Extension of Bode's Constant Resistance Lattice Synthesis of Transfer Impedance Function*

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Bode developed some explicit formulas in terms of the poles and zeros of the transfer impedance function for each element of the first and second degree constant resistance lattice structures. To extend work in this area, we derive explicit formulas for two of Bode's structures using coupled coils; we give two new structures which avoid coupled coils. Illustrative examples show the usage of these formulas. Finally, we include a general procedure for synthesizing any physically realizable, rational transfer impedance function by a constant resistance lattice network. A flow chart aids in detailing this procedure.

With the addition of these results, a general method for synthesizing any physically realizable, rational transfer impedance function with explicit formulas is complete. The explicit formulas method developed in this paper gives more rapid results and introduces fewer round-off errors than the step-by-step procedures used in the past.

I. INTRODUCTION

An important characteristic of constant resistance lattice networks is the absence of reflection effects when such two-port lattice networks are connected in tandem. The synthesis of a given transfer impedance function is simplified by representing the function by a partial product expansion. Thus the transfer impedance may be represented by a tandem connection of a number of constant resistance structures (one for each partial product). This process will result in

* Some results presented in this paper are based upon the author's thesis, "Explicit Formulas for Constant Resistance Lattice Synthesis of Transfer Impedance," presented to the Moore School of Electrical Engineering, University of Pennsylvania in December 1965 in partial fulfillment of the requirements for the degree of Master of Science in engineering.

realizable, simpler, transfer impedance functions provided that the constant multiplier of the given transfer impedance function is made large enough to permit each of the constituent networks to have non-negative loss on the real axis.[1] In general, then, there will be additional fixed loss for the overall two-port network.

For physical realizability, it is required that both members of any conjugate complex pair of zeros or poles be retained within a given partial product. Hence, each of the elementary constituent networks must be represented by a biquadratic factor. When there are single zero and single pole pairs on the σ axis, the partial product factor for each pair is reduced to the bilinear form. It is recognized that the expansion can be performed with the zeros and the poles collected in a variety of ways and assigned to the individual networks. The elementary lattice networks for these bilinear and biquadratic factors are first and second degree constant resistance lattice structures, respectively. Therefore, one can realize a complicated rational transfer impedance function, to within a constant loss, by a combination of elementary structures in tandem.

Bode[1] developed the basic first and second degree structures, which are given in Fig. 2. They cover all the possible pole-zero combinations. Furthermore, he derived the explicit formula for each element of structures I to VI in terms of the poles and the zeros of the transfer impedance function. The object of this paper is to extend work in this area. Explicit formulas are obtained for structures VII and VIII and for two additional structures (Fig. 7). The structures of Fig. 7 avoid coupled coils. Illustrative examples are given to show the usage of these formulas. Finally, a general procedure for synthesizing any physically realizable, rational transfer impedance function by a constant resistance lattice network is included. This procedure is detailed with the aid of a flow chart. The appendix gives a method of obtaining the physical realizability conditions for one of the struc-

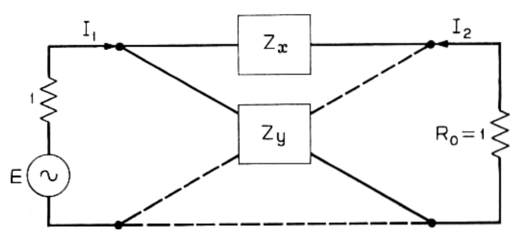


Fig. 1 — General constant resistance lattice network.

tures as an illustration. Reference 2 supplies the derivations of the physical realizability conditions of other structures.

II. DEVELOPMENT OF SECOND DEGREE CONSTANT RESISTANCE LATTICE STRUCTURE INTO GENERAL FORMULAS

The transfer impedance function exp θ of the constant resistance lattice given in Fig. 1 of second degree can be written as the biquadratic factor

$$\exp\theta = \frac{E_1}{2I_2} = K\frac{(s-a_1)(s-a_2)}{(s-b_1)(s-b_2)} \tag{1}$$

where exp θ is related to the series branch impedance z_x by the expression

$$z_x = \frac{[\exp\theta]-1}{[\exp\theta]+1} \quad \text{for} \quad z_x z_y = R_0^2 = 1 \tag{2}$$

or

$$\exp\theta = \frac{1+z_x}{1-z_x}. \tag{3}$$

Hence z_x is a biquadratic of the form

$$z_x = \frac{A_5 s^2 + A_3 s + A_1}{A_6 s^2 + A_4 s + A_2} \tag{4}$$

The solution for A_j's can be expressed in terms of the zeros and the poles of the transfer impedance function exp θ by setting (1) and (3) equal

$$K\frac{s^2-(a_1+a_2)s+a_1a_2}{s^2-(b_1+b_2)s+b_1b_2} = \frac{(A_6+A_5)s^2+(A_4+A_3)s+(A_2+A_1)}{(A_6-A_5)s^2+(A_4-A_3)s+(A_2-A_1)}. \tag{5}$$

For convenience let

$$\alpha_1 = a_1 + a_2\,, \qquad \alpha_2 = a_1a_2 \tag{6}$$

and

$$\beta_1 = b_1 + b_2\,, \qquad \beta_2 = b_1b_2\,. \tag{7}$$

Then equating coefficients in (5) and solving for A_j's yields

$$A_1 = \tfrac{1}{2}[\alpha_2(A_6+A_5) - \beta_2(A_6-A_5)] \tag{8}$$

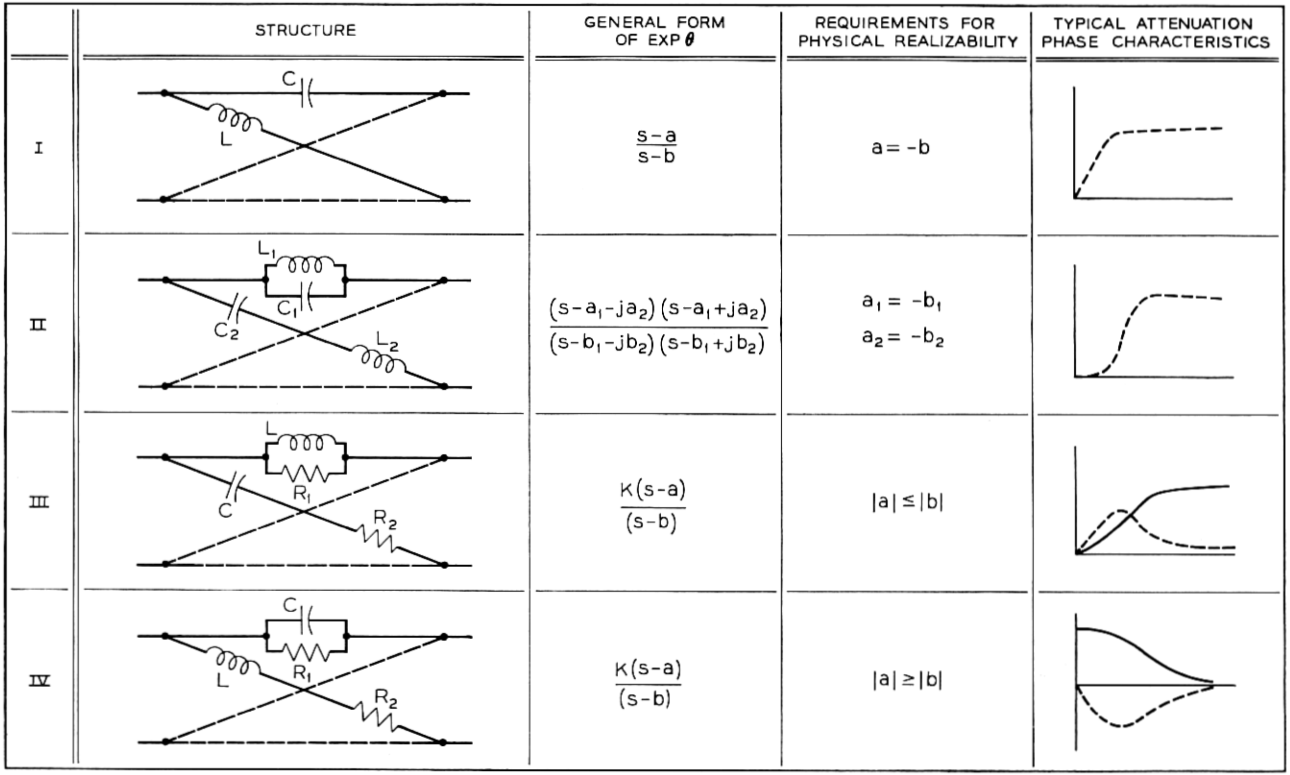
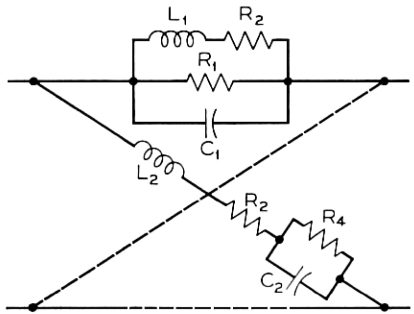
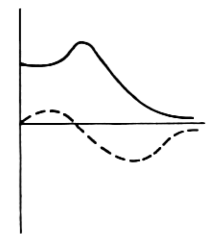
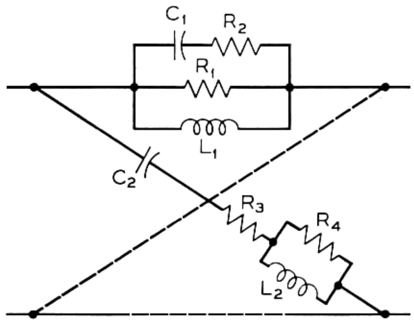
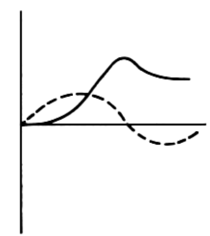
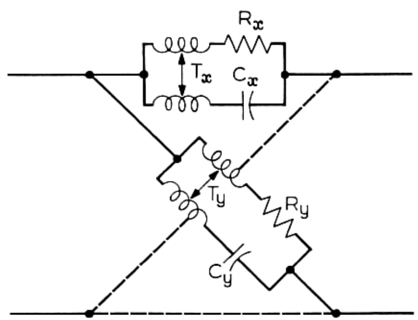
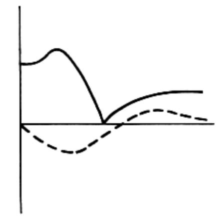
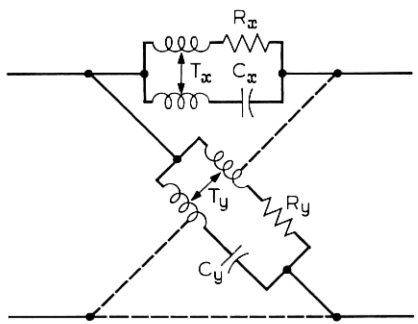
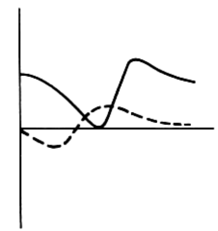


Fig. 2 — Constant resistance lattice equalizers.

	STRUCTURE	GENERAL FORM OF EXP θ	REQUIREMENTS FOR PHYSICAL REALIZABILITY	TYPICAL ATTENUATION PHASE CHARACTERISTICS
V		$\frac{K(s-a_1)(s-a_2)}{(s-b_1)(s-b_2)}$	$\lvert a_1 a_2\rvert \geq \lvert b_1 b_2\rvert$ $b_1^2 + b_2^2 \leq a_1^2 + a_2^2$ THE b's ARE COMPLEX THE a's MAY BE REAL OR COMPLEX	
VI		$\frac{K(s-a_1)(s-a_2)}{(s-b_1)(s-b_2)}$	$\lvert a_1 a_2\rvert \leq \lvert b_1 b_2\rvert$ $\frac{1}{b_1^2} + \frac{1}{b_2^2} \leq \frac{1}{a_1^2} + \frac{1}{a_2^2}$ THE b's ARE COMPLEX THE a's MAY BE REAL OR COMPLEX	

	STRUCTURE	GENERAL FORM OF EXP θ	REQUIREMENTS FOR PHYSICAL REALIZABILITY	TYPICAL ATTENUATION PHASE CHARACTERISTICS
VII		$\frac{K(s-a_1)(s-a_2)}{(s-b_1)(s-b_2)}$	$\lvert a_1 a_2 \rvert \geq \lvert b_1 b_2 \rvert$ $b_1^2 + b_2^2 \geq a_1^2 + a_2^2$ THE a's ARE COMPLEX THE b's MAY BE REAL OR COMPLEX	
VIII		$\frac{K(s-a_1)(s-a_2)}{(s-b_1)(s-b_2)}$	$\lvert a_1 a_2 \rvert \leq \lvert b_1 b_2 \rvert$ $\frac{1}{b_1^2} + \frac{1}{b_2^2} \geq \frac{1}{a_1^2} + \frac{1}{a_2^2}$ THE a's ARE COMPLEX THE b's ARE REAL OR COMPLEX	

$$A_2 = \tfrac{1}{2}[\alpha_2(A_6 + A_5) + \beta_2(A_6 - A_5)] \tag{9}$$

$$A_3 = \tfrac{1}{2}[\alpha_1(A_6 + A_5) - \beta_1(A_6 - A_5)] \tag{10}$$

$$A_4 = \tfrac{1}{2}[\alpha_1(A_6 + A_5) + \beta_1(A_6 - A_5)] \tag{11}$$

$$K = \frac{A_6 + A_5}{A_6 - A_5}. \tag{12}$$

Expressing exp θ on the real frequency axis we have

$$\exp(\alpha + j\beta) = \frac{K[(a_1a_2 - \omega^2) - (a_1 + a_2)j\omega]}{[(b_1b_2 - \omega^2) - (b_1 + b_2)j\omega]} \tag{13}$$

where $\theta = \alpha + j\beta$ may be called the transfer loss and phase. From this the expression for the transfer loss is obtained as

$$\exp(2\alpha) = \frac{K^2[(-\omega^2 + a_1a_2)^2 + (a_1 + a_2)^2\omega^2]}{(-\omega^2 + b_1b_2)^2 + (b_1 + b_2)^2\omega^2} \tag{14}$$

by letting

$$k = K^2 \qquad \text{and} \qquad x = \omega^2 \tag{15}$$

and from (6) and (7), equation (14) becomes

$$\exp(2\alpha) = \frac{k(\alpha_2 - x)^2 + \alpha_1^2 x}{(\beta_2 - x)^2 + \beta_1^2 x}. \tag{16}$$

It can be shown that in general the attenuation characteristic of a lattice for which exp θ is a biquadratic function exhibits a minimum at a real frequency.[1] One can shift this minimum loss to have zero loss at that particular frequency; thus the transfer impedance obtained will be within a constant loss. By doing this the attenuation characteristics of all elementary structures will have zero transfer loss at one frequency ω_0. Corresponding to ω_0, exp $(2\alpha) = 1$. Thus k can be determined in terms of the zeros and the poles of the transfer impedance function by (16). If A_6 is equated to unity, A_5 is obtained by (12). With the relationships (8) to (11) one can determine A_j's in terms of the zeros and the poles of the transfer impedance function. Hence from (4) and (2) z_x and z_y can be obtained respectively.

III. EXPLICIT FORMULAS FOR STRUCTURE VII

The physical realizability conditions and the typical attenuation characteristic of this structure are given in Fig. 2. Zero attenuation occurs at a frequency ω_0; therefore we must choose k such that the attenuation becomes zero at the same frequency.

For zero attenuation, exp (2α) must be equal to 1; thus from (16) after rearranging and letting $\omega_0^2 = x$, we obtain a quadratic in x_0

$$(k - 1)x_0^2 + (k\alpha_1^2 - 2\alpha_2 k + 2\beta_2 - \beta_1^2)x_0 + (k\alpha_2^2 - \beta_2^2) = 0 \tag{17}$$

which can be written compactly as

$$ax_0^2 + bx_0 + c = 0 \tag{18}$$

where

$$a = k - 1 \tag{19}$$

$$b = (\alpha_1^2 - 2\alpha_2)k + 2\beta_2 - \beta_1^2 \tag{20}$$

$$c = k\alpha_2^2 - \beta_2^2 . \tag{21}$$

In order that the frequency be real and the attenuation equal to zero, the solution of (18) must have a double root at x_0. This condition holds only when the discriminant $b^2 - 4ac = 0$. First we find from (18)

$$x_0 = \frac{-b}{2a}. \tag{22}$$

Secondly we obtain the following quadratic in k

$$Ak^2 + 2Bk + C = 0 \tag{23}$$

where

$$A = \alpha_1^2(\alpha_1^2 - 4\alpha_2) \tag{24}$$

$$B = (\alpha_1^2 - 2\alpha_2)(2\beta_2 - \beta_1^2) + 2(\beta_2^2 + \alpha_2^2) \tag{25}$$

$$C = \beta_1^2(\beta_1^2 - 4\beta_2). \tag{26}$$

It can be shown that the larger root of (23) must be used to insure that z_x is a positive real function.* Denote this larger real root by k_m. Thus, from (15) and (23)

$$k_m = K_m^2 = \left\{\frac{-2B \pm [(2B)^2 - 4AC]^{\frac{1}{2}}}{2A}\right\}_{\max} \tag{27}$$

Hence K_m can be obtained quite easily and it is in terms of the zeros and the poles of exp θ.

* Notice that k_m as well as the discriminant of (23) must be positive and real because of the physical realizability conditions. By considering all the possible sign combinations for A, B and C, one of the positive roots of K_m is greater than, or equal to, unity.

By substituting K_m into (12) and letting $A_6 = 1$ we obtain

$$A_5 = \frac{K_m - 1}{K_m + 1}. \tag{28}$$

Notice that from (12) and (28), the root K_m must be greater or equal to unity in order for z_x to be positive real functions.

Using (8), (9), (10) and (11) we can determine A_1, A_2, A_3 and A_4. Then multiplying each coefficient by $K_m + 1$ we get

$$A_1 = \alpha_2 K_m - \beta_2 \tag{29}$$

$$A_2 = \alpha_2 K_m + \beta_2 \tag{30}$$

$$A_3 = \beta_1 - K_m \alpha_1 \tag{31}$$

$$A_4 = -\beta_1 - K_m \alpha_1 \tag{32}$$

$$A_5 = K_m - 1 \tag{33}$$

$$A_6 = K_m + 1. \tag{34}$$

Thus the coefficients of z_x and K_m are expressed implicitly in terms of the zeros and the poles of the transfer impedance function exp θ. Furthermore from (33) and (34), we must have the positive root $K_m > 1$ for A_j's to be positive.

Before considering the realization of the biquadratic z_x in (4) with its coefficients given from (29) to (34), we will show that z_x is a minimum resistance function.

Rewriting (3) as

$$\exp \theta = \frac{1 + z_x}{1 - z_x} = \frac{(1 + R_x) + jX_x}{(1 - R_x) - jX_x} \tag{35}$$

where

$$z_x = R_x + jX_x , \tag{36}$$

the corresponding magnitude is

$$\exp (2\alpha) = \frac{(1 + R_x)^2 + X_x^2}{(1 - R_x)^2 + X_x^2}. \tag{37}$$

Since we require that

$$\exp (2\alpha) = 1 \text{ at } \omega_0 \tag{38}$$

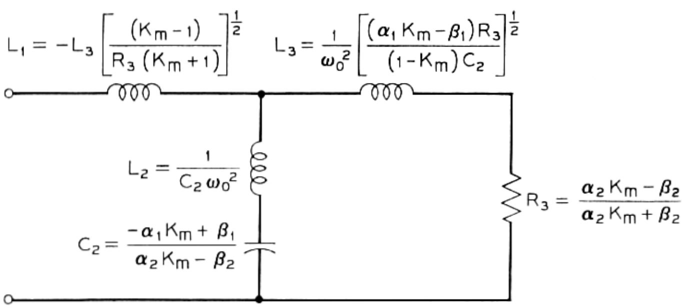
then R_x must equal zero and hence z_x must be a minimum resistance function.

Now we can determine the element values for structure VII by using the results given in Chapter 4 of Boghosian and Bedrosian, in that the element values of a Brune network were expressed explicitly in terms of the coefficients of a minimum resistance biquadratic impedance function.[3] Since we have shown that the biquadratic function in (4) is also minimum resistance, it is a simple matter to express the element values in terms of the zeros and the poles of exp θ. The case when the coefficients of z_x satisfy the inequality

$$[A_5A_2]^{\frac{1}{2}} - [A_1A_6]^{\frac{1}{2}} < 0 \tag{39}$$

is suitable for structure VII. Thus the Brune network for the series arm z_x of structure VII is shown in Fig. 3, where the equivalent-T is used instead of the transformer. Since $z_xz_y = 1$, we have for the cross arm of these lattices

$$z_y = \frac{A_6s^2 + A_4s + A_2}{A_5s^2 + A_3s + A_1}. \tag{40}$$



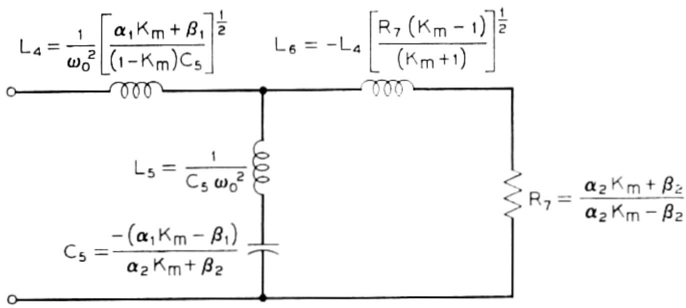
$$\omega_0^2 = \left[\frac{(\alpha_2K_m)^2-\beta_2^2}{K_m^2-1}\right]^{\frac{1}{2}} = \left[\frac{(a_1a_2K_m)^2-(b_1b_2)^2}{K_m^2-1}\right]^{\frac{1}{2}}$$

$$L_3 = \frac{1}{\omega_0^2}\left[\frac{[(a_1+a_2)K_m-(b_1+b_2)]R_3}{(1-K_m)C_2}\right]^{\frac{1}{2}}$$

$$C_2 = -\frac{(a_1+a_2)K_m+(b_1+b_2)}{a_1a_2K_m-b_1b_2}$$

$$R_3 = \frac{a_1a_2K_m-b_1b_2}{a_1a_2K_m+b_1b_2}$$

Fig. 3 — Series arm z_x for structure VII of Fig. 2.



$$\omega_0^2 = \left[\frac{(a_1 a_2 K_m)^2 - (b_1 b_2)^2}{K_m^2 - 1}\right]^{\frac{1}{2}}$$

$$L_4 = \frac{1}{\omega_0^2}\left[\frac{(a_1 + a_2) K_m + (b_1 + b_2)}{(1 - K_m) C_2}\right]^{\frac{1}{2}}$$

$$C_5 = -\frac{(a_1 + a_2) K_m - (b_1 + b_2)}{a_1 a_2 K_m + b_1 b_2}$$

$$R_7 = \frac{a_1 a_2 K_m + b_1 b_2}{a_1 a_2 K_m - b_1 b_2}$$

Fig. 4 — Cross arm z_y for structure VII of Fig. 2.

Then z_y is given by the network in Fig. 4 where the values of K_m, the zeros and the poles are the same as those given in Fig. 3.

IV. EXPLICIT FORMULAS FOR STRUCTURE VIII

The physical realizability conditions and the typical attenuation characteristic of this structure are given in Fig. 2. It can be shown that general formulas in terms of the zeros and the poles of exp θ for K_m and the A_j's are the same as those for structure VII, with the exception that for structure VIII the coefficients of z_x satisfy the following inequality

$$[A_5 A_2]^{\frac{1}{2}} - [A_1 A_6]^{\frac{1}{2}} > 0 \tag{41}$$

yielding a positive sign for the reactance $j\omega_0 L_1$ and a negative sign for the reactance $j\omega_0 L_3$ of Fig. 3. Similarly, the cross arm z_y of structure VIII yielding a negative sign for reactance $j\omega_0 L_4$ and a positive sign for reactance $j\omega_0 L_6$ of Fig. 4. Thus the Brune network for the series arm z_x and the cross arm z_y of structure VIII is shown in Figs. 5 and 6 respectively.

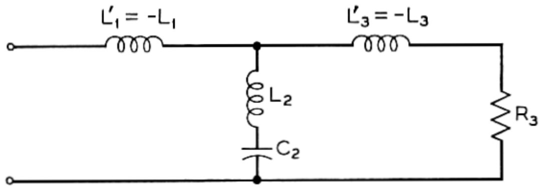


Fig. 5 — Series arm z_x for structure VIII of Fig. 2. (L's, C_2, and R_3 are same as for Fig. 3.)

V. EQUIVALENT NETWORKS TO STRUCTURES VII AND VIII

To avoid the need for coupled coils in the lattices developed previously, we can introduce the Bott-Duffin impedance arms in structures IX and X to obtain equivalent networks to structures VII and VIII respectively in Fig. 2.[4] The series and cross arm of structure IX have the same configuration as the cross and series arm respectively of structure X. Hence only the series arm of each lattice is shown in Fig. 7. These new lattice structures necessarily have the same physical realizability requirements and exhibit the same typical characteristics as sketched in Fig. 2 for structures VII and VIII. The element values for the general case of these lattice networks without mutual inductance are given in Table I.

VI. EXAMPLE

We now illustrate by an example the methods we have developed to obtain realization of constant resistance lattice networks. Let us find such a realization given the transfer impedance function

$$\exp\,\theta = \frac{K(s^4 + 2.268s^3 + 6.517s^2 + 3.302s + 4.905)}{s^4 + 2s^3 + 4.778s^2 + 5.556s + 5.556} \tag{42}$$

In order to represent the given function as tandem lattices, we ob-

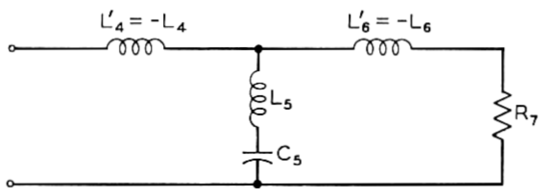


Fig. 6 — Cross arm z_y for structure VIII of Fig. 2. (L's, C_5 and R_7 are same as for Fig. 4.)

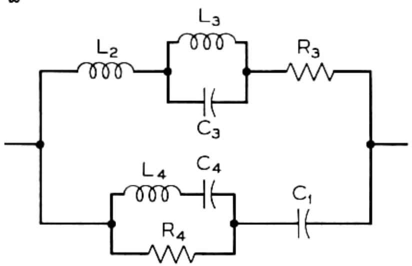
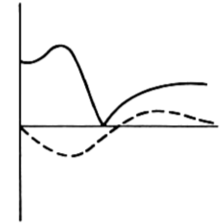
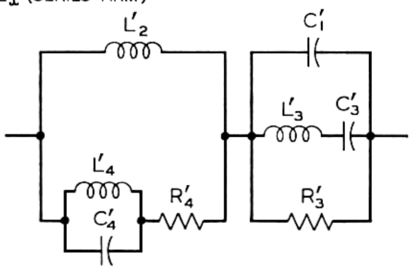
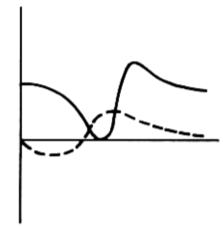
	STRUCTURE	GENERAL FORM OF EXP θ	REQUIREMENTS FOR PHYSICAL REALIZABILITY	TYPICAL ATTENUATION PHASE CHARACTERISTICS
IX	Z_x (SERIES ARM)  Z_y HAVE THE SAME CONFIGURATION AS Z_x IN (X)	$\frac{K(s-a_1)(s-a_2)}{(s-b_1)(s-b_2)}$	$\lvert a_1 a_2 \rvert \geq \lvert b_1 b_2 \rvert$ $b_1^2 + b_2^2 \geq a_1^2 + a_2^2$ THE a's ARE COMPLEX THE b's MAY BE REAL OR COMPLEX	
X	Z_x (SERIES ARM)  Z_y HAVE THE SAME CONFIGURATION AS Z_x IN (IX)	$\frac{K(s-a_1)(s-a_2)}{(s-b_1)(s-b_2)}$	$\lvert a_1 a_2 \rvert \leq \lvert b_1 b_2 \rvert$ $\frac{1}{b_1^2} + \frac{1}{b_2^2} \geq \frac{1}{a_1^2} + \frac{1}{a_2^2}$ THE a's ARE COMPLEX THE b's ARE REAL OR COMPLEX	

Fig. 7 — Constant resistance lattice equalizers.

TABLE I(a)—ELEMENT VALUES FOR STRUCTURE IX

Z_x	Z_y
$L_2 = \dfrac{R_3 R_4}{C_1}$	$L_2' = \dfrac{1}{C_1}$
$L_3 = \dfrac{M}{[(\alpha_2 K_m + \beta_2)^3 (K_m + 1)]^{\frac{1}{2}}} = \dfrac{M}{[(a_1 a_2 K_m + b_1 b_2)^3 (K_m + 1)]^{\frac{1}{2}}}$	$C_2' = L_2$
$C_3 = \dfrac{R_3 R_4}{L_4}$	$L_3' = \dfrac{1}{C_3}$
$R_3 = \dfrac{\alpha_2 K_m - \beta_2}{\alpha_2 K_m + \beta_2} = \dfrac{a_1 a_2 K_m - b_1 b_2}{a_1 a_2 K_m + b_1 b_2}$	$C_3' = L_3$
$L_4 = \dfrac{[(K_m - 1)^3 (\alpha_2 K_m - \beta_2)]^{\frac{1}{2}}}{M} = \dfrac{[(K_m - 1)^3 (a_1 a_2 K_m - b_1 b_2)]^{\frac{1}{2}}}{M}$	$R_3' = \dfrac{1}{R_3}$
$C_4 = \dfrac{R_3 R_4}{L_3}$	$L_4' = \dfrac{1}{C_4}$
$R_4 = \dfrac{K_m - 1}{K_m + 1}$	$C_4' = L_4$
$C_1 = \left[\dfrac{(\alpha_2 K_m - \beta_2)(\beta_1 - K_m \alpha_1)}{(-\beta_1 - K_m \alpha_1)(K_m + 1)}\right]^{\frac{1}{2}} = \left\{\dfrac{(a_1 a_2 K_m - b_1 b_2)[(b_1 + b_2) - K_m(a_1 + a_2)]}{[-(b_1 + b_2) - K_m(a_1 + a_2)](K_m + 1)}\right\}^{\frac{1}{2}}$	$R_4' = \dfrac{1}{R_4}$

where

$$
\begin{aligned}
M &= (\beta_1 - K_m \alpha_1)[(\alpha_2 K_m + \beta_2)(K_m + 1)]^{\frac{1}{2}} \\
&\quad - (\beta_1 + K_m \alpha_1)[(\alpha_2 K_m - \beta_2)(K_m - 1)]^{\frac{1}{2}} \\
&= [(b_1 + b_2) - K_m(a_1 + a_2)][(a_1 a_2 K_m + b_1 b_2)(K_m + 1)]^{\frac{1}{2}} \\
&\quad - [(b_1 + b_2) + K_m(a_1 + a_2)][(a_1 a_2 K_m - b_1 b_2)(K_m - 1)]^{\frac{1}{2}}
\end{aligned}
$$

TABLE I(b)—ELEMENT VALUES FOR STRUCTURE X

Z_x	Z_y
$L_2' = \left\{\dfrac{(K_m - 1)(\beta_1 - K_m\alpha_1)}{(\alpha_2 K_m + \beta_2)(-\beta_1 - K_m\alpha_1)}\right\}^{\frac{1}{2}}$	$L_2 = C_1'$
$= \left\{\dfrac{(K_m - 1)[(b_1 + b_2) - K_m(a_1 + a_2)]}{(a_1 a_2 K_m + b_1 b_2)[-(b_1 + b_2) - K_m(a_1 + a_2)]}\right\}^{\frac{1}{2}}$	
$C_1' = \dfrac{L_2'}{R_3 R_4}$	$L_3 = C_3'$
$L_3' = R_3' R_4' C_4'$	$C_3 = \dfrac{1}{L_3'}$
$C_3' = \dfrac{M}{[(\alpha_2 K_m - \beta_2)^3 (K_m - 1)]^{\frac{1}{2}}}$	$R_3 = \dfrac{1}{R_3'}$
$= \dfrac{M}{[(a_1 a_2 K_m - b_1 b_2)^3 (K_m - 1)]^{\frac{1}{2}}}$	
$R_3' = \dfrac{\alpha_2 K_m - \beta_2}{\alpha_2 K_m + \beta_2} = \dfrac{a_1 a_2 K_m - b_1 b_2}{a_1 a_2 K_m + b_1 b_2}$	$L_4 = C_4'$
$L_4' = R_3' R_4' C_3'$	$C_4 = \dfrac{1}{L_4'}$
$C_4' = \dfrac{[(K_m + 1)^3 (\alpha_2 K_m + \beta_2)]^{\frac{1}{2}}}{M}$	$R_4 = \dfrac{1}{R_4'}$
$= \dfrac{[(K_m + 1)^3 (a_1 a_2 K_m + b_1 b_2)]^{\frac{1}{2}}}{M}$	
$R_4' = \dfrac{K_m - 1}{K_m + 1}$	$C_1 = \dfrac{1}{L_2'}$

where

$$
\begin{aligned}
M &= (\beta_1 - K_m\alpha_1)[(\alpha_2 K_m + \beta_1)(K_m + 1)]^{\frac{1}{2}} \\
&\quad - (\beta_1 + K_m\alpha_1)[(\alpha_2 K_m - \beta_2)(K_m - 1)]^{\frac{1}{2}} \\
&= [(b_1 + b_2) - K_m(a_1 + a_2)][(a_1 a_2 K_m + b_1 b_2)(K_m + 1)]^{\frac{1}{2}} \\
&\quad - [(b_1 + b_2) + K_m(a_1 + a_2)][(a_1 a_2 K_m - b_1 b_2)(K_m - 1)]^{\frac{1}{2}}
\end{aligned}
$$

tain a partial product expansion of exp θ wherein the factors are bilinear or biquadratic forms. In the present example, we find

$$\exp \theta = K\left(\frac{s^2 + 2s + 5}{s^2 + 2s + 2}\right)\left(\frac{s^2 + 0.268s + 0.981}{s^2 + 2.778}\right) \tag{43}$$

or

$$\exp \theta = \left[K_1\left(\frac{s^2 + 2s + 5}{s^2 + 2s + 2}\right)\right]\left[K_2\left(\frac{s^2 + 0.268s + 0.981}{s^2 + 2.778}\right)\right] \tag{44}$$

where K or K_1K_2 are constant multipliers to allow for any corresponding net increase in loss required by the overall network. Each biquadratic factor must be physically realizable if it is to be synthesized using one of the basic structures. For the first factor we get the following zeros for the polynomials

$$\begin{aligned} a_1 &= -1 + j2, & a_2 &= -1 - j2, \\ b_1 &= -1 + j, & b_2 &= -1 - j. \end{aligned} \tag{45}$$

From these a's and b's we determine

$$\begin{aligned} \alpha_1 &= -2, & \alpha_2 &= 5, \\ \beta_1 &= -2, & \beta_2 &= 2, \\ a_1^2 + a_2^2 &= -6, & b_1^2 + b_2^2 &= 0. \end{aligned} \tag{46}$$

Substituting into the physical realizability conditions of structures VII and IX, we find that these conditions are satisfied. The second factor in (44) has the following values

$$\begin{aligned} \alpha_1 &= -0.268, & \alpha_2 &= 0.981, \\ \beta_1 &= 0, & \beta_2 &= 2.778, \\ \frac{1}{a_1^2} + \frac{1}{a_2^2} &= -1.964, & \frac{1}{b_1^2} + \frac{1}{b_2^2} &= -0.72. \end{aligned} \tag{47}$$

With these values the second factor satisfies the requirements for structures VIII and X. Hence, the given transfer function (42) can be represented by two second degree lattices in tandem with or without mutual coupled coils.

Using (27), K_1 and K_2 of (44) become

$$K_1 = K_{m1} = 1.2895, \tag{48}$$

$$K_2 = K_{m2} = 7.035, \tag{49}$$

and the constant multiplier K is

$$K = K_1 K_2 = 9.0716. \tag{50}$$

Thus by substituting (46) and (48) into the explicit formulas for structures VII and IX, and substituting (47) and (49) into the explicit formulas for structures VIII and X, the element values for each corresponding structure can be obtained. These element values are summarized in Tables II, III, IV and V. It should be apparent that another realization may be obtained for (42) by interchanging the numerators of the two biquadratic factors given in (43). Then the counterparts would have to be re-examined to see which basic structure would be realizable.

VII. GENERAL SYNTHESIS PROCEDURE

The flow chart shown in Fig. 8 is a guide for the general synthesis procedure of any physical realizable, rational transfer impedance function exp θ. This flow chart can be summarized as follows:

(*i*) Factor the given transfer impedance function into first and second degree functions with both members of any conjugate complex pair of zeros and poles retained in each given partial product.

(*ii*) Synthesize all first degree functions by structure III or IV according to their physical realizability conditions.

(*iii*) Examine the poles and zeros of these second degree functions to see whether they are real or complex; then use the appropriate group of structures indicated. If the poles and zeros are real, factor the second degree function into first degree functions.

(*iv*) Examine the physical realizability conditions further to determine to which sub-group of structures the function belongs.

TABLE II—ELEMENT VALUES OF STRUCTURES VII FOR (43)

z_x (series arm)	z_y (cross arm)
$L_1 = -0.0658$	$L_4 = 2.0178$
$L_2 = 0.1290$	$L_5 = 1.9379$
$C_2 = 1.0296$	$C_5 = 0.0685$
$L_3 = 0.1344$	$L_6 = -0.9876$
$R_3 = 0.5265$	$R_7 = 1.8994$

TABLE III—ELEMENT VALUES OF STRUCTURE IX FOR (43) (EQUIVALENT TO STRUCTURE VII)

z_x (series arm)		z_y (cross arm)	
$C_1 = 0.4956$	$L_3 = 0.2084$	$C_1' = 0.1342$	$C_3' = 0.2084$
$L_2 = 0.1342$	$R_4 = 0.1264$	$L_2' = 2.0178$	$R_4' = 7.9113$
$R_3 = 0.5265$	$L_4 = 0.0424$	$R_3' = 1.8994$	$L_4' = 3.1319$
$C_3 = 1.5695$	$C_4 = 0.3193$	$L_3' = 0.6371$	$C_4' = 0.0424$

(*v*) Connect the synthesized elementary structures in tandem.
(*vi*) Raise the impedance level to the desired R_0 .

The realizability conditions of structures VII and IX are the same and also those for structure VIII and X. If one wishes to avoid having coupled coils, he should use structures IX and X. Since structures IX and X are generally more complex, one may elect to use the structures VII and VIII to save on the number of elements in the final network.

VIII. CONCLUSION

In the field of classical network theory, Bode developed explicit formulas in terms of the poles and zeros of the transfer impedance function for synthesizing constant resistance lattice structures of the types I, II, III, IV, V, and VI. This paper has shown the detailed development and derivations of explicit formulas in terms of the poles and zeros of the transfer impedance function for synthesizing types VII and VIII, with coupled coils by Brune Method, and for types IX and X which are new types of structures that may be used to avoid having coupled coils by Bott-Duffin Procedure. With the addition of these results, a general method for synthesizing any physically realizable, rational transfer impedance function with explicit formulas is complete. The explicit formulas method developed in this

TABLE IV—ELEMENT VALUES OF STRUCTURE VIII FOR (43)

z_x (series arm)	z_y (cross arm)
$L_1' = 0.7896$	$L_4' = -1.3958$
$L_2 = 2.4106$	$L_5 = 5.6767$
$C_2 = 0.4572$	$C_5 = 0.1948$
$L_3' = -0.5949$	$L_6' = 1.8564$
$R_3 = 0.4266$	$R_7 = 2.3475$

TABLE V—ELEMENT VALUES OF STRUCTURE X FOR (43) (EQUIVALENT TO STRUCTURE VIII)

z_x (series arm)		z_y (cross arm)	
$C_1' = 2.4677$	$C_3' = 1.2658$	$C_1 = 1.2665$	$C_3 = 1.1481$
$L_2' = 0.7896$	$R_4' = 0.7511$	$L_2 = 2.4677$	$L_4 = 2.7222$
$R_3' = 0.4260$	$C_4' = 2.7222$	$R_3 = 2.3470$	$C_4 = 2.4691$
$L_3' = 0.8710$	$L_4' = 0.4050$	$L_3 = 1.2658$	$R_4 = 1.3314$

paper gives more rapid results and introduces fewer round-off errors than the step-by-step procedures used in the past.

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APPENDIX

Derivation of Physical Realizability Conditions

We now develop the physical realizability conditions for second degree lattices in terms of the poles and the zeros of exp θ. We shall find that the requirement for non-negative loss at real frequencies for such two-ports leads to both a product and a summation condition on the poles and zeros of the transfer impedance function. This analysis is carried out in terms of an example utilizing structures VII and IX. These structures exhibit zero loss at a finite frequency ω_0 (see Figs. 2 and 7). To evaluate the constant multiplier for these structures we set exp $(2\alpha) = 1$ at ω_0, and let $x_0 = \omega_0^2$. Then from (16), k becomes

$$k = \frac{(\beta_2 - x_0)^2 + \beta_1^2 x_0}{(\alpha_2 - x_0)^2 + \alpha_1^2 x_0} \tag{51}$$

where α's and β's are given by (6) and (7) respectively.

Substituting (51) back into (16) and applying the non-negative loss condition, that is, exp $(2\alpha) \geqq 1$ for all frequencies we have

$$\left[\frac{(\beta_2 - x_0)^2 + \beta_1^2 x_0}{(\alpha_2 - x_0)^2 + \alpha_1^2 x_0}\right]\left[\frac{(\alpha_2 - x)^2 + \alpha_1^2 x}{(\beta_2 - x)^2 + \beta_1^2 x}\right] \geqq 1. \tag{52}$$

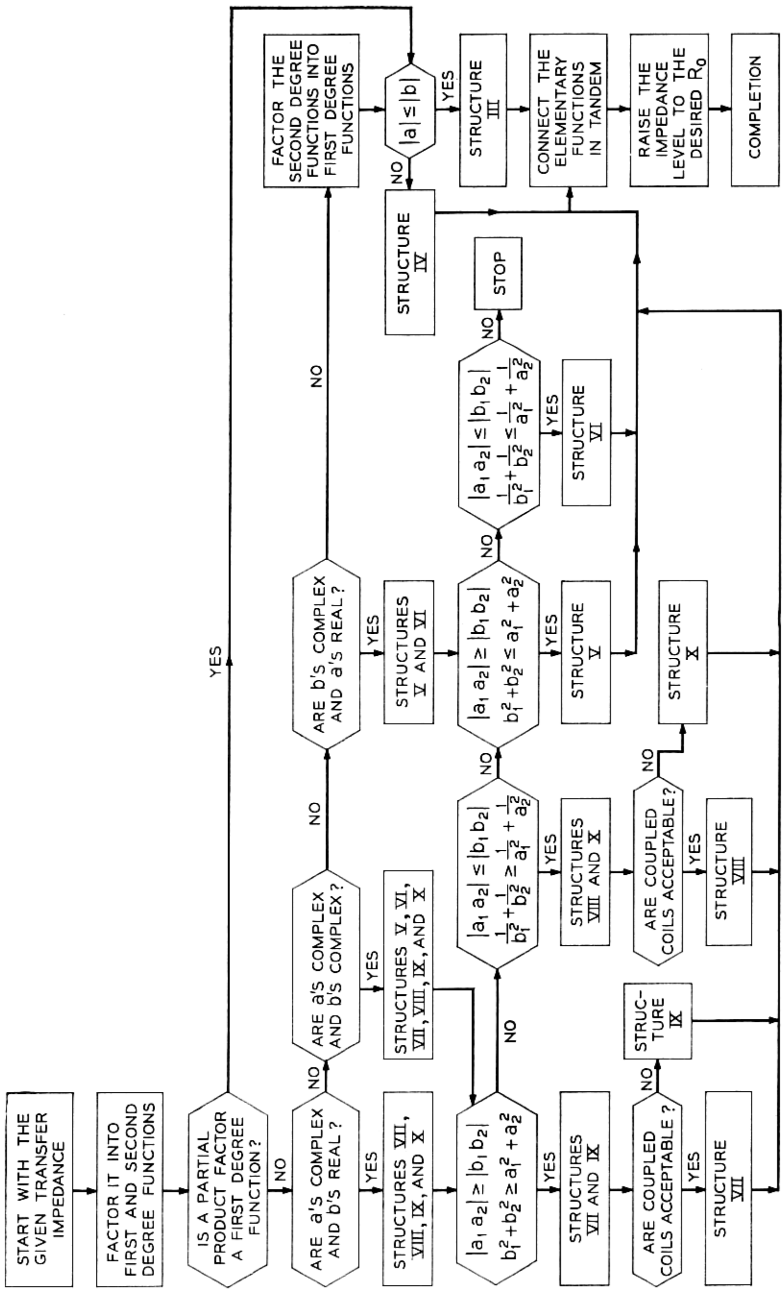

Fig. 8 — Flow chart for synthesizing transfer impedance functions with constant resistance lattice networks.

We shall obtain one of the realizability conditions by letting the frequency approach infinity. The result is the expression

$$(\beta_2 - x_0)^2 + \beta_1^2 x_0 \geqq (\alpha_2 - x_0)^2 + \alpha_1^2 x_0 \,. \tag{53}$$

Expanding this expression and substituting for α's and β's from (6) and (7), we obtain

$$(b_1 b_2)^2 + (b_1^2 + b_2^2)x_0 \geqq (a_1 a_2)^2 + (a_1^2 + a_2^2)x_0 \,. \tag{54}$$

For

$$| \, a_1 a_2 \, | \geqq | \, b_1 b_2 \, | \tag{55}$$

then

$$(a_1 a_2)^2 \geqq (b_1 b_2)^2. \tag{56}$$

Thus (56) can be rewritten as

$$(a_1 a_2)^2 = (b_1 b_2)^2 + \epsilon \tag{57}$$

where ϵ is a positive quantity. Substituting (57) into (54) we get

$$(b_1 b_2)^2 + (b_1^2 + b_2^2)x_0 \geqq (b_1 b_2)^2 + \epsilon + (a_1^2 + a_2^2)x_0 \,. \tag{58}$$

Simplifying and dividing both sides by x_0 yields

$$b_1^2 + b_2^2 \geqq a_1^2 + a_2^2 + \frac{\epsilon}{x_0}. \tag{59}$$

Since ϵ and x_0 are positive quantities their ratio may be deleted without altering the inequality of (59). Thus we have shown that the realizability requirements for second degree structures VII and IX are given by the pair of expressions

$$b_1^2 + b_2^2 \geqq a_1^2 + a_2^2 \tag{60}$$

and

$$| \, a_1 a_2 \, | \geqq | \, b_1 b_2 \, |.$$

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