Error Rate Considerations for Coherent Phase-Shift Keyed Systems with Co-Channel Interference

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(Manuscript received July 23, 1968)

In this paper we present a theoretical analysis of the performance of an m-phase coherent phase-shift keyed system in the presence of random gaussian noise and interference. An explicit expression is given for the probability of error of the phase angle of the received signal; we show that this probability of error can be expressed as a converging power series. We show that the coefficients of this series are expressible in terms of well-known and well-tabulated functions, and we give methods of evaluating the character error rates of the systems. We also show that this error rate is minimum when all the interference power is concentrated in a single interferer, and that it attains its maximum $[P_m]_{max}$ *when the total interference power is equally distributed amongst the K interferers. The limiting case when* K *goes to infinity is considered. The cases of* K = 1, *and* m = 2, 4, 8, *and* 16 *are treated in some detail, and the results are given graphically. The usefulness of the results presented in this paper is that the designer can have at his disposal very simple expressions with which to evaluate the performance of any given Coherent Phase-Shift Keyed system when the received signal is corrupted by both interference and random gaussian noise.*

I. INTRODUCTION

The performance of coherent phase-shift keyed (CPSK) systems has been investigated by many authors;[1–5] in the transmission of information the CPSK system has been shown to be one of the most efficient techniques for trading bandwidth for signal-to-noise ratio. However, the type of noise considered by these authors is almost always limited to be random gaussian noise although most authors admit that interference other than normal noise must be considered in the design of any modulation scheme for digital transmission.

Consider the following situation. In the frequency bands above 10 GHz where the signal attenuation resulting from rain storms could be very severe, close spacings of the repeaters are almost always mandatory for reliable communication from point to point and for all periods of time.[6] In such cases the problem of interference may be much more important than the problem of noise in the optimum detection of the desired signal; hence it is very desirable to evaluate the performance of a CPSK system with co-channel and adjacent channel interference so that, for the selection of an optimum transmission scheme, comparative advantages of CPSK over other broadband modulation techniques (like FM) in combating interference can be determined.

We consider in this paper the performance of a CPSK system when the received signal is corrupted by both interference and random gaussian noise.* We first discuss binary (2-phase) and quaternary (4-phase) CPSK systems and show that exact expressions can be obtained for their probability of error P_m. These expressions are in the form of infinite power series which are shown to converge for all values of signal-to-noise ratio and for all signal-to-interference ratios above a certain level determined by the system. For $m = 2$ and 4, these error rates are calculated and the results are given in graphical form.

For $m = 3$ and for $m > 4$ we show that exact expressions for P_m are very complicated functions of signal-to-noise ratio, and signal-to-interference ratios; in this paper we only indicate how these expressions can be obtained. However, we do obtain expressions for upper and lower bounds to P_m and show that the difference between these two bounds is a monotonically decreasing function of signal-to-noise ratio, signal-to-interference ratios, and the number m of phases used in the system. For $m \geqq 4$, signal-to-noise ratio $\rho^2 \geqq 5$ dB,† and for signal-to-interference ratio $1/L^2 \geqq 20$ dB, we show that this difference is less than 5 percent, and that the upper bound can be used as a good approximation to P_m. For $m = 8$ and 16, we calculate these upper bounds and we present the results graphically.

For a given amount of interference power, we show that the character error rate is minimum when all the power is concentrated in a single interferer. If the total number of interferers is K we also show that the error rate P_m reaches its maximum $[P_m]_{\max}$ when the interference power is equally distributed among all the interferers. It

* The word "noise" indicates random gaussian noise corrupting the desired received signal.

† We use the notation $b = a$ dB if $10 \log_{10} b = a$.

follows that $[P_m]_{\max}$ is a monotonically increasing function of K and attains its maximum when K goes to infinity. We show that the case of K going to infinity can be treated in a simple manner.

For the computation of error rates P_m (or upper bounds to P_m, $m > 2$) it is necessary to calculate the central moments μ_{2n}'s of a certain random variable η defined in terms of the K interfering carriers. For large values of K the conventional method of evaluating μ_{2n}'s can be rather tedious; we give some simple methods of evaluating these moments.

In conclusion, this paper determines the performance of m-phase CPSK systems for the important case of signals corrupted by random gaussian noise and interference. The cases of $m = 2$, 4, 8, and 16 are treated in some detail.

II. PHASE ANGLE DISTRIBUTION IN CPSK SYSTEMS

Let us consider an m-phase CPSK system. We assume that there is a steady received signal* which is corrupted by random gaussian noise and interference. The gaussian noise is assumed to have zero mean and variance σ^2. The signals under consideration consist of phase-modulation pulses of specified width transmitted at a known repetition rate; we assume that there are K interferers, each interferer having the same form as the signal.

If we assume that each signal transmitted has a duration T, the received signal waveform in the absence of noise during the Nth interval can be represented as

$$s_N(t) = (2S)^{\frac{1}{2}} \cos (\omega_0 t + \theta), NT \leqq t \leqq (N + 1)T, \qquad (1)$$

where S is the received signal power, ω_0 is the angular frequency of the signal, and θ will have some value in the discrete set $2\pi k/m$, $0 \leqq k \leqq m - 1$, corresponding to the Nth message. All m messages are assumed to be equally likely. In the absence of noise and interference, the set of m possible received signals is described by a set of m equally-spaced vectors in the complex plane as shown in Fig. 1. The noise and interference corrupting the signal distort the signal both in amplitude and in phase; a zero-phase signal (corresponding to $k = 0$), as disturbed by noise and interference, is also shown in Fig. 1.

If we now assume that powei in the jth interferer is I_j, the jth inter-

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* In this paper we do not consider the effects of fading on the error rates of CPSK systems. The effects of fading can usually be accounted for by a further integration of error rates obtained in this paper.[7]

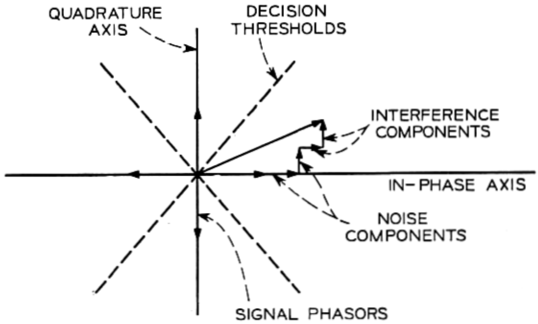

Fig. 1 — Phasor representation of CPSK signals for $m = 4$.

ferer as received during the Nth interval can be represented as*

$$i_{jN}(t) = (2I_j)^{\frac{1}{2}} \cos \{\omega_j t + \theta_j + \mu_j\}, NT \leqq t \leqq (N+1)T \qquad (2)$$

where ω_j is the angular frequency of the jth interferer, θ_j is some value in the discrete set $(2\pi/m)\, k$, $0 \leqq k \leqq m-1$, and the probability density $\pi_{\mu_j}(\mu_j)$ of μ_j is given by

$$\pi_{\mu_j}(\mu_j) = \begin{cases} \dfrac{1}{2\pi}, & 0 \leqq \mu_j < 2\pi \\ 0, & \text{otherwise.} \end{cases} \qquad (3)$$

Since the K interferers are assumed to originate from K different sources, it is reasonable to assume that all μ_j's are statistically independent of each other and are also independent of gaussian noise $n(t)$.

The total received signal during the Nth interval can then be written as

$$r_N(t) = (2S)^{\frac{1}{2}} \cos(\omega_0 t + \theta) + \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \cos(\omega_j t + \theta_j + \mu_j) + n(t),$$
$$NT \leqq t \leqq (N+1)T \qquad (4)$$

where $n(t)$ has zero mean and variance σ^2.

Assuming that the receiver used in the system detects only the phase angle Φ of $r_N(t)$ and does not respond to its amplitude variations,† we can write[8]

* We assume that all i_{jN}'s, $1 \leqq j \leqq K$, are in the passband of the CPSK receiver used in the system.

† This can be achieved in practice by using an ideal limiter at the front end of the receiver. If $A(t)e^{j\varphi(t)}$ is the input to an ideal limiter, its output is given by $A_0 e^{j\varphi(t)}$ where A_0 is a constant.

$$\Phi = \tan^{-1} \frac{\hat{r}_N(t)}{r_N(t)} - \omega_0 t \tag{5}$$

where $\hat{r}_N(t)$ is the Hilbert transform of $r_N(t)$ and is given by

$$\hat{r}_N(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{r_N(\tau)}{t - \tau} \, d\tau. \tag{6}$$

Let us write

$$n(t) = I_c \cos (\omega_0 t + \theta) - I_s \sin (\omega_0 t + \theta). \tag{7}$$

We can show[9] that I_c and I_s are two independent gaussian random variables each distributed with mean zero and variance σ^2.* From (4)–(7), we can now show that

$$\Phi = \theta + \tan^{-1} \cdot \frac{I_s + \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \sin [(\omega_j - \omega_0)t + \theta_j - \theta + \mu_j]}{(2S)^{\frac{1}{2}} + I_c + \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \cos [(\omega_j - \omega_0)t + \theta_j - \theta + \mu_j]} . \tag{8}$$

Let us now write

$$\rho = \frac{S^{\frac{1}{2}}}{\sigma} , \tag{9}$$

$$\frac{I_s}{(2S)^{\frac{1}{2}}} = v, \tag{10}$$

$$\frac{I_c}{(2S)^{\frac{1}{2}}} = u, \tag{11}$$

$$\delta = \sum_{j=1}^{K} R_j \sin \lambda_j , \tag{12}$$

$$\eta = \sum_{j=1}^{K} R_j \cos \lambda_j , \tag{13}$$

where

$$R_j = \left(\frac{I_j}{S}\right)^{\frac{1}{2}}, \tag{14}$$

and

$$\lambda_j = (\omega_j - \omega_0)t + \theta_j - \theta + \mu_j . \tag{15}$$

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* It is assumed that the spectrum of gaussian noise is symmetrical around the frequency $\omega = \omega_0$.

Let us also denote the set $\{\lambda_1, \lambda_2, \cdots, \lambda_i, \cdots, \lambda_K\}$ of random variables λ_i's by $\underset{\sim}{\lambda}$.

We can now write eq. (8) as

$$\Phi = \theta + \tan^{-1} \frac{v + \delta}{1 + u + \eta} \tag{16}$$

where δ and η are functions of $\underset{\sim}{\lambda}$.

If K is a finite number, we can show[10] that the probability density $p_\eta(\eta)$ can be represented as*

$$p_\eta(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta t} \prod_{j=1}^{K} J_0(tR_j)\, dt, \tag{17}$$

where $J_0(x)$ is the Bessel function of the first kind and of order zero. For $K = 1$, we can show that[10]

$$p_\eta(\eta) = \begin{cases} \dfrac{1}{\pi} \dfrac{1}{(R_1^2 - \eta^2)^{\frac{1}{2}}}, & |\eta| \leqq R_1 \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

For $K = 2$, $p_\eta(\eta)$ can be expressed in terms of elliptic functions, and for $K > 2$, no closed form expressions can be obtained for $p_\eta(\eta)$. In Ref. 10 $p_\eta(\eta)$ has been expressed as a converging sum and has been evaluated for $K = 10$. It is easy to show that

$$p_\eta(\eta) = 0, \quad \text{for} \quad |\eta| > \sum_{j=1}^{K} R_j\,, \tag{19}$$

and

$$\int_{-\infty}^{\infty} p_\eta(\eta) e^{-i\eta t}\, d\eta = \prod_{j=1}^{K} J_0(tR_j). \tag{20}$$

III. CPSK RECEIVER

An ideal CPSK receiver is shown in Fig. 2. The ideal limiter removes all the amplitude variations of the received signal before it reaches the ideal phase detector of the system. We shall assume maximum likelihood detection for our analysis of the receiver. Let us assume that the receiver shown in Fig. 2 has zero-width decision thresholds as shown in Fig. 1.

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* We can also write similar expressions for $p_\delta(\delta)$.

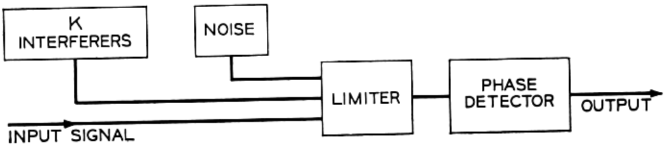

Fig. 2 — CPSK receiver.

3.1 *Error Rates for Binary CPSK Systems*

For a binary CPSK system the set of two possible received signals in the absence of noise and interference is shown in Fig. 3. The noise and interference corrupting the desired signal distort the signal both in amplitude and in phase; a zero-phase signal (corresponding to $k = 0$) as disturbed by noise and interference is also shown in Fig. 3.

When the message $k = 0$ is sent, and when the phase angle Φ of the received signal lies in the second and third quadrants of the complex plane shown in Fig. 3, an error is made in detecting the received signal. For a given ρ^2, and for an arbitrary set of λ_i's let us assume that the origin of the gaussian noise vector is at point G in Fig. 3. When the terminus or tip of the gaussian noise vector lies in the left half of the complex plane (the shaded portion of Fig. 3) an error is made by the receiver. Since I_c and I_s are two independent gaussian random variables and since they are distributed independently of λ_i's, the probability $P_2(\lambda)$ that the terminus of the gaussian noise vector lies in the

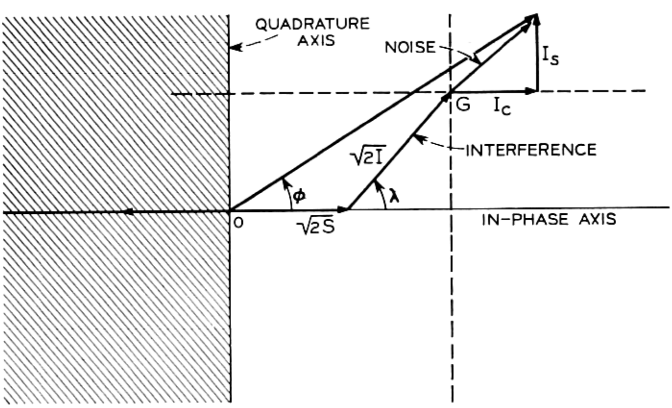

Fig. 3 — Phasor representation of CPSK signals for $m = 2$. I_c and I_s are the in-phase and quadrature components of gaussian noise corrupting the desired received signal.

left half of the complex plane is given by*

$$\begin{aligned} P_2(\lambda) &= \Pr[-\infty < I_s < \infty] \\ &\quad \cdot \Pr\left[-\infty < I_c < -\left((2S)^{\frac{1}{2}} + \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \cos \lambda_j\right)\right] \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} \int_{-\infty}^{-\left\{(2S)^{\frac{1}{2}} + \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \cos \lambda_j\right\}} \exp(-t^2/2\sigma^2)\, dt. \end{aligned} \tag{21}$$

We can show from Equation (21) that

$$P_2(\lambda) = \tfrac{1}{2} \operatorname{erfc}[\rho + \rho\eta], \tag{22}$$

where

$$\operatorname{erf}(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_0^x \exp(-u^2)\, du \tag{23}$$

and

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x). \tag{24}$$

The character error rate P_2 for a binary CPSK system is, therefore, given by

$$P_2 = E[P_2(\lambda)], \tag{25}$$

where $E[P_2(\lambda)]$ represents the mathematical expectation of the random function $P_2(\lambda)$.

From Equations (22) and (25) we have

$$P_2 = \tfrac{1}{2} E[\operatorname{erfc}\{\rho + \rho\eta\}]. \tag{26}$$

We now note that we can write[11, 12]

$$\operatorname{erfc}[x + z] = \operatorname{erfc}[x] + \frac{2}{\pi^{\frac{1}{2}}} \exp(-x^2) \sum_{\ell=1}^{\infty} (-1)^{\ell} H_{\ell-1}(x) \frac{z^{\ell}}{\ell!}, \tag{27}$$

where $H_n(x)$ represents the Hermite polynomial of order n. The series converges for all values of $x + z$ such that

$$x + z \geqq 0. \tag{28}$$

From Equations (26) and (27) we have

$$P_2 = \tfrac{1}{2} \operatorname{erfc}(\rho) + \frac{1}{\pi^{\frac{1}{2}}} \exp(-\rho^2) \sum_{\ell=1}^{\infty} (-1)^{\ell} H_{\ell-1}(\rho) \frac{\rho^{\ell}}{\ell!} E(\eta^{\ell}). \tag{29}$$

* The notation $\Pr[a < x < b]$ denotes the probability that the random variable x satisfies the inequality $a < x < b$. It may also be noted that $P_2(\underline{\lambda})$ is a conditional probability conditioned on $\underline{\lambda}$.

Let us denote by μ_n the nth central moment of η.[8] It can then be shown that[10]

$$\mu_{2\ell+1} = 0, \qquad \ell = 0, 1, 2, \cdots . \tag{30}$$

We, therefore, have

$$P_2 = \tfrac{1}{2} \operatorname{erfc}(\rho) + \frac{1}{\pi^{\frac{1}{2}}} \exp(-\rho^2) \sum_{\ell=1}^{\infty} H_{2\ell-1}(\rho) \frac{\rho^{2\ell}}{(2\ell)!} \mu_{2\ell} . \tag{31}$$

The series given in Equation (31) converges for all values of ρ and R_j's such that

$$\rho + \rho\eta \geqq 0 \qquad \text{for all } \lambda. \tag{32}$$

From Equations (13) and (32) we can show that the series converges when

$$\Omega \leqq 1, \tag{33}$$

where

$$\Omega = \sum_{j=1}^{K} R_j . \tag{34}$$

Equation (34) states that the sum of the normalized amplitudes of all the interfering carriers may not exceed the normalized amplitude of the desired signal. This is not a very stringent requirement and it is almost always satisfied when low error rates are desired.

Since we also know that*

$$\left\{\frac{1}{K} \sum_{j=1}^{K} R_j\right\} \geqq \prod_{j=1}^{K} R_j^{1/K}, \tag{35}$$

when Equation (33) is satisfied, we have

$$\prod_{j=1}^{K} \left(\frac{I_j}{S}\right) \leqq \left(\frac{1}{K}\right)^{2K}. \tag{36}$$

The expression S/I_j denotes the signal-to-interference ratio of the jth interfering carrier.

When there is only one interfering carrier we can show that,

$$\mu_{2\ell} = R^{2\ell} \frac{(2\ell)!}{2^{2\ell}\{\ell!\}^2}, \tag{37}$$

* Equation (35) states that the arithmetic mean of a set of real variables is always greater than or equal to its geometric mean.

and equation (31) can be written as

$$P_2 = \tfrac{1}{2} \operatorname{erfc}(\rho) + \frac{1}{\pi^{\frac{1}{2}}} \exp(-\rho^2) \sum_{\ell=1}^{\infty} H_{2\ell-1}(\rho) \frac{\left[\frac{\rho R}{2}\right]^{2\ell}}{[\ell!]^2}. \tag{38}$$

The series in equation (38) converges for all signal-to-interference ratios such that

$$I/S \leqq 1. \tag{39}$$

The values of P_2 have been calculated from equation (38) and the results are given in graphical form in Fig. 4.*

Notice that we need to calculate only the even order moments μ_{2n}'s of the random variable η in determining P_2 from equation (31). Some methods of calculating these moments are given in Appendix A.

3.2 *Error Rates for Quaternary CPSK Systems*

Let us now consider a 4-phase CPSK system. For this system the set of four possible signal phasors and the four optimum decision thresholds are shown in Fig. 5. A signal phasor (corresponding to $k = 1$) as disturbed by noise and interference is also shown in Fig. 5.

For a given set of λ_j's let us assume that the gaussian noise is represented by a vector from the point G. If the message $k = 1$ is transmitted, an error is made if the received phase angle lies in areas marked 1, 2, and 3. The phase angle of the received signal will lie in areas marked 1, 2, and 3 if the terminus of the gaussian noise vector lies in this area of the plane.[14]

We notice that

$$GA = (2S)^{\frac{1}{2}} \sin \frac{\pi}{4} + \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \sin \left(\frac{\pi}{4} + \lambda_j\right), \tag{40}$$

and

$$GB = (2S)^{\frac{1}{2}} \sin \frac{\pi}{4} + \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \cos \left(\frac{\pi}{4} + \lambda_j\right). \tag{41}$$

Let us denote by $\Pi_{k1,k2,\cdots,kn}(\lambda)$ the probability that the terminus of the gaussian noise vector lies in area

$$\bigcup_{i=1}^{n} k_i.\dagger$$

* The results obtained in Fig. 4 indicate that the error rates obtained in Refs. 13, 14, and 15 agree well with those obtained in this paper.

† The notation $\cup_{i=1}^{n} k_i$ denotes the union of all elements of the set $\{k_1, k_2, \cdots, k_n\}$.

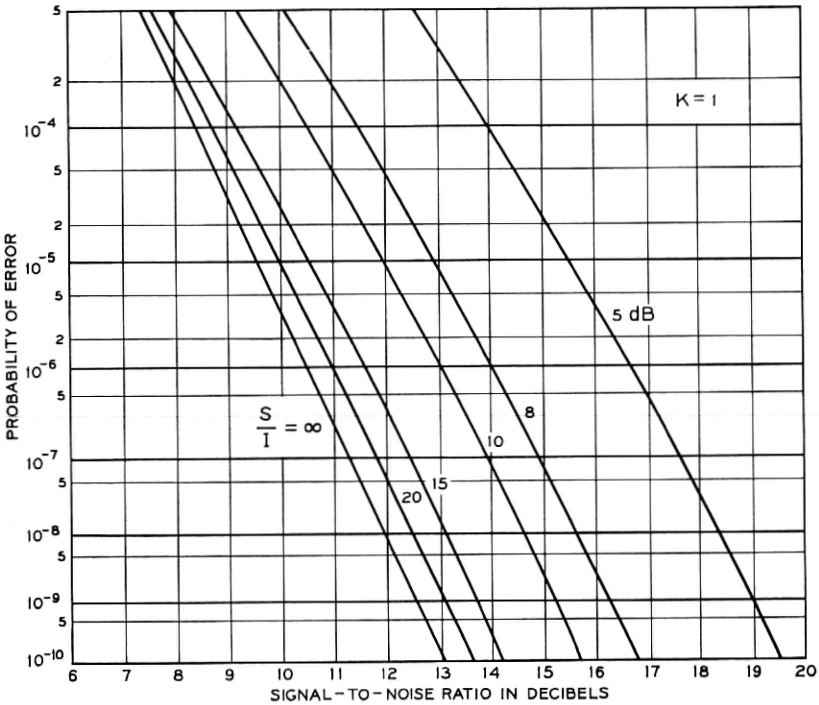

Fig. 4 — Error rates for a 2-phase CPSK system with one interferer.

We can show from Fig. 5 that

$$\Pi_{1,2}(\lambda) = \tfrac{1}{2} \operatorname{erfc}\left[\rho \sin\frac{\pi}{4} + \rho \sum_{j=1}^{K} R_j \cos\left(\frac{\pi}{4} + \lambda_j\right)\right], \tag{42}$$

$$\Pi_{2,3}(\lambda) = \tfrac{1}{2} \operatorname{erfc}\left[\rho \sin\frac{\pi}{4} + \rho \sum_{j=1}^{K} R_j \sin\left(\frac{\pi}{4} + \lambda_j\right)\right], \tag{43}$$

and

$$\Pi_2(\lambda) = \tfrac{1}{4} \operatorname{erfc}\left[\rho \sin\frac{\pi}{4} + \rho \sum_{j=1}^{K} R_j \cos\left(\frac{\pi}{4} + \lambda_j\right)\right] \cdot \operatorname{erfc}\left[\rho \sin\frac{\pi}{4} + \rho \sum_{j=1}^{K} R_j \sin\left(\frac{\pi}{4} + \lambda_j\right)\right]. \tag{44}$$

The probability $P_4(\lambda)$ of an error due to noise is, therefore, given by

$$P_4(\lambda) = \Pi_{1,2}(\lambda) + \Pi_{2,3}(\lambda) - \Pi_2(\lambda). \tag{45}$$

The probability of an error due to noise and interference is therefore

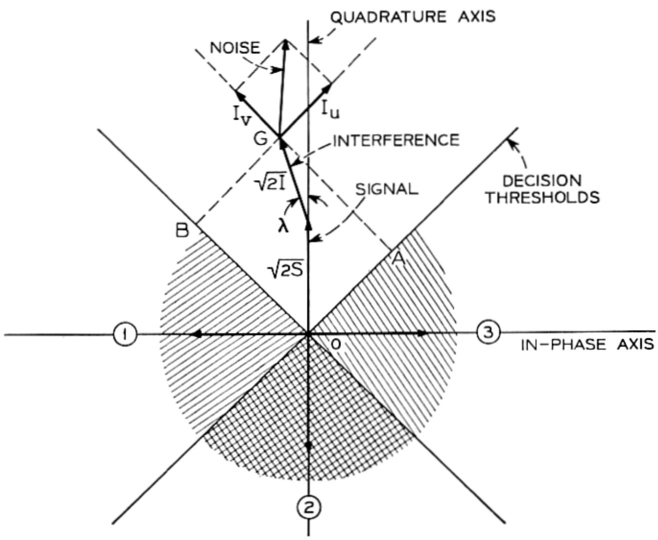

Fig. 5 — Phasor representation of CPSK signals for $m = 4$. I_u and I_v are two orthogonal components of gaussian noise.

given by

$$P_4 = E[P_4(\lambda)]. \tag{46}$$

From equations (27), and (42) through (46) we can show that

$$\begin{aligned} P_4 = {} & \operatorname{erfc}\left[\rho \sin \frac{\pi}{4}\right] - \frac{1}{4} \operatorname{erfc}^2\left[\rho \sin \frac{\pi}{4}\right] \\ & + \frac{1}{(\pi)^{\frac{1}{2}}} \exp\left(-\rho^2 \sin^2 \frac{\pi}{4}\right)\left\{2 - \operatorname{erfc}\left[\rho \sin \frac{\pi}{4}\right]\right\} \\ & \cdot \sum_{\ell=1}^{\infty} H_{2\ell-1}\left(\rho \sin \frac{\pi}{4}\right) \frac{\rho^{2\ell}}{(2\ell)!} \mu_{2\ell} - \frac{1}{\pi} \exp\left(-2\rho^2 \sin^2 \frac{\pi}{4}\right) \\ & \cdot \sum_{\ell=1}^{\infty} \sum_{j=1}^{\infty} \frac{H_{2\ell-1}\left(\rho \sin \frac{\pi}{4}\right) H_{2j-1}\left(\rho \sin \frac{\pi}{4}\right)}{(2\ell)!\,(2j)!} \rho^{2(\ell+j)} \mu^*_{2\ell,2j} \end{aligned} \tag{47}$$

where $\mu^*_{2\ell,2j}$'s are given by

$$\begin{aligned} \mu^*_{2\ell,2j} = {} & \frac{1}{(2\pi)^K} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_K \left\{\sum_{j=1}^{K} R_j \cos \theta_j\right\}^{2\ell} \\ & \cdot \left\{\sum_{\ell=1}^{K} R_\ell \sin \theta_\ell\right\}^{2j} . \end{aligned} \tag{48}$$

For a given set of R_j's, $\mu^*_{2\ell,2j}$'s may be evaluated from equation (48). For $K = 1$, we can show that

$$\mu^*_{2\ell,2s} = R^{2(\ell+s)} \frac{(2\ell)!\,(2s)!}{2^{2(\ell+s)}\ell!\,s!\,(\ell+s)!}. \tag{49}$$

For $K = 1$, we have calculated P_4 from equation (47) and the results are presented in Fig. 6.

We can again show that the series given in equation (47) converges for all values of ρ and R_j's such that

$$\Omega \leqq \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}. \tag{50}$$

For $K = 1$, equation (50) becomes

$$S/I \geqq 2. \tag{51}$$

Equation (50) is usually satisfied by systems encountered in practice.

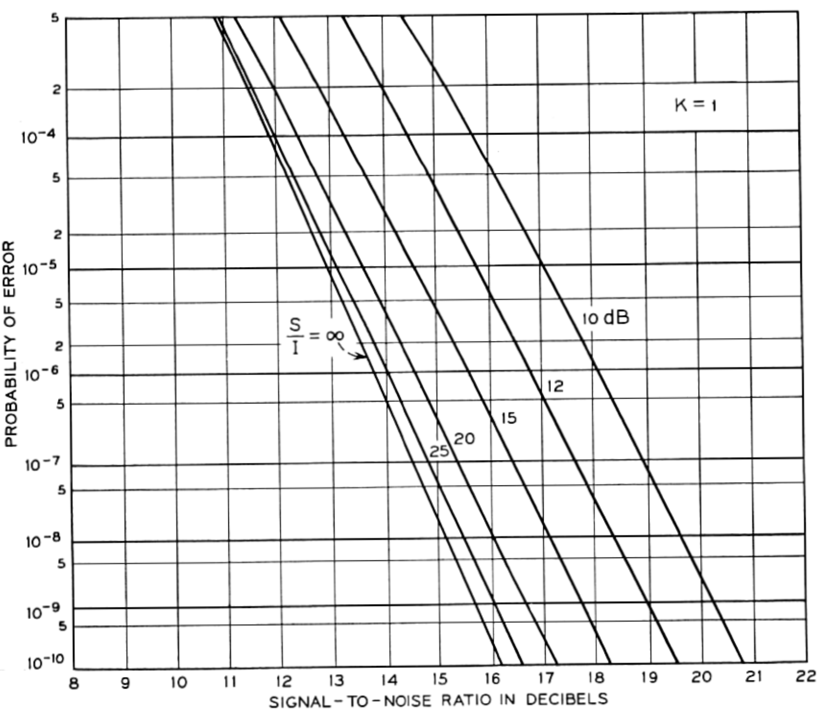

Fig. 6 — Error rates for a 4-phase CPSK system with one interferer.

3.3 *Error Rates for Multilevel CPSK Systems*

In this section we shall investigate the performance of a multilevel ($m \geqq 3$) CPSK Systems and indicate a method in which an exact expression can be obtained for the probability of error of the system. This exact expression is a very complicated function of signal-to-noise ratio and R_i's; we do not obtain this expression in this paper. However, we obtain upper and lower bounds to P_m and show that the difference between these two bounds is a monotonically decreasing function of ρ, m, and signal-to-interference ratios. For $K = 1$, $m \geqq 4$, $\rho^2 \geqq 5$ dB, and $S/I \geqq 20$ dB, we show that this difference is less than 5 percent of the lower bound, and hence the upper bound is a good approximation to P_m when low error rates are desired.

A signal phasor corresponding to $k = 0$ as disturbed by noise and interference is shown in Fig. 7. For a given set of λ_i's let us again assume that random gaussian noise is represented by a vector from the point G shown in Fig. 7. If the message $k = 0$ is transmitted, an error is made if the terminus of the noise vector lies in areas marked 1, 2, and 3.

We can show that *

$$\Pi_{1,2}(\lambda) = \tfrac{1}{2}\,\text{erfc}\left[\rho \sin\frac{\pi}{m} + \rho \sum_{i=1}^{K} R_i \sin\left(\frac{\pi}{m} - \lambda_i\right)\right] \tag{52}$$

and

$$\Pi_{2,3}(\lambda) = \tfrac{1}{2}\,\text{erfc}\left[\rho \sin\frac{\pi}{m} + \rho \sum_{i=1}^{K} R_i \sin\left(\frac{\pi}{m} + \lambda_i\right)\right]. \tag{53}$$

The probability of error due to noise is, therefore, given by

$$P_m(\lambda) = \Pi_{1,2}(\lambda) + \Pi_{2,3}(\lambda) - \Pi_2(\lambda). \tag{54}$$

By looking at Fig. 7 we can see that no simple expression can be obtained for $\Pi_2(\lambda)$ (except when $m = 4$). $\Pi_2(\lambda)$ denotes the probability that the terminus of the gaussian noise vector lies in area 2; we shall now obtain upper and lower bounds to $\Pi_2(\lambda)$. Assume that

$$\rho \sin\frac{\pi}{m} - \rho \sum_{i=1}^{K} R_i \geqq 0 \tag{55}$$

* Note that

$$GA = (2S)^{\frac{1}{2}} \sin\frac{\pi}{m} + \sum_{i=1}^{K} (2I_i)^{\frac{1}{2}} \sin\left(\frac{\pi}{m} - \lambda_i\right)$$

and

$$GB = (2S)^{\frac{1}{2}} \sin\frac{\pi}{m} + \sum_{i=1}^{K} (2I_i)^{\frac{1}{2}} \sin\left(\frac{\pi}{m} + \lambda_i\right).$$

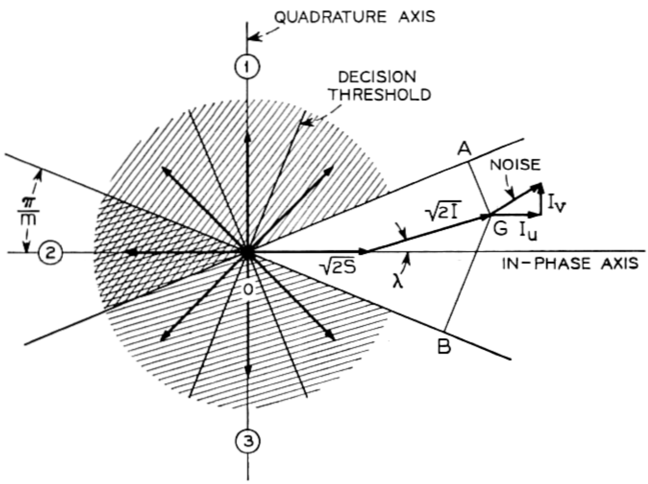

Fig. 7 — Phasor representation of CPSK signals for $m = 8$. I_u and I_v are the in-phase and quadrature components of gaussian noise.

so that $\Pi_{1,2}(\lambda)$ and $\Pi_{2,3}(\lambda)$ are nonnegative for all values of λ. If equation (55) is satisfied, it is easy to see (see Fig. 7) that

$$\Pi_2(\lambda) \geqq 0 \qquad \text{for all } \lambda, \tag{56}$$

and $\Pi_2(\lambda)$ reaches its maximum when*

$$\eta = -\sum_{j=1}^{K} R_j = -\Omega. \tag{57}$$

For this value of η it can be shown (see Fig. 8) that

$$\Pi_2(\lambda) = \frac{1}{\pi\sigma^2} \int_{-\infty}^{-y_0} \exp(-y^2/2\sigma^2)\, dy \int_0^{-(y+y_0)\tan \pi/m} \exp(-x^2/2\sigma^2)\, dx$$

or

$$\Pi_2(\lambda) = \frac{1}{\pi\sigma^2} \int_{y_0}^{\infty} \exp(-y^2/2\sigma^2)\, dy \int_0^{(y-y_0)\tan \pi/m} \exp(-x^2/2\sigma^2)\, dx \tag{58}$$

where

$$y_0 = (2S)^{\frac{1}{2}}[1 - \Omega]. \tag{59}$$

Since we always have

$$0 \leqq \exp(-x^2/2\sigma^2) \leqq 1 \text{ for all real } x, \tag{60}$$

* We can show that $\eta = -\Omega$ when all λ_j's are odd multiples of π.

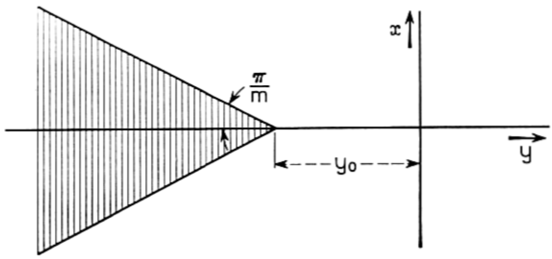

Fig. 8 — Computation of lower bound to P_m.

we have

$$\Pi_2(\lambda) \leqq \frac{\tan \pi/m}{\pi\sigma^2} \int_{y_0}^{\infty} (y - y_0) \exp(-y^2/2\sigma^2)\, dy. \tag{61}$$

Equation (61) can be simplified to*

$$\Pi_2(\lambda) \leqq Q_{m0} = \frac{\tan \pi/m}{\pi} \exp[-\rho^2(1 - \Omega)^2] \cdot [1 - (\pi)^{\frac{1}{2}}\rho(1 - \Omega) \exp[\rho^2(1 - \Omega)^2] \operatorname{erfc}\{\rho(1 - \Omega)\}]. \tag{62}$$

From equations (54), (56), and (62) we have

$$\Pi_{1,2}(\lambda) + \Pi_{1,3}(\lambda) - Q_{m0} \leqq P_m(\lambda) \leqq \Pi_{1,2}(\lambda) + \Pi_{1,3}(\lambda). \tag{63}$$

Since

$$P_m = E[P_m(\lambda)], \tag{64}$$

we can show from equations (52), (53), (55), and (63) that

$$Q_m - Q_{m0} \leqq P_m \leqq Q_m \tag{65}$$

where

$$Q_m = \operatorname{erfc}\left(\rho \sin \frac{\pi}{m}\right) + \frac{2}{(\pi)^{\frac{1}{2}}} \exp\left(-\rho^2 \sin^2 \frac{\pi}{m}\right) \sum_{\ell=1}^{\infty} \frac{H_{2\ell-1}\left(\rho \sin \frac{\pi}{m}\right)}{(2\ell)!} \rho^{2\ell}\mu_{2\ell} . \tag{66}$$

---

* For large values of ρ and small values of Ω, Q_{m0} is approximately equal to

$$\frac{\tan \pi/m}{2\pi} \frac{\exp[-\rho^2(1 - \Omega)^2]}{\rho^2(1 - \Omega)^2}.$$

The series given in equation (66) converges if equation (55) is satisfied, or if

$$\Omega \leqq \sin \frac{\pi}{m}. \tag{67}$$

When low error rates are desired, equation (67) must be satisfied.

Equation (65) gives an upper and a lower bound to P_m; as can be seen from equation (62) the difference Q_{m0} between these two bounds is a monotonically decreasing function of ρ, m, and signal-to-interference ratios. From equation (65) we have

$$-\frac{Q_{m0}}{Q_m - Q_{m0}} \leqq \frac{P_m - Q_m}{P_m} \leqq 0. \tag{68}$$

For $K = 1$, $R_1 = \frac{1}{10}$, and for $m = 4$, 8, and 16, we have plotted in Fig. 9 $Q_{m0}/(Q_m - Q_{m0})$ as a function of ρ^2. From Fig. 9 we see that $Q_{m0}/(Q_m - Q_{m0})$ is less than 5 percent for $\rho^2 \geqq 5$ dB and for $m \geqq 4$. We can, therefore, use Q_m as a good approximation to P_m for high values of signal-to-noise ratio ($\rho^2 \geqq 5$ dB) and for high values of signal-to-interference ratio ($1/R_1 \geqq 10$ dB).

In these cases we then have

$$P_m \approx \operatorname{erfc}\left(\rho \sin \frac{\pi}{m}\right) + \frac{2}{(\pi)^{\frac{1}{2}}} \exp\left(-\rho^2 \sin^2 \frac{\pi}{m}\right) \sum_{k=1}^{\infty} \frac{H_{2k-1}\left(\rho \sin \dfrac{\pi}{m}\right)}{(2k)!} \rho^{2k} \mu_{2k} . \tag{69}$$

For $K = 1$, and for $m = 8$ and 16, the values of P_m obtained from equation (69) are given in Figs. 10 and 11. The error made in this approximation can be estimated from equation (68).

IV. ERROR RATE AS A FUNCTION OF NUMBER OF INTERFERERS

Let us now investigate how P_m varies as a function of K for a total given interference power. Let us assume that the total interference power is some number SL^2 where

$$\sum_{i=1}^{K} I_i = SL^2. \tag{70}$$

This power SL^2 can be distributed among the K interferers in a variety of ways; every one of these distributions will in general lead to a different character error rate of the system. Let us find out those

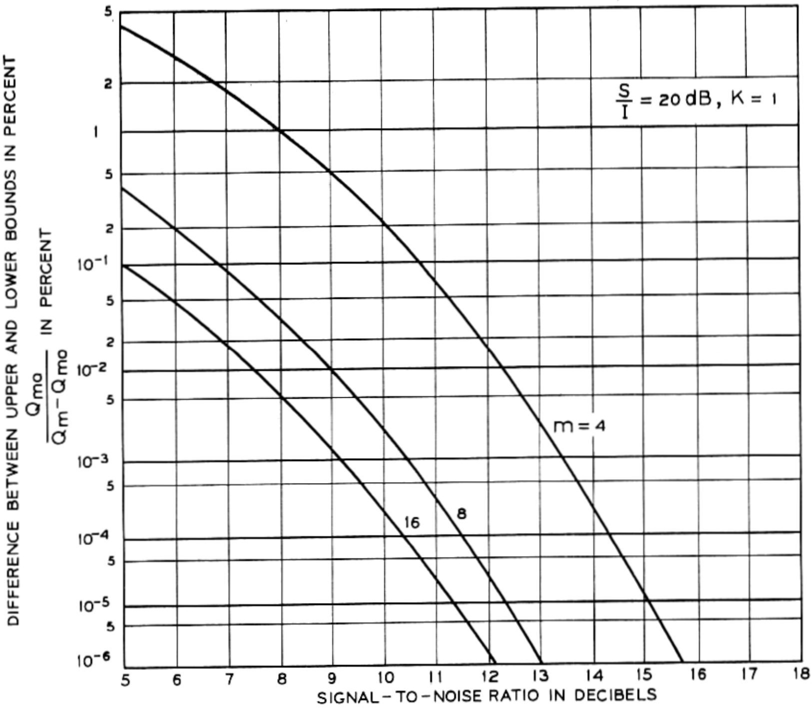

Fig. 9 — $Q_{m0}/(Q_m - Q_{m0})$ as a function of ρ.

distributions of power (if they exist) which make this character error rate a maximum or a minimum.

4.1 *Error Rates for K Interferers*

Let us first consider the case when $\rho \gg 1$ and $\Omega \ll 1$. In this case the series corresponding to P_m (or Q_m) converges very rapidly; let us say that the first N terms of the series are sufficient to evaluate P_m to the desired degree of accuracy.

For all ℓ and z, we have

$$H_{2\ell-1}(z) = 2zH_{2\ell-2}(z) - 2\,(2\ell - 2)H_{2\ell-3}(z). \tag{71}$$

From equation (71) we can show that

$$H_{2j-1}\left(\rho \sin \frac{\pi}{m}\right) \geqq 0, \qquad 1 \leqq j \leqq N, \tag{72}$$

if

$$\rho \geqq \frac{2N - \frac{3}{2}}{\sin \frac{\pi}{m}}, \qquad N > 1. \tag{73}$$

If Equations (72) and (73) are satisfied notice from Equations (31) and (69) that P_m's are monotonically increasing functions of $\mu_{2\ell}$'s, $\ell \geqq 1$. For a given μ_2, it can be shown from Equation (13) that $\mu_{2\ell}$'s $\ell \geqq 2$, reach their minimum when Ω is minimum and they reach their maximum when Ω is maximum.

We can then say that P_m's (or Q_m's) attain their minimum when Ω is minimum and that they are at their maximum when Ω is maximum.

From Figs. 3, 5, and 7 this seems to be true for all values of ρ and Ω which satisfy Equation (55).

Let us now find out when Ω is minimum for a given value of signal-

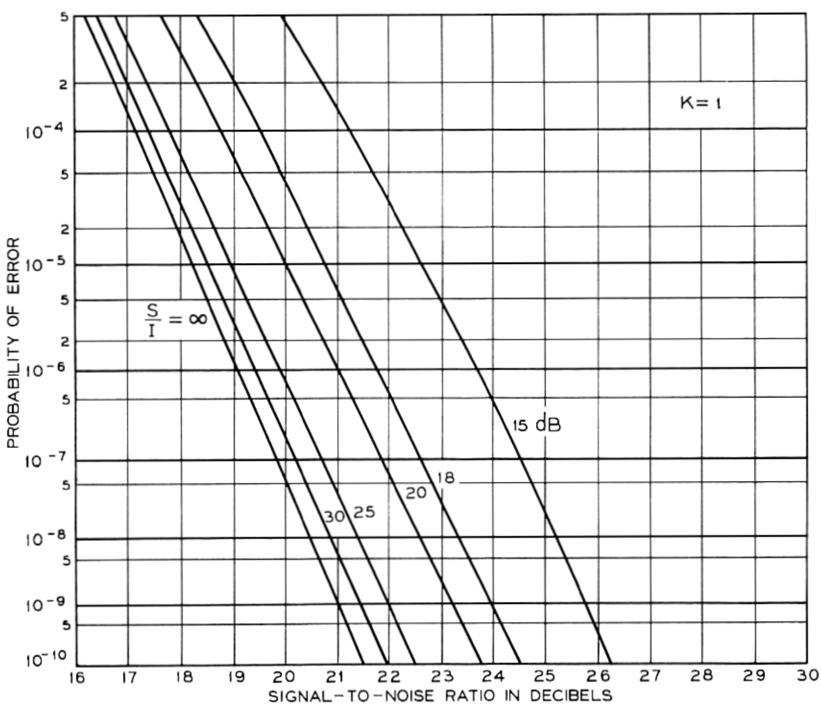

Fig. 10 — Error rates for an 8-phase CPSK system with one interferer.

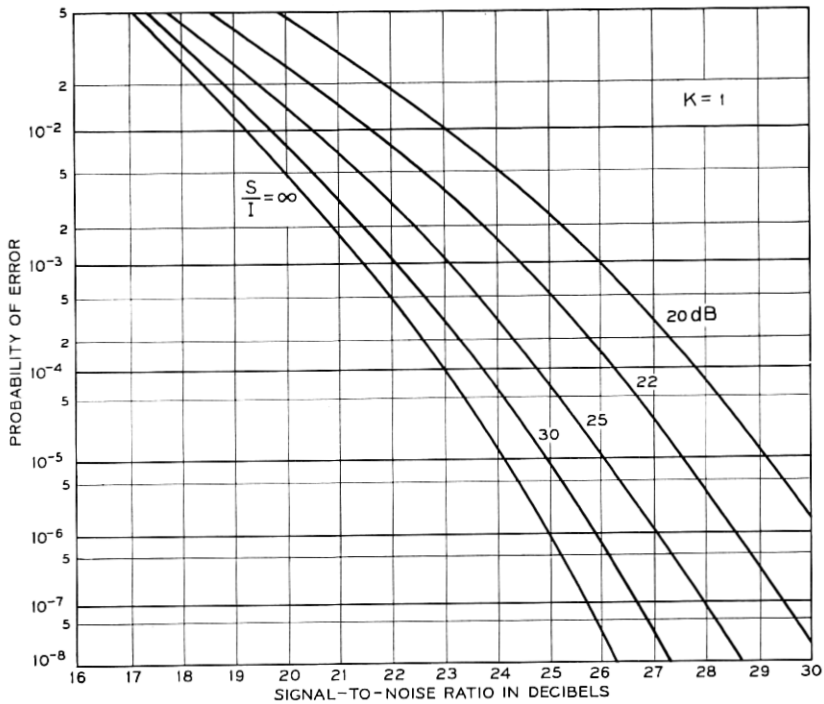

Fig. 11 — Error rates for a 16-phase CPSK system with one interferer.

to-interference ratio. The signal-to-interference ratio $1/L^2$ is given by

$$L^2 = \sum_{j=1}^{K} \frac{I_j}{S}. \tag{74}$$

Clearly Ω is minimum when

$$I_j = SL^2, \qquad 1 \leqq j \leqq K, \tag{75}$$

and

$$I_\ell = 0, \qquad 1 \leqq \ell \leqq K, \ell \neq j. \tag{76}$$

We can then say that the character error rate P_m is minimum when the total interference power is concentrated in a single interferer.

Now from equations (14), (34) and (74), Ω is a maximum when

$$\frac{\partial}{\partial I_j}\left[\sum_{j=1}^{K}\left(\frac{I_j}{S}\right)^{\frac{1}{2}} - \epsilon \sum_{j=1}^{K}\left(\frac{I_j}{S}\right)\right] = 0, \qquad 1 \leqq j \leqq K. \tag{77}$$

ϵ is a constant and is the Lagrange multiplier used in finding the extremum of Ω.

Solving equation (77) we observe that Ω is a maximum or that P_m is a maximum when

$$I_j = \frac{SL^2}{K}, \qquad 1 \leqq j \leqq K \tag{78}$$

or that the total interference power is equally distributed among the K interferers.

Let us now assume that K is a variable number. It is clear from equation (78) that $[P_m]_{\max}$ is a monotonically increasing function of K.

4.2 *Error Rates for a Large Number of Interferers**

Let us now consider the limiting case when K goes to infinity and

$$\sum_{j=1}^{K} I_j = SL^2. \tag{79}$$

We can show[10] that the probability distribution function of†

$$y(t) = \sum_{j=1}^{K} (2I_j)^{\frac{1}{2}} \cos \{\omega_j t + \theta_j + \mu_j\} \tag{80}$$

as K goes to infinity approaches that of gaussian noise with mean zero and variance SL^2 under certain conditions.

In this case we have from equation (4)

$$r_N(t) = (2S)^{\frac{1}{2}} \cos (\omega_0 t + \theta) + y(t) + n(t). \tag{81}$$

Since $y(t)$ and $n(t)$ are independent gaussian random variables their sum

$$b(t) = y(t) + n(t) \tag{82}$$

is also a random gaussian variable with mean zero and variance $SL^2 + \sigma^2$.

From equations (81) and (82) we can write

$$r_N(t) = (2S)^{\frac{1}{2}} \cos (\omega_0 t + \theta) + b(t) \tag{83}$$

where $b(t)$ is a gaussian random variable.

* The results of this section are applicable for any signal-to-noise ratio and any signal-to-interference ratio.

† Ruthroff has shown that for $K \geqq 50$ the distribution of $y(t)$ can be considered to be gaussian in practice for the computation of distortion in PM systems.[16]

The case where $r_N(t)$ can be described by equation (83) has been considered in detail in Ref. 17;* we can easily determine the deterioration in performance produced by interference from the results presented in that paper. For example, suppose that $m = 4$, $S/\sigma^2 = 16$ dB, and $L^2 = -16$ dB. Clearly

$$\frac{\sigma^2}{S} + L^2 = -13 \text{ dB} \tag{84}$$

and P_4 from Ref. 17 is given by

$$P_4 = 7.9 \times 10^{-6}. \tag{85}$$

For the calculation of the effect of interference in CPSK systems, we note that we have not shown the validity of the gaussian approximation of $y(t)$ for $K \gg 1$. However, this assumption seems to be justified for large signal-to-noise ratios and small interference-to-signal ratios.[15]

In conclusion, this section gives methods of evaluating character error rates of CPSK systems for all values of m and for all values of K. It shows that the error rate P_m is minimum when all the interference power is concentrated in a single interferer and that it attains its maximum value $[P_m]_{\max}$ when the interference power is equally distributed amongst all the interferers. We further show that $[P_m]_{\max}$ is a monotonically increasing function of the number K of interferers. We also show that the case, K going to infinity, can be treated and that the deterioration in performance produced by interference can be determined.

V. CONCLUSIONS

A method to evaluate the character error rates of CPSK systems has been presented in this paper. The received signal is assumed to be corrupted by both interference and random gaussian noise. When the number of interferers is very large it can be shown that the interference and random gaussian noise can be combined together to give rise to an equivalent noise source having gaussian properties. The variance of this random variable is the sum of variance of random gaussian noise and total interference power. In this case the analysis of the CPSK system can be done by methods presented in Ref. 17.

When K is a finite number and when $m = 2$ or 4, exact expressions

* The results presented in this paper for $S/I = \infty$ are also sufficient to determine P_m for a large number of interferers.

are given for the probability of error P_m. When $m \geqq 3$, upper and lower bounds to P_m are derived. We show that the difference between these two bounds is a monotonically decreasing function of signal-to-noise ratio ρ^2, signal-to-interference ratio $1/L^2$, and the number m of phases used in the system. For $K = 1$, $m \geqq 4$, $\rho^2 \geqq 5$ dB, and $1/R_1 \geqq 10$ dB we show that this difference is less than 5 per cent, and that the upper bound can be used as a good approximation to P_m.

We then show that for any m-phase CPSK system the character error rates can be expressed in terms of the central moments of a certain random variable η and that they can be calculated to any desired degree of accuracy by using a set of tables or by using a digital computer.

For a total given interference power we show that the character error rate P_m attains its minimum when all the power is concentrated in a single interferer, and that it reaches its maximum $[P_m]_{\max}$ when the power is equally distributed among all the K interferers. It is also shown that $[P_m]_{\max}$ is a monotonically increasing function of K.

The cases of $K = 1$, $m = 2$, 4, 8, and 16, have been treated in some detail and the results are given in graphical form. The required signal-to-noise ratio for any value of signal-to-interference ratio can be determined from these figures.

The usefulness of the presented results is that they provide the designer with some relatively simple expressions with which to evaluate the performance of any given CPSK system with interference and random gaussian noise. The only quantities he must have at his disposal are the central moments of a certain random variable η defined in terms of the K interfering carriers.

VI. ACKNOWLEDGMENT

I would like to express my very sincere thanks to Clyde L. Ruthroff for his help and advice in the preparation of this paper, as well as, for suggesting the approach taken in this paper.

APPENDIX

Evaluation of Central Moments of η

In the computation of character error rates for CPSK systems it is necessary to calculate the even order moments of the random variable η; we shall give in this section two alternate methods to evaluate these moments.

By definition μ_{2n} is given by

$$\mu_{2n} = \frac{1}{(2\pi)^K} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_K \left[\sum_{j=1}^{K} R_j \cos \theta_j \right]^{2n}. \tag{86}$$

By the multinomial theorem

$$\left[\sum_{j=1}^{K} R_j \cos \theta_j \right]^{2n} = \sum \frac{(2n)!}{\prod_{j=1}^{K} n_j!} \prod_{j=1}^{K} (R_j)^{n_j} \cos^{n_j} \theta_j \tag{87}$$

where n_j's are positive integers such that

$$\sum_{j=1}^{K} n_j = 2n. \tag{88}$$

Since θ_j's are statistically independent of each other, and since $\mu_{2\ell+1} = 0$ for all ℓ, we have from Equations (86) and (87)*

$$\mu_{2n} = \sum \frac{(2n)!}{\prod_{\ell=1}^{K} n_\ell!} \prod_{\ell=1}^{K} (R_\ell)^{n_\ell} \frac{(n_\ell)!}{2^{n_\ell}\left[\left(\frac{n_\ell}{2}\right)!\right]^2}, \tag{89}$$

where n_ℓ's are a set of even positive integers satisfying Equation (88).

Even though equation (89) gives an exact expression to evaluate μ_{2n}'s, it can be rather tedious to evaluate μ_{2n}'s from equation (89) for large values of n and K. We shall therefore give an alternate method to evaluate the central moments of the random variable η.

It can be shown that the probability density function $p_\eta(\eta)$ of the random variable η can be expressed as[10]

$$p_\eta(\eta) = \frac{1}{2\Omega}\left[1 + 2 \sum_{s=1}^{\infty} \cos \frac{s\pi\eta}{\Omega} \prod_{j=1}^{K} J_0\left(\frac{s\pi R_j}{\Omega}\right)\right]. \tag{90}$$

The $2n$th moment of η can be represented as

$$\mu_{2n} = \int_{-\Omega}^{\Omega} z^{2n} p_\eta(z)\, dz. \tag{91}$$

From equations (90) and (91) we can show that

$$\mu_{2n} = \Omega^{2n}\left(\frac{1}{2n+1} + 2 \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \cdot \left\{\left[\prod_{j=1}^{K} J_0\left(\frac{\ell\pi R_j}{\Omega}\right)\right] \sum_{k=1}^{n} (-1)^k \frac{(2n)!}{[2n-2k+1]!\,(\ell\pi)^{2k}}\right\}\right). \tag{92}$$

* For $K = 1$, equation (89) reduces to equation (37).

It can be seen that the infinite series appearing in equation (92) converges rapidly for all values of R_j's; we need take only a finite number of terms from equation (92) to estimate μ_{2n}'s. It is, therefore, easier to evaluate μ_{2n}'s from equation (92) than from equation (89) when there are a large number of interferers, and we have to take a large number of terms in estimating P_m.

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