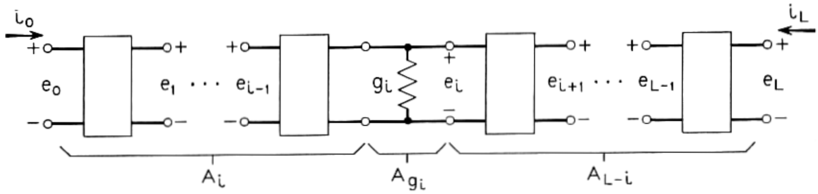


Fig. 8 — RC line model for loading analysis.

A.3 Pole Sensitivity to Tap Loading

To calculate the sensitivity of a pole λ_j to the loading at the ith tap by a conductance g_i, we assume that all other taps are not loaded, that is, $g_j = 0$ for $j \neq i$, and calculate the voltage transfer function from the input to the various taps. Having found these, we calculate the denominator of the system transfer function, $D(s)$, and proceed to calculate $\partial\lambda_j/\partial g_i$ in the same way as was done in Section *A.2*.

To calculate the voltage transfer function we will use the chain matrix description of the line which is

$$A_k = \begin{bmatrix} \cosh k(\tau s)^{\frac{1}{2}} & Z_0 \sinh k(\tau s)^{\frac{1}{2}} \\ Z_0^{-1} \sinh k(\tau s)^{\frac{1}{2}} & \cosh k(\tau s)^{\frac{1}{2}} \end{bmatrix}$$

where $Z_0 = (r/cs)^{1/2}$ and kd_0 is the length of the line. The line, loaded by g_i at the ith tap, can be considered as the cascade connection of an RC line of length id_0 connected to a two-port consisting solely of g_i which in turn is connected to an RC line of length $(L - i)d_0$ as seen in Fig. 8. The chain matrix of g_i is

$$A_{gi} = \begin{bmatrix} 1 & 0 \\ g_i & 1 \end{bmatrix}.$$

From Fig. 8 and the properties of the chain matrix we have for $0 \leqq k < i$

$$\begin{bmatrix} e_k \\ i_k \end{bmatrix} = A_{i-k} A_{gi} A_{L-i} \begin{bmatrix} e_L \\ -i_L \end{bmatrix} \triangleq B_k \begin{bmatrix} e_L \\ -i_L \end{bmatrix}$$

and for $i \leqq k \leqq L$

$$\begin{bmatrix} e_k \\ i_k \end{bmatrix} = A_{L-k} \begin{bmatrix} e_L \\ -i_L \end{bmatrix} \triangleq B_k \begin{bmatrix} e_L \\ -i_L \end{bmatrix}.$$

With $i_L = 0$ the above relations give the voltage transfer functions from the line input to the kth tap as

$$G_k(s) = e_k(s)/e_0(s) = B_{k11}/B_{011}$$

where B_{k11} is the (1, 1) element of the matrix B_k.

The matrices B_k for $0 \leqq k < i$ and $i \leqq k \leqq L$, are

$$B_k = \begin{bmatrix} \left\{ \begin{array}{l} \cosh (L-k)(\tau s)^{\frac{1}{2}} \\ \qquad + g_i Z_0 \sinh (i-k)(\tau s)^{\frac{1}{2}} \cosh (L-i)(\tau s)^{\frac{1}{2}} \end{array} \right\} & \{\cdots\} \\ \{ \qquad \cdots \qquad \} & \{\cdots\} \end{bmatrix}$$

and

$$B_k = \begin{bmatrix} \cosh (L-k)(\tau s)^{\frac{1}{2}} & \{\cdots\} \\ \{\cdots\} & \{\cdots\} \end{bmatrix},$$

respectively, which give the gains $G_k(s)$ as

$$G_k(s) = \frac{\cosh (L-k)(\tau s)^{\frac{1}{2}} + g_i Z_0 \sinh (i-k)(\tau s)^{\frac{1}{2}} \cosh (L-i)(\tau s)^{\frac{1}{2}}}{\cosh L(\tau s)^{\frac{1}{2}} + g_i Z_0 \sinh i(\tau s)^{\frac{1}{2}} \cosh (L-i)(\tau s)^{\frac{1}{2}}},$$

for $0 \leqq k < i$, and for $i \leqq k \leqq L$

$$G_k(s) = \frac{\cosh (L-k)(\tau s)^{\frac{1}{2}}}{\cosh L(\tau s)^{\frac{1}{2}} + g_i Z_0 \sinh i(\tau s)^{\frac{1}{2}} \cosh (L-i)(\tau s)^{\frac{1}{2}}}.$$

By using these equations in the expression for the gain of the feedback structure and multiplying the numerator and denominator of this expression by B_{011}, we have the following expression for the denominator, $D(s)$.

$$\begin{aligned} D(s) &= \sum_{k=0}^{L} c_k B_{k11} \\ &= \sum_{k=0}^{L} c_k \cosh (L-k)(\tau s)^{\frac{1}{2}} \\ &\qquad\qquad + g_i Z_0 \cosh (L-i)(\tau s)^{\frac{1}{2}} \sum_{k=0}^{i-1} c_k \sinh (i-k)(\tau s)^{\frac{1}{2}}. \end{aligned}$$

Now proceeding as in the previous sections, let $D(s_j) = 0$ and differentiate with respect to g_i to get

$$\begin{aligned} &\frac{\partial s_j}{\partial g_i} \sum_{k=0}^{L} \frac{c_k (L-k)\tau}{2(\tau s_j)^{\frac{1}{2}}} \sinh (L-k)(\tau s_j)^{\frac{1}{2}} \\ &\quad + \frac{R}{(\tau s_j)^{\frac{1}{2}}} \cosh (L-i)(\tau s_j)^{\frac{1}{2}} \sum_{k=0}^{i-1} c_k \sinh (i-k)(\tau s_j)^{\frac{1}{2}} + g_i(\cdots) = 0. \end{aligned}$$

With $g_i = 0$, $s_j = \lambda_j$ and the above gives with (18)

$$S_{g_i}^{\lambda_j} = -\frac{g_i R \cosh (L - i)(\tau\lambda_j)^{\frac{1}{2}} \sum_{k=0}^{i-1} c_k \sinh (i - k)(\tau\lambda_j)^{\frac{1}{2}}}{2^{L-2}\tau\lambda_j \sinh (\tau\lambda_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ \neq j}}^{L} (P_j - P_k)}.$$

A. 4 *Sensitivity to RC Product Changes*

It is assumed that the product RC $= \tau$ of the line is uniform but not correct. The sensitivity of the poles to changes in τ is easily seen to be

$$S_{\tau}^{\lambda_j} = -1$$

since τ always appears multiplying s in the transfer function and is a frequency scale factor.

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