# Spectral Density Bounds of a PM Wave

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(Manuscript received November 5, 1968)

*In this paper we derive upper and lower bounds of the spectrum of a sinusoidal carrier phase modulated by gaussian noise having a rectangular power spectrum. It has been found in practice that such a random process adequately simulates for some purposes, a frequency division multiplex signal, a composite speech signal, and so on. We show that these upper and lower bounds of the spectrum are very close to each other if the root mean square phase deviation of the carrier is even moderately high. Also, a simple method called the saddle-point method can be used at all frequencies* f *to estimate the spectrum with less than ten percent error. We also show that the results obtained from the quasistatic approximation, often used in such cases, are too small for large* f, *and that this low-frequency approximation cannot be used in cases where the behavior on the tails is of importance.*

## I. INTRODUCTION

It has been found in practice that a bandlimited random gaussian noise having a rectangular power spectrum adequately simulates for some purposes a wideband composite speech signal, a frequency division multiplex baseband signal consisting of a group of single sideband carrier telephone channels, and so on.[1] In the design of communication systems, the spectral characteristics of a sinusoidal carrier phase modulated by such a baseband signal are of great interest; various methods have been used in recent years to estimate this spectrum for large and small values of mean square phase deviation of the wave, both close to and far from the carrier frequency (that is, in the principal part of the spectrum and far down on the tails of the spectrum respectively).[1–8]

It has been shown that the spectrum may be expanded as an infinite series of weighted convolution terms.[2,5,7,8] This series may be used to estimate the principal part of the spectrum (close to the carrier) for small or moderate index (that is, small or moderate values of rms phase deviation). However, for large index, or far down on the tails for small index, too many terms would have to be included if this series is to be used directly.

The simplest analysis—often called the quasistatic approximation—yields a gaussian spectrum for large-index angle-modulated waves[2, 4–8] in most cases.* This approximation fails far out on the tails of the spectrum; a careful investigation has been given in only a few cases.[7] We obtain below upper and lower bounds for the spectrum of an angle-modulated wave with white, band-limited phase modulation; far out on the tails the spectrum far exceeds that predicted by the quasistatic approximation.

This problem is of interest in considering interference between two (or more) phase modulation (PM) systems in neighboring locations. Consider the following situation. In the frequency bands above 10 GHz, where the signal attenuation due to rain storms could be very severe, close spacings of the repeaters are almost mandatory for reliable communication from point to point.[9] In such cases the problem of interference between neighboring systems may be much more important than the problem of noise; the system may thus be interference limited. In order to combat this interference it is very likely that broadband modulation techniques like PM [or frequency modulation (FM)] or pulse code modulation (PCM) will have to be used. In order to investigate the effect of this interference between two co-channel PM (or FM) waves it is necessary to evaluate the spectrum of a PM wave, so that the parameters (such as rms phase deviations) of the two PM systems can be properly chosen to keep the interchannel interference below a certain desired level.

We first obtain an expression for the covariance function of the PM wave. From this covariance function we then derive an expression for the spectrum of the PM wave and show that it can be expressed as an infinite series. This series has been evaluated for certain values of rms phase deviation $N$.[6]

We then show that the autocorrelation function of the PM wave is analytic at all points in the finite part of the complex plane determined by the argument of the autocorrelation function. In determining the Fourier transform of the autocorrelation function we change the path of integration† so that the contour is very close to the path

* For exceptional cases see Ref. 7, Ch. 4, pp. 131–135.
† The method used in this paper to evaluate the spectrum is a close relative of the method of steepest descent (or the saddle-point method) used in evaluating certain kinds of integrals.[1, 7, 10, 11, 12]

of steepest descent of the integrand. We then divide this contour into four (or five) disjoint sections and show that the major contribution to the integral comes from one of these sections.

We next derive upper and lower bounds to the spectrum $S_V(f)$ of the PM wave and show that these bounds are very close for all $f$ and for all values of rms phase deviation $N \geqq 5$. For $N \geqq (10)^{\frac{1}{2}}$ we show that the spectrum can be evaluated by this saddle-point method in a very simple manner with a very small fractional error (less than 10 percent), and we give this method.

We finally compare the quasistatic approximation to the saddle-point approximation. For large values of frequency $f$, we show that the quasistatic approximation gives too small a value for the spectrum, and that it cannot be used in cases where the spectral behavior on the tails is of importance. However, as we show, the saddle-point method can be used in all cases in which $N$ is moderately high.

In conclusion, this paper gives a simple method of evaluating the spectrum of a sinusoidal carrier phase modulated by gaussian noise having a rectangular power spectrum and having a moderately high modulation index.

## II. SPECTRAL ANALYSIS OF PM WAVES WITH RANDOM PHASE MODULATION

A sinusoidal wave of constant-amplitude phase modulated by a signal $n(t)$ may be written as

$$W(t) = A \cos [\omega_0 t + n(t) + \theta], \tag{1}$$

where $A$ is the amplitude of the wave, $f_0 = \omega_0/2\pi$ is the carrier frequency of the wave, and $\theta$ is a random variable with probability density*

$$\pi_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \leqq \theta < 2\pi \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Assume that $n(t)$ is a stationary bandlimited white gaussian random process with mean zero and variance $N^2$.† Its spectral density

* If $n(t)$ is a stationary random process the introduction of random variable $\theta$ in equation (1) makes $W(t)$ a random process which is at least wide-sense stationary so that its spectrum can be calculated from the Wiener-Khintchine theorem.[2, 7] If we do not have $\theta$ in equation (1), $W(t)$ is no longer (even wide-sense) stationary, and the spectrum of $W(t)$ is usually calculated from the time autocorrelation function of $W(t)$.[7] The results obtained in the two cases are identical.

† The parameter $N$ represents the rms phase deviation (or modulation index) of the PM wave given in equation (1).

$S_n(f)$ can be represented (see Fig. 1) as

$$S_n(f) = \begin{cases} N^2/2W, & |f| < W, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Such a random process $n(t)$, is often used to simulate a multiplex signal, a composite speech signal, and so on.[1, 5]

We can show from equation (3) that the covariance function $R_n(\tau)$ of $n(t)$ is given by

$$R_n(\tau) = N^2 \frac{\sin 2\pi W\tau}{2\pi W\tau} ; \tag{4}$$

this function $R_n(\tau)$ is shown in Fig. 2. Since $n(t)$ is a stationary gaussian random process it can be shown that $W(t)$ is at least wide-sense stationary and that its covariance function $R_W(\tau)$ can be represented as[2, 7]

$$R_W(\tau) = \frac{A^2}{2} \exp\left[-R_N(0)\right] \exp\left[R_N(\tau)\right] \cos \omega_0 \tau. \tag{5}$$

From the Wiener-Khintchine theorem, and from equation (5), the spectrum $S_W(f)$ of $W(t)$ can be written as

$$S_W(f) = \int_{-\infty}^{\infty} R_W(\tau) \exp\left[-j2\pi f\tau\right] d\tau, \tag{6}$$

or

$$S_W(f) = \frac{A^2}{4} \left[S_V(f - f_0) + S_V(f + f_0)\right], \tag{7}$$

where

$$S_V(f) = \int_{-\infty}^{\infty} \exp\left[-R_N(0)\right] \exp\left[R_N(\tau)\right] \exp\left[-j2\pi f\tau\right] d\tau. \tag{8}$$

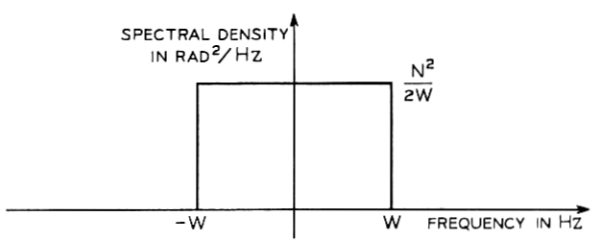


Fig. 1 — Spectral density of phase modulation.

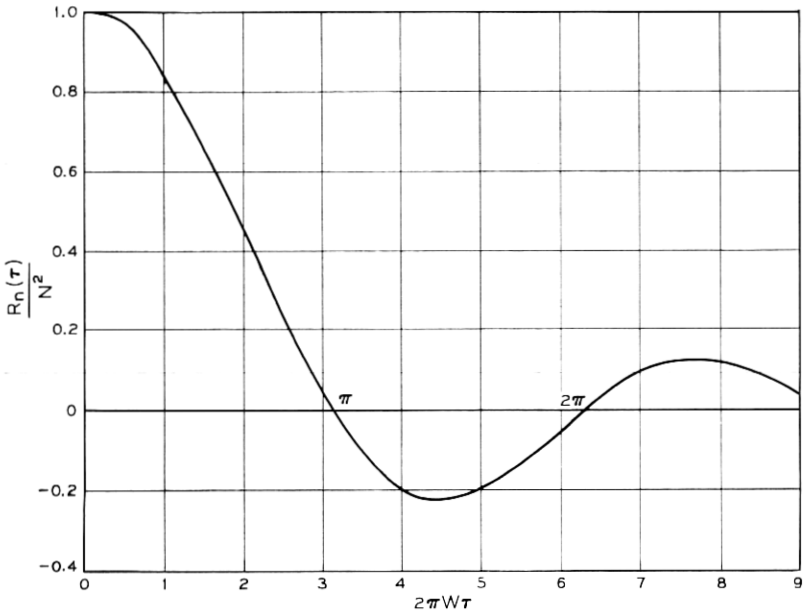


Fig. 2 — Covariance function of $n(t)$. Since $R_n(\tau)$ is an even function of $\tau$ we only show $R_n(\tau)$ for $\tau \geqq 0$.

From equations (4) and (8) we have

$$S_V(f) = \frac{1}{2\pi W} \int_{-\infty}^{\infty} \exp\left\{-N^2\left[1 - \frac{\sin p}{p}\right]\right\} \exp\left[-j\lambda p\right] dp, \tag{9}$$

where

$$\lambda = \frac{f}{W}. \tag{10}$$

III. SERIES EXPANSION OF SPECTRUM FOR GAUSSIAN MODULATION

The integral in Equation (9) can be evaluated by expanding

$$\exp\left\{-N^2\left[1 - \frac{\sin p}{p}\right]\right\}$$

into a Taylor series; integrating term by term we can write*

$$\exp\left\{-N^2\left[1 - \frac{\sin p}{p}\right]\right\} = \exp\left[-N^2\right] \sum_{\ell=0}^{\infty} \frac{N^{2\ell}}{\ell!}\left(\frac{\sin p}{p}\right)^{\ell}. \tag{11}$$

* We note that $\sum_{n=0}^{\infty} x^n/n!$ converges uniformly to $\exp[x]$ for all finite values of $x$.

From equations (9) and (11) we have*

$$S_V(f) = \exp[-N^2] \cdot \left\{ \delta(f) + \frac{1}{2\pi W} \sum_{\ell=1}^{\infty} \frac{N^{2\ell}}{\ell!} \int_{-\infty}^{\infty} \left(\frac{\sin p}{p}\right)^{\ell} \exp[-j\lambda p] \right\} dp, \tag{12}$$

where $\delta(f)$ is the Dirac delta (unit impulse) function.

We now note that

$$\int_{-\infty}^{\infty} \frac{\sin p}{p} \exp[-j\lambda p]\, dp = F_1(\lambda) = \begin{cases} \pi, & |\lambda| < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

or

$$F_1(\lambda) = \pi[u_{-1}(\lambda + 1) - u_{-1}(\lambda - 1)], \tag{14}$$

where $u_{-1}(x)$ is the unit step function defined by

$$u_{-1}(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{15}$$

and that[1, 13]

$$\int_{-\infty}^{\infty} \left(\frac{\sin p}{p}\right)^{\ell} \exp[-j\lambda p]\, dp = F_\ell(\lambda)$$

$$= \begin{cases} \dfrac{\ell\pi}{2^{\ell-1}} \displaystyle\sum_{k=0}^{M} (-1)^k \frac{(|\lambda| + \ell - 2k)^{\ell-1}}{k! \cdot (\ell - k)!}, & 0 \leqq |\lambda| < \ell,\ \ell \geqq 2, \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where

$$M = INT\left[\frac{\ell + |\lambda|}{2}\right], \tag{17}$$

and $INT[x]$ represents the largest integer not greater than $x$.

It can be shown that $F_\ell(\lambda)$, $\ell \geqq 2$ is a continuous function of $\lambda$ and that $F_1(\lambda)$ is discontinuous at $\lambda = 1$. For large $\ell$, we can show from the central-limit theorem† that[2]

$$F_\ell(\lambda) \sim \left(\frac{6\pi}{\ell}\right)^{\frac{1}{2}} \exp\left[\frac{-3\lambda^2}{2\ell}\right]. \tag{18}$$

* The term containing $\delta(f)$ in equation (12) represents the dc component of $S_V(f)$.

† See pp. 362–366 of Ref. 2. It can be shown that $\left(\frac{\sin p}{p}\right)^{\ell}$ can be interpreted as the characteristic function of the sum $\Omega$ of $\ell$ independent random variables with identical uniform probability distributions.[5] The function $F_\ell(\lambda)/2\pi$ therefore represents the probability density function of $\Omega$. Alternatively $F_\ell(\lambda)$ is the $(\ell - 1)$-fold convolution of the flat spectrum with itself.

From equations (12), (13), and (16) we can write

$$S_V(f) = \exp[-N^2]\left[\delta(f) + \frac{1}{2\pi W}\sum_{\ell=1}^{\infty}\frac{N^{2\ell}}{\ell!}F_\ell(f/W)\right]. \tag{19}$$

For $N^2 = 6$, we have calculated the spectrum from equation (19) and the results are shown in Fig. 3. Notice that the spectral density is discontinuous at $f/W = 1$.

For $N^2 \ll 1$ (low-index modulation), we have from (19)

$$S_V(f) \approx S_{V0}(f) = \exp[-N^2]\left[\delta(f) + \frac{N^2}{2\pi W}F_1(f/W)\right], \tag{20}$$

and the error in this approximation may be investigated from equations (9) and (20).* The approximation given in equation (20) represents the low-index approximation for the spectrum; this result has been obtained by many authors.[3-6]

The series given in equation (19) may be used to estimate the principal part of the spectrum (close to the carrier) for small or moderate index, since only a small number of terms need to be included in the partial sum. However, for large $N^2$, or far down on the tails of the spectrum for small $N^2$, too many terms would have to be included to estimate the spectral density. In fact for $N^2 \gg 1$, or for $f/W \gg 1$, the degree of complexity required in estimating $S_V(f)$ from equation (19) becomes inordinately high.

When $N^2 \gg 1$, and for low frequencies, several authors have given[1-7] the quasistatic approximation†

$$S_V(f) \approx \exp(-N^2)\,\delta(f) + \frac{1}{NW}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} \exp\left[-\frac{3}{2N^2}\left(\frac{f}{W}\right)^2\right] \tag{21}$$

for the spectrum. The question arises whether equation (21) can be used for large $f$. Since $R_W(\tau)$ is infinitely differentiable there is no simple way (known to the authors) of investigating, for large $f$, the error is this approximation.[7]

In the problem of interference between two neighboring channels it is necessary to evaluate $S_V(f)$ for large $f$ so that the effect of this interference can be determined. As we shall show later on in this paper equation (21) gives too small values to $S_V(f)$ for large $f$; it is therefore essential to have a simple and elegant method (different from the series method) to evaluate $S_V(f)$ for large $f$ and for large $N^2$.

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* At times the low-index approximation for the spectrum is written as $\exp[-N^2]\delta(f) + (N^2/2\pi W)F_1(f/W)$. For $N^2 \ll 1$, $\exp[-N^2] \approx 1$.

† Note that mean square frequency deviation is $N^2W^2/3$.

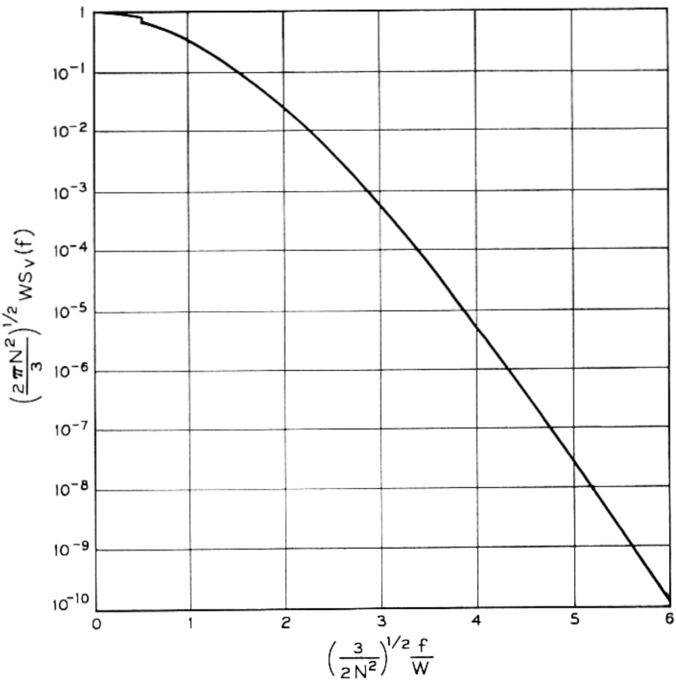


Fig. 3 — Spectral density of a PM wave for $N^2 = 6$. The discrete part of $S_V(f)$ for $f = 0$ is not shown in this figure. Note the discontinuity in the spectrum at $f/W = 1$.

Readers who might be interested in the final results without wishing to work through the detailed analysis, might skip Section IV of this paper.

## IV. SPECTRUM EVALUATION BY CONTOUR INTEGRATION

Let us now consider the integral given in equation (9). Since $\delta(f)$ and $F_1(f/W)$ are discontinuous functions of $f$, let us define an integral*

$$S(f) = \int_{-\infty}^{\infty} \exp\,[-N^2]\left\{\exp\left[N^2 \frac{\sin p}{p}\right] - \left(1 + N^2 \frac{\sin p}{p}\right)\right\} \cos \lambda p \, dp$$

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* In Ref. 14 this integral has been studied by Lewin for $\lambda = 0$ and $\lambda = 1$. It also occurs in several limiting cases in Ref. 1. It is sometimes referred to by the name Lewin's integral.[1]

or

$$S(f) = \int_{-\infty}^{\infty} \exp\,[-N^2]\left\{\exp\left[N^2 \frac{\sin p}{p}\right] - \left(1 + N^2 \frac{\sin p}{p}\right)\right\} \cdot \exp\,(-j\lambda p)\,dp. \qquad (22)$$

Since all $F_\ell(f)$, $\ell \geqq 2$ are continuous, it can be shown from equations (19), and (22) that $S(f)$ is a continuous function of $f$.

From equations (9), (19), and (22) we can then write*

$$S_V(f) = \exp\,(-N^2)\left\{\delta(f) + \frac{N^2}{2W}\,[u_{-1}(f + W) - u_{-1}(f - W)]\right\} + \frac{1}{2\pi W}\,\mathrm{Re}\,S(f). \qquad (23)$$

Notice from equation (22) that the integration is carried out along the real axis, and that for large $|\lambda|$ (or $|f|/W$), the final factor of the integrand $\exp\,(-j\lambda p)$ is a very rapidly oscillating function of $p$. From Refs. 7, 10–13 notice that in such circumstances the method of steepest descent (or saddle-point method), or one of its close relatives, is often useful to get an approximate expression for the integral; we shall now apply such a method to evaluate $S_V(f)$.

In applying this method to the evaluation of an integral with a real variable of integration, we must first be able to regard the integral as a contour integral along the real axis of the complex plane, with an analytic integrand. We note that the integrand in equation (22) is an analytic function of $p$, and that it has no singularities in the finite part of the complex plane (defined by $p$). From Cauchy's theorem it therefore follows that the contour of integration can be arbitrarily deformed as long as one end is at $p = -\infty + j0$ and the other at $p = \infty + j0$.[11]

In making use of the method of steepest descent the contour must be deformed so that the phase of the integrand remains constant (or almost so), while the magnitude of the integrand is small except in one or more localized regions, where it varies rapidly. This is usually accomplished by deforming the contour so that it goes through one or more saddle points. In other cases there may be some additional constraints on the contour;[7] then only a portion of the path of steep-

* Re $z$ and Im $z$ denote respectively the real and imaginary parts of complex number $z$.

est descent through a saddle point may be used in finding the integral, and the deformed contour may not actually reach the saddle point. In this case the original integral is usually reduced to a virtually real integral whose integrand behaves sufficiently simply on the modified path of integration so that an approximate evaluation of the integral with rigorous (upper and lower) bounds on the error may be obtained.

Departures from the strict method of steepest descent occur in this paper in that approximate paths of steepest descent are chosen. Although not quite optimum, they are analytically tractable and serve to give useful bounds on the integral under consideration.

Consider equation (22). Since the integrand in equation (22) is an analytic function of $p$, let us assume that $p = x + jy$ is a complex variable, and let us deform the contour so as to obtain the path of steepest descent. Since the integrand behaves properly on the contour for large $|p|$, it is clear that the contour of integration can be deformed in quite a general way in the complex $p$-plane without modifying the value of the integral.

From equation (22) it can be shown that the major portion of the integrand

$$R(p) \equiv \exp[-N^2] \exp\left[N^2 \frac{\sin p}{p}\right] \exp[j\lambda p] \tag{24}$$

has a saddle point on the imaginary axis, with the path of steepest descent parallel to the real axis at this point. The location $p_s = jy_s$ of this saddle point is given by

$$\frac{\cosh y_s}{y_s} - \frac{\sinh y_s}{y_s^2} = \frac{\lambda}{N^2} = \frac{f}{N^2W}; \tag{25}$$

for a given $f/N^2W$, equation (25) can be solved numerically to give the required $y_s$. We plot $y_s$ as a function of $f/N^2W$ in Fig. 4, and ln $R(jy_s)$ in Fig. 5.

Let us now deform the contour so that it passes through the point $p_s = jy_s$ and is parallel to the real axis at this point. From equation (22) we then have

$$S(f) = \exp\left[-2N^2\left(\cosh^2 \frac{y_s}{2} - \frac{\sinh y_s}{y_s}\right)\right] \operatorname{Re} I \tag{26}$$

where

$$I = \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]$$
$$\cdot \int_{-\infty}^{\infty} \left\{\exp\left[N^2 \frac{\sin(x + jy_s)}{x + jy_s}\right] - \left[1 + N^2 \frac{\sin(x + jy_s)}{x + jy_s}\right]\right\}$$
$$\cdot \exp[j\lambda x]\, dx. \tag{27}$$

Rewriting equation (27)

$$I = \int_{-\infty}^{\infty} G(x, y_s)\, dx, \tag{28}$$

where

$$G(x, y_s) \equiv \exp[-Q_R(x, y_s)] \exp[jQ_I(x, y_s)]$$
$$- \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left\{1 + N^2 \frac{\sin(x + jy_s)}{x + jy_s}\right\} \exp(j\lambda x), \tag{29}$$

where $Q_R(x, y_s)$ and $Q_I(x, y_s)$ are real and

$$Q_R(x, y_s) = N^2\left\{\frac{\sinh y_s}{y_s} - \mathrm{Re}\left[\frac{\sin(x + jy_s)}{x + jy_s}\right]\right\} \tag{30}$$

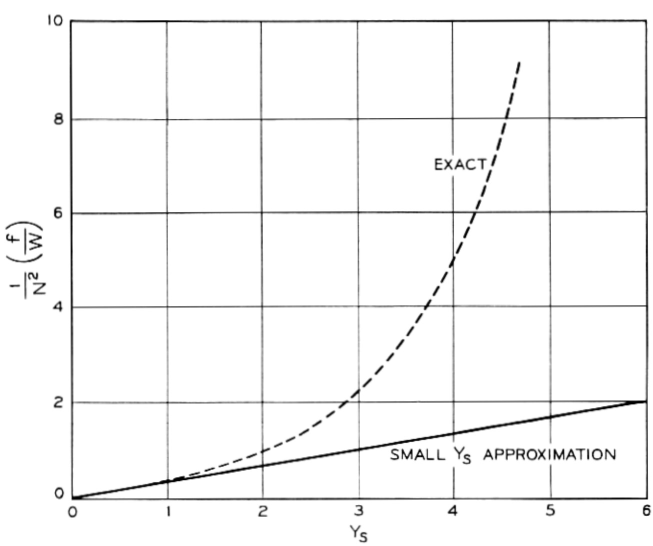


Fig. 4 — Location of saddle-point $y_s$.

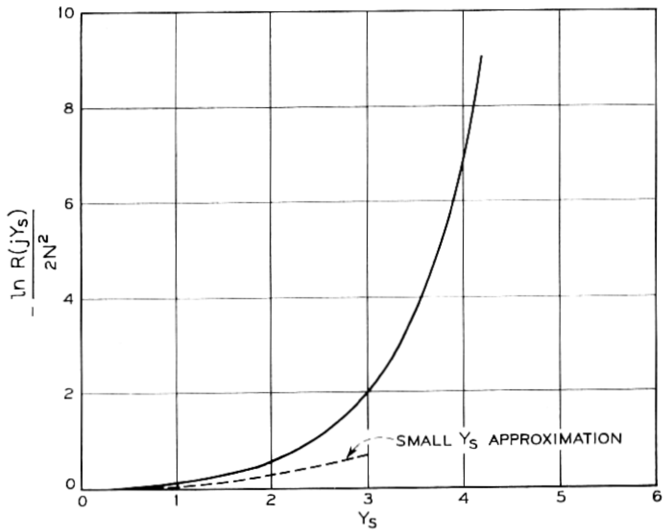


Fig. 5 — Value of $-[\ln R(jy_s)]/(2N^2)$.

or

$$Q_R(x, y_s) = N^2 \frac{\sinh y_s}{y_s} x^2 \frac{\dfrac{1 - \dfrac{y_s}{\tanh y_s} \dfrac{\sin x}{x}}{y_s^2} + \dfrac{1 - \cos x}{x^2}}{1 + (x/y_s)^2} \tag{31}$$

and

$$Q_I(x, y_s) = N^2 \operatorname{Im} \left[ \frac{\sin (x + jy_s)}{x + jy_s} \right] + \lambda x, \tag{32}$$

or

$$Q_I(x, y_s)$$
$$= N^2 \left\{ \frac{\sinh y_s \cos x - y_s \cosh y_s \dfrac{\sin x}{x}}{x^2 + y_s^2} + \left[ \frac{\cosh y_s}{y_s} - \frac{\sinh y_s}{y_s^2} \right] \right\} x. \tag{33}$$

The functions $Q_R(x, y_s)$ and $Q_I(x, y_s)$ have been plotted in Fig. 6 for a set of values of $y_s$.

Since we are primarily interested in the high-index case let us assume that $N^2 \gg 1$. From equations (27)–(33) we now observe that the principal contribution to the integral $I$ comes from small $x$. For small $x$,

$$G(x, y_s) \approx \exp\left[-Q_R(x, y_s)\right]. \tag{34}$$

It can be shown (see Figs. 6, 7, and Appendix A) that $Q_R(x, y_s)$ is a monotonically increasing function of $x$ for $0 \leqq x \leqq \pi$ and that it oscillates for values of $x > \pi$. For large $x$,

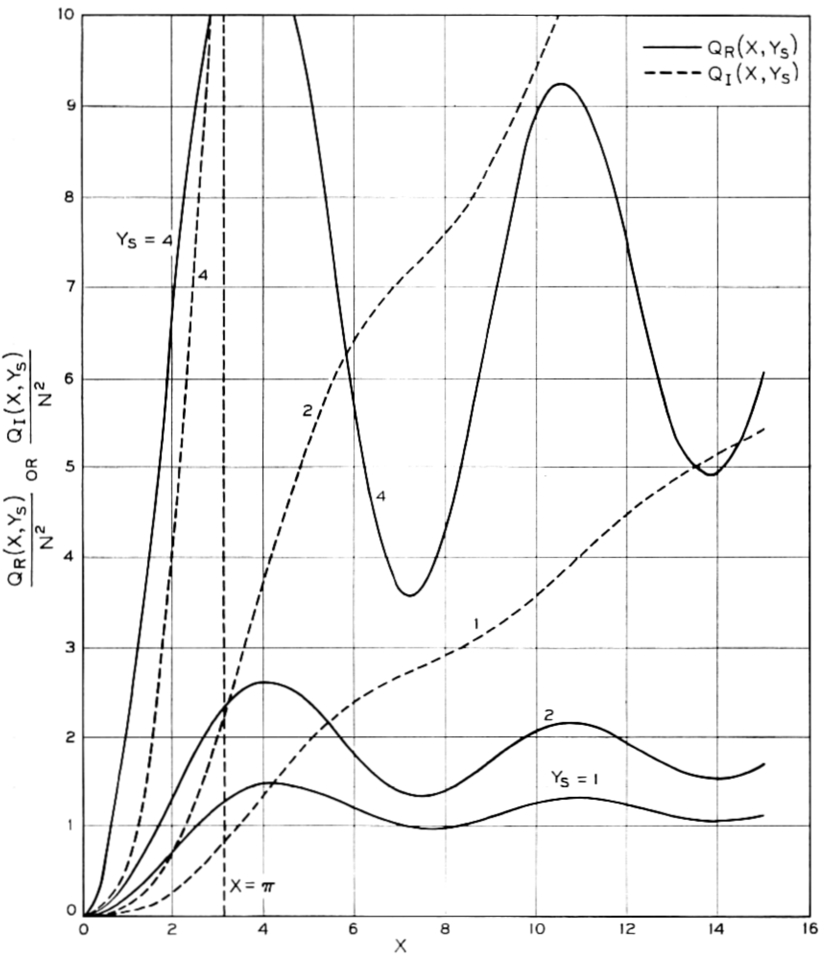


Fig. 6 — Functions $Q_R(x, y_s)$ and $Q_I(x, y_s)$.

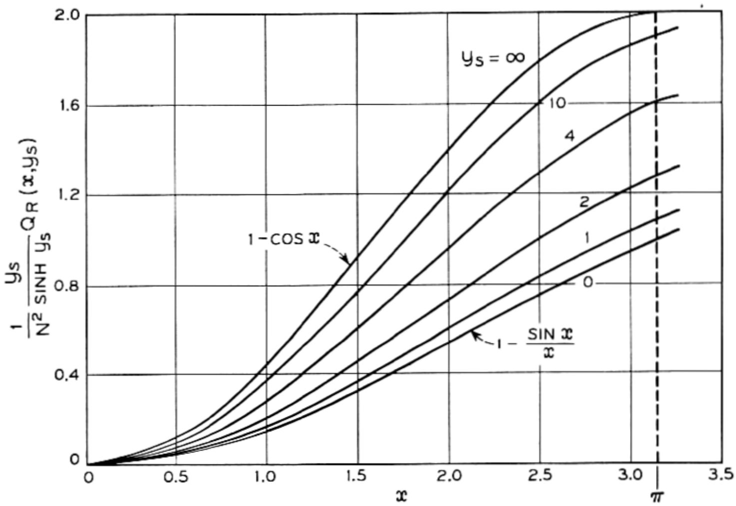


Fig. 7 — Function $Q_R(x,\ y_s)$. From this figure, it can be seen that $Q_R(x,\ y_s)$ is a monotonically increasing function of $x$ for $0 \leqq x \leqq \pi$.

$$G(x,\ y_s) \approx \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]$$
$$\cdot\left\{\exp\left[N^2 \cosh y_s \frac{\sin x}{x}\right] \exp\left[jN^2 \sinh y_s \frac{\cos x}{x}\right]\right.$$
$$\left. - \left[1 + N^2\left(\cosh y_s \frac{\sin x}{x} + j \sinh y_s \frac{\cos x}{x}\right)\right]\right\} e^{j\lambda x}, \qquad (35)$$

and we note that the first and second terms both have small and almost equal magnitude, and almost opposite phase angle, so that they almost cancel. As $|x| \to \infty$ the cancellation becomes exact. For these reasons it is convenient to divide the range of integration in equation (27) into at least four regions:

$$\begin{aligned} 0 < |x| < x_1, &\qquad \text{small } |x|, \\ x_1 < |x| < \pi, &\qquad \text{intermediate } |x|, \\ \pi < |x| < x_2, &\qquad \text{intermediate } |x|, \qquad (36) \\ x_2 < |x| < \infty, &\qquad \text{large } |x|. \end{aligned}$$

From equations (25), (30), and (32) we can show that*

* These expressions for $Q_R$ and $Q_I$ may be obtained from the Taylor series expansion of the function $[\sin (x + jy_s)]/(x + jy_s)$.

$$Q_R(x, y_s) = N^2 \sum_{\ell=1}^{\infty} (-1)^{\ell-1} A_{2\ell} \frac{x^{2\ell}}{(2\ell)!}, \tag{37}$$

and

$$Q_I(x, y_s) = N^2 \sum_{\ell=1}^{\infty} (-1)^{\ell+1} A_{2\ell+1} \frac{x^{2\ell+1}}{(2\ell+1)!}, \tag{38}$$

where

$$A_0 = \frac{\sinh y_s}{y_s}, \tag{39}$$

$$A_1 = \frac{\cosh y_s}{y_s} - \frac{\sinh y_s}{y_s^2} = \frac{\lambda}{N^2}, \tag{40}$$

$$A_{2\ell} = \frac{\sinh y_s}{y_s} - 2\ell \frac{A_{2\ell-1}}{y_s}, \qquad \ell = 1, 2, 3, \cdots, \tag{41}$$

and

$$A_{2\ell+1} = \frac{\cosh y_s}{y_s} - (2\ell + 1) \frac{A_{2\ell}}{y_s}, \qquad \ell = 1, 2, 3, \cdots. \tag{42}$$

It can also be shown that

$$A_{2k-1} = \sum_{\ell=0}^{\infty} \frac{1}{2\ell + 2k + 1} \frac{y_s^{2\ell+1}}{(2\ell+1)!}, \qquad k = 1, 2, 3, \cdots, \tag{43}$$

and

$$A_{2k} = \sum_{\ell=0}^{\infty} \frac{1}{2\ell + 2k + 1} \frac{y_s^{2\ell}}{(2\ell)!}, \qquad k = 1, 2, 3, \cdots. \tag{44}$$

Since the spectrum is an even function of $f$ we can assume without loss of generality that

$$y_s \geqq 0. \tag{45}$$

For $y_s \geqq 0$, from equations (43) and (44) all $A_\ell$'s are monotonically increasing functions of $y_s$, and we can further show that

$$0 < A_{2(k+1)} < A_{2k}, \qquad k = 0, 1, 2, \cdots, \tag{46}$$

and

$$0 \leqq A_{2\ell+1} \leqq A_{2\ell-1}, \qquad \ell = 1, 2, 3, \cdots. \tag{47}$$

For large $y_s$ (for large $f/W$), it can also be proved that

$$A_{2\ell} \approx A_{2\ell-1} \approx \frac{\exp(y_s)}{2y_s}. \tag{48}$$

Since it appears that the main contribution to the integral $I$ comes from the region of small $|x|$, assume that $|x_1|$ is small and that

$$I = I_1 + I_R , \tag{49}$$

where

$$I_1 = \int_{-x_1}^{x_1} G(x, y_s)\, dx, \tag{50}$$

and

$$I_R = \int_{-\infty}^{-x_1} G(x, y_s)\, dx + \int_{x_1}^{\infty} G(x, y_s)\, dx. \tag{51}$$

For small $|x|$, we have from equations (37) and (38)

$$Q_R(x) \approx N^2 \tfrac{1}{2} A_2 x^2, \tag{52}$$

and

$$Q_I(x) \approx N^2 \tfrac{1}{6} A_3 x^3. \tag{53}$$

Let us choose* $x_1$ so that $\exp(-\frac{1}{2}N^2A_2x^2)$ falls to $\exp(-5) \approx 0.0067$ for $x = x_1$ ,† or that

$$x_1 = \left(\frac{10}{N^2 A_2}\right)^{\frac{1}{2}}. \tag{54}$$

Since it can be shown from equations (39)–(44) that the minimum value of $A_2$ is $\frac{1}{3}$ (at $y_s = 0$),

$$x_1 \leqq \left(\frac{30}{N^2}\right)^{\frac{1}{2}}. \tag{55}$$

Assume that

$$x_1 \leqq \pi, \tag{56}$$

so that $\exp\{-Q_R(x, y_s)\}$ is a monotonically decreasing function of $x$ for $0 \leqq x \leqq x_1$ . Equation (56) will be satisfied for all $y_s$ if

$$N^2 \geqq \frac{30}{\pi^2} \approx 3.039. \tag{57}$$

Since we are primarily interested in the high-index case, this is not a significant restriction.

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* See equation (34).

† While there is some degree of arbitrariness in such choices, the bounds obtained are not too sensitive to small variations in the value of $x_1$, $x_2$, and so on.

Noticing that

$$\sin x \leqq x, \qquad 0 \leqq x < \infty, \tag{58}$$

$$\left| \frac{\sin (x + jy_s)}{x + jy_s} \right| = \frac{(\sin^2 x + \sinh^2 y_s)^{\frac{1}{2}}}{(x^2 + y_s^2)^{\frac{1}{2}}} \leqq \left( \frac{x^2 + \sinh^2 y_s}{x^2 + y_s^2} \right)^{\frac{1}{2}}. \tag{59}$$

Since $(\sinh y_s)/(y_s) \geqq 1$, it can be proved from equation (59) that

$$\left| \frac{\sin (x + jy_s)}{x + jy_s} \right| \leqq \frac{\sinh y_s}{y_s}. \tag{60}$$

We can show from equations (29), (50), and (60) that

$$| I_1 | < 2 \int_0^{x_1} \{\exp [-Q_R(x, y_s)] + H(x, y_s)\} \, dx, \tag{61}$$

where

$$H(x, y_s) = \exp \left[ -N^2 \frac{\sinh y_s}{y_s} \right] \left[ 1 + N^2 \frac{\sinh y_s}{y_s} \right]. \tag{62}$$

From equation (37)

$$\begin{aligned} Q_R(x, y_s) = N^2 \bigg\{ & \frac{1}{2} A_2 x^2 - \frac{1}{24} A_4 x^4 + \frac{1}{6!} A_6 x^6 \left[ 1 - \frac{6!}{8!} \frac{A_8}{A_6} x^2 \right] \\ & + \frac{1}{10!} A_{10} x^{10} \left[ 1 - \frac{10!}{12!} \frac{A_{12}}{A_{10}} x^2 \right] + \cdots \bigg\}. \end{aligned} \tag{63}$$

From equations (46) and (63) it can be shown that for all $y_s$

$$Q_R(x, y_s) \geqq N^2 \left[ \frac{1}{2} A_2 x^2 - \frac{1}{24} A_4 x^4 \right], \qquad 0 \leqq x \leqq x_1 < (56)^{\frac{1}{2}}. \tag{64}$$

We then have

$$| I_1 | \leqq 2 \int_0^{x_1} \exp \left\{ -N^2 \left[ \frac{1}{2} A_2 x^2 - \frac{1}{24} A_4 x^4 \right] \right\} + 2 \int_0^{x_1} H(x, y_s) \, dx. \tag{65}$$

One can show that

$$e^t \leqq 1 + \frac{e^R - 1}{R} t, \qquad 0 \leqq t \leqq R. \tag{66}$$

Since we have

$$0 \leqq \frac{N^2}{24} A_4 x^4 \leqq \frac{25}{6} \frac{A_4}{A_2} \frac{1}{N^2 A_2}, \qquad 0 \leqq x \leqq x_1, \tag{67}$$

$$| I_1 | \leqq 2 \int_0^{x_1} \exp [-N^2 \tfrac{1}{2} A_2 x^2] \left\{ 1 + \frac{\exp \left[ \frac{25}{6} \frac{A_4}{A_2} \frac{1}{N^2 A_2} \right] - 1}{x_1^4} x^4 \right\} dx$$

$$+ 2 \int_0^{x_1} H(x, y_s) \, dx \qquad < 2 \int_0^{\infty} \exp [-N^2 \tfrac{1}{2} A_2 x^2]$$

$$+ 2 \frac{\exp \left[ \frac{25}{6} \frac{A_4}{A_2} \frac{1}{N^2 A_2} \right] - 1}{x_1^4} \int_0^{\infty} x^4 \exp [-N^2 \tfrac{1}{2} A_2 x^2] \, dx$$

$$+ 2 \int_0^{x_1} H(x, y_s) \, dx. \tag{68}$$

From equation (68) it can be shown that

$$| I_1 | < \left( \frac{2\pi}{N^2 A_2} \right)^{\frac{1}{2}} [1 + E_1], \tag{69}$$

where

$$E_1 = \frac{3}{100} \left\{ \exp \left[ \frac{25}{6} \frac{A_4}{A_2} \frac{1}{N^2 A_2} \right] - 1 \right\}$$

$$+ 2 \left( \frac{5}{\pi} \right)^{\frac{1}{2}} \exp \left[ -N^2 \frac{\sinh y_s}{y_s} \right] \left[ 1 + N^2 \frac{\sinh y_s}{y_s} \right]. \tag{70}$$

Since we know that

$$-| p | \leqq \operatorname{Re} p \leqq | p |, \qquad p \text{ any arbitrary complex number,} \tag{71}$$

equation (69) gives an upper bound to Re $I_1$. Let us now find a lower bound to Re $I_1$.

From equations (29) and (50) we have

$$\operatorname{Re} I_1 = 2 \int_0^{x_1} \exp [-Q_R(x, y_s)] \cos [Q_I(x, y_s)] \, dx$$

$$- 2 \operatorname{Re} \int_0^{x_1} \exp \left[ -N^2 \frac{\sinh y_s}{y_s} + j\lambda x \right] \left[ 1 + N^2 \frac{\sin (x + jy_s)}{x + jy_s} \right] dx. \tag{72}$$

As shown earlier in this paper

$$2 \operatorname{Re} \int_0^{x_1} \exp \left[ -N^2 \frac{\sinh y_s}{y_s} + j\lambda x \right] \left[ 1 + N^2 \frac{\sin (x + jy_s)}{x + jy_s} \right] dx$$

$$\leqq 2x_1 \exp \left[ -N^2 \frac{\sinh y_s}{y_s} \right] \left\{ 1 + N^2 \frac{\sinh y_s}{y_s} \right\}. \tag{73}$$

One can also show that for $z$ real

$$\cos z \geqq 1 - \frac{z^2}{2}, \qquad -\infty < z < \infty. \tag{74}$$

Using equation (74) we can write

$$2 \int_0^{x_1} \exp\,[-Q_R(x, y_s)] \cos\,[Q_I(x, y_s)]\, dx$$
$$\geqq 2 \int_0^{x_1} \exp\,[-Q_R(x, y_s)] \left[1 - \frac{Q_I^2(x, y_s)}{2}\right] dx. \tag{75}$$

Now from equations (37), and (38) we have

$$Q_R(x, y_s) = N^2 \left[\frac{1}{2} A_2 x^2 - \frac{1}{4!} A_4 x^4 \left(1 - \frac{4!}{6!} \frac{A_6}{A_4} x^2\right) - \frac{1}{8!} A_8 x^8 \left(1 - \frac{8!}{10!} \frac{A_{10}}{A_8} x^2\right) - \cdots\right], \tag{76}$$

and

$$Q_I(x, y_s) = N^2 \left[\frac{1}{6} A_3 x^3 - \frac{1}{5!} A_5 x^5 \left\{1 - \frac{5!}{7!} \frac{A_7}{A_5} x^2\right\} - \frac{1}{9!} A_9 x^9 \left\{1 - \frac{9!}{11!} \frac{A_{11}}{A_9} x^2\right\} - \cdots\right]. \tag{77}$$

From equations (76), and (77) one can show that

$$Q_R(x, y_s) \leqq \tfrac{1}{2} N^2 A_2 x^2, \qquad 0 \leqq x \leqq x_1 < (30)^{\frac{1}{2}}, \tag{78}$$

and

$$Q_I(x, y_s) \leqq \tfrac{1}{6} N^2 A_3 x^3, \qquad 0 \leqq x \leqq x_1 < (42)^{\frac{1}{2}}. \tag{79}$$

Equations (75), (78), and (79) yield

$$2 \int_0^{x_1} \exp\,[-Q_R(x, y_s)] \cos\,[Q_I(x, y_s)]\, dx$$
$$> 2 \int_0^{x_1} \exp\,[-\tfrac{1}{2} N^2 A_2 x^2] \left[1 - \frac{N^4}{72} A_3^2 x^6\right] dx$$
$$= 2 \int_0^{\infty} \exp\,[-\tfrac{1}{2} N^2 A_2 x^2]\, dx - 2 \int_{x_1}^{\infty} \exp\,[-\tfrac{1}{2} N^2 A_2 x^2]\, dx$$
$$- \frac{N^4}{36} A_3^2 \int_0^{x_1} x^6 \exp\,[-\tfrac{1}{2} N^2 A_2 x^2]\, dx. \tag{80}$$

Notice that

$$\int_0^{x_1} x^6 \exp\left[-\tfrac{1}{2}N^2A_2x^2\right] dx < \int_0^{\infty} x^6 \exp\left[-\tfrac{1}{2}N^2A_2x^2\right] dx$$

$$= \frac{1}{2}\left(\frac{2\pi}{N^2A_2}\right)^{\frac{1}{2}} \frac{15}{(N^2A_2)^3}. \tag{81}$$

We can also show that

$$\int_{a^2}^{\infty} \exp(-p^2t^2)\, dt < \frac{\exp(-p^2a^4)}{2a^2p^2}, \qquad a^2p^2 \neq 0. \tag{82}$$

We can, therefore, write

$$2\int_{x_1}^{\infty} \exp\left[-\tfrac{1}{2}N^2A_2x^2\right] dx < \left(\frac{2}{5N^2A_2}\right)^{\frac{1}{2}} e^{-5}. \tag{83}$$

From equations (72)–(73), (75), (80)–(83) it can now be shown that

$$\text{Re}\, I_1 > \left(\frac{2\pi}{N^2A_2}\right)^{\frac{1}{2}} [1 - E_1'], \tag{84}$$

where

$$E_1' = \frac{1}{(5\pi)^{\frac{1}{2}}} e^{-5} + \frac{5}{24}\left(\frac{A_3}{A_2}\right)^2 \frac{1}{N^2A_2} + 2\left(\frac{5}{\pi}\right)^{\frac{1}{2}}$$

$$\cdot \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[1 + N^2 \frac{\sinh y_s}{y_s}\right]. \tag{85}$$

We shall now obtain upper and lower bounds to Re $I_R$ in equation (51). According to equation (36) let

$$I_R = I_2 + I_T \tag{86}$$

where

$$I_2 = \int_{-\pi}^{-x_1} G(x, y_s)\, dx + \int_{x_1}^{\pi} G(x, y_s)\, dx, \tag{87}$$

and

$$I_T = \int_{-\infty}^{-\pi} G(x, y_s)\, dx + \int_{\pi}^{\infty} G(x, y_s)\, dx. \tag{88}$$

From equation (87) we have

$$I_2 \mid \leqq 2\int_{x_1}^{\pi} \mid G(x, y_s) \mid dx. \tag{89}$$

Now it can be shown from equations (29) and (60) that

$$| G(x, y_s) | \leqq \exp [-Q_R(x, y_s)] + \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[1 + N^2 \frac{\sinh y_s}{y_s}\right]. \qquad (90)$$

From equations (89) and (90) we can write

$$| I_2 | \leqq 2 \int_{x_1}^{\pi} \exp [-Q_R(x, y_s)]\, dx + 2(\pi - x_1) \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[1 + N^2 \frac{\sinh y_s}{y_s}\right]. \qquad (91)$$

From equation (64)

$$Q_R(x, y_s) \geqq N^2\left[\frac{1}{2} A_2 x^2 - \frac{1}{24} A_4 x^4\right] \equiv N^2 v, \qquad x_1 \leqq x \leqq \pi \qquad (92)$$

where

$$v = \frac{1}{2} A_2 x^2 - \frac{1}{24} A_4 x^4. \qquad (93)$$

It can be shown (see Fig. 8) that $v$ and $dv/dx$ are positive for $x_1 \leqq x \leqq x_m = (6)^{1/2}(A_2/A_4)^{1/2} \geqq (6)^{1/2}$ and that $dv/dx$ is a monotonically increasing function of $x$ for $x_1 \leqq x \leqq x_n$ where

$$x_n = \sqrt{2}\left(\frac{A_2}{A_4}\right)^{1/2} \geqq \sqrt{2}. \qquad (94)$$

We also know that $Q_R(x, y_s)$ is a monotonically increasing function of $x$ for $x_1 \leqq x \leqq \pi$. Let us now assume that $x_1 < \sqrt{2}$.

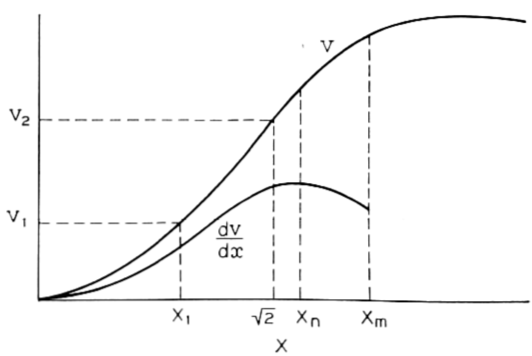


Fig. 8—Functions $v$ and $\frac{dv}{dx}$ appearing in equation (95).

Equations (92)–(94) therefore yield

$$2 \int_{x_1}^{\pi} \exp [-Q_R(x, y_s)] \, dx$$

$$< 2 \int_{v_1}^{v_2} \exp [-N^2 v] \frac{dx}{dv} \, dv + 2(\pi - \sqrt{2}) \exp [-Q_R(\sqrt{2}, y_s)], \quad (95)$$

where

$$v_1 = \frac{5}{N^2} \left[ 1 - \frac{5}{6} \frac{A_4}{A_2} \frac{1}{N^2 A_2} \right], \quad (96)$$

$$v_2 = A_2 - \frac{A_4}{6}, \quad (97)$$

and

$$0 \leqq v_1 \leqq v_2 . \quad (98)$$

Since we know that

$$0 \leqq \frac{dx}{dv} = \frac{1}{\dfrac{dv}{dx}} \leqq \frac{1}{\left[ \dfrac{dv}{dx} \right]_{x=x_1}} = \frac{N^2 x_1}{10 \left[ 1 - \dfrac{5}{3} \dfrac{A_4}{A_2} \dfrac{1}{N^2 A_2} \right]}, \quad (99)$$

we can write

$$2 \int_{v_1}^{v_2} \exp [-N^2 v] \frac{dx}{dv} \, dv < \frac{N^2 x_1}{5 \left[ 1 - \dfrac{5}{3} \dfrac{A_4}{A_2} \dfrac{1}{N^2 A_2} \right]} \int_{v_1}^{v_2} \exp (-N^2 v) \, dv$$

$$< \frac{N^2 x_1}{5 \left[ 1 - \dfrac{5}{3} \dfrac{A_4}{A_2} \dfrac{1}{N^2 A_2} \right]} \int_{v_1}^{\infty} \exp (-N^2 v) \, dv. \quad (100)$$

Since it can be shown that

$$\int_{a^2}^{\infty} \exp [-p^2 t] \, dt = \frac{\exp [-p^2 a^2]}{p^2}, \qquad p^2 \neq 0, \quad (101)$$

from equations (91), (95) and (100), we have

$$| I_2 | < \left( \frac{2\pi}{N^2 A_2} \right)^{\frac{1}{2}} E_2 , \quad (102)$$

where

$$E_2 = \frac{1}{(5\pi)^{\frac{1}{2}}} \frac{1}{1 - \frac{5}{3}\frac{A_4}{A_2}\frac{1}{N^2 A_2}} \exp\left[-5\left(1 - \frac{5}{6}\frac{A_4}{A_2}\frac{1}{N^2 A_2}\right)\right]$$
$$+ 2(\pi - \sqrt{2})\left(\frac{N^2 A_2}{2\pi}\right)^{\frac{1}{2}} \exp\left[-Q_R(\sqrt{2}, y_s)\right]$$
$$+ 2(\pi - x_1)\left(\frac{N^2 A_2}{2\pi}\right)^{\frac{1}{2}} \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[1 + N^2 \frac{\sinh y_s}{y_s}\right],$$
$$x_1 < \sqrt{2}. \qquad (103)$$

Similarly, if $x_1 > \sqrt{2}$, one can show that

$$E_2 = 2(\pi - x_1)\left(\frac{N^2 A_2}{2\pi}\right)^{\frac{1}{2}}\left\{\exp\left[-Q_R(x_1, y_s)\right] + \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[1 + N^2 \frac{\sinh y_s}{y_s}\right]\right\}. \qquad (104)$$

Let us now consider the range of integration $\pi < x < \infty$. For $x \gg y_s$,

$$\exp\left[-Q_R(x, y_s) + jQ_I(x, y_s)\right] \approx \exp\left[-\left(N^2 \frac{\sinh y_s}{y_s} - N^2 \cosh y_s \frac{\sin x}{x}\right) + j\left(\lambda x + N^2 \sinh y_s \frac{\cos x}{x}\right)\right], \qquad (105)$$

and

$$\exp\left[-N^2 \frac{\sinh y_s}{y_s} + j\lambda x\right] \cdot \left[1 + N^2 \frac{\sin(x + jy_s)}{x + jy_s}\right] \approx \exp\left[-N^2 \frac{\sinh y_s}{y_s} + j\lambda x\right] \cdot \left\{1 + N^2\left[\frac{\sin x}{x}\cosh y_s + j\frac{\cos x}{x}\sinh y_s\right]\right\}. \qquad (106)$$

Let us choose the point $x_2 + jy_s$ along the path of integration so that the amplitudes of the two terms in equations (105) and (106) differ by less than 10.5 percent $[(N^2 \cosh y_s)/x_2 \leqq 0.1]$ and their relative angle departs from 180° by less than 0.1 radian $[(N^2 \sinh y_s)/x_2 \leqq 0.1]$.

Such a point $x_2 + jy_s$ is given by

$$x_2 = 10N^2 \cosh y_s \,. \tag{107}$$

We assume that $x_2 > \pi \geqq x_1$, or that

$$N^2 > \frac{\pi}{10} \approx 0.31416. \tag{108}$$

Since from equation (57) $N^2 \geqq 30/\pi^2$, this inequality is always satisfied.

We shall now write

$$I_T = I_3 + I_4 \,, \tag{109}$$

where

$$I_3 = \int_{-x_2}^{-\pi} G(x, y_s)\, dx + \int_{\pi}^{x_2} G(x, y_s)\, dx, \tag{110}$$

and

$$I_4 = \int_{-\infty}^{-x_2} G(x, y_s)\, dx + \int_{x_2}^{\infty} G(x, y_s)\, dx. \tag{111}$$

Noticing that

$$\frac{|\sin (x + jy_s)|}{|x + jy_s|} \leqq \frac{\cosh y_s}{(x^2 + y_s^2)^{\frac{1}{2}}} \leqq \frac{\cosh y_s}{x}, \tag{112}$$

we can show that

$$| I_3 | \leqq 2 \int_{\pi}^{x_2} \exp\,[-Q_R(x, y_s)]\, dx$$
$$+ 2(x_2 - \pi) \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[1 + N^2 \cosh y_s \frac{\ln\left(\dfrac{x_2}{\pi}\right)}{x_2 - \pi}\right], \tag{113}$$

and

$$\int_{\pi}^{x_2} \exp\,[-Q_R(x, y_s)]\, dx \leqq \sum_{\ell=1}^{K} \int_{(2\ell-1)\pi}^{2\ell\pi} \exp\,[-Q_R(x, y_s)]\, dx$$
$$+ \sum_{\ell=1}^{K} \int_{2\ell\pi}^{(2\ell+1)\pi} \exp\,[-Q_R(x, y_s)]\, dx, \tag{114}$$

where $K$ is an integer such that

$$(2K + 1)\pi > x_2 \geqq (2K - 1)\pi. \tag{115}$$

One can show that equation (115) is satisfied if

$$K = INT\left[\frac{x_2}{2\pi} + \frac{1}{2}\right]$$

or

$$K = INT\left[\frac{5N^2 \cosh y_s}{\pi} + \frac{1}{2}\right]. \tag{116}$$

We now have

$$0 < \frac{(n\pi)^2}{(n\pi)^2 + y_s^2} \leqq \frac{x^2}{x^2 + y_s^2} \leqq \frac{(n+1)^2\pi^2}{(n+1)^2\pi^2 + y_s^2},$$

$$n\pi \leqq x \leqq (n+1)\pi, \qquad n = 1, 2, 3, \cdots, \tag{117}$$

and

$$1 - \frac{y_s}{\tanh y_s}\frac{\sin x}{x} + y_s^2 \frac{1 - \cos x}{x^2} \geqq 1,$$

$$(2\ell - 1)\pi \leqq x \leqq 2\ell\pi, \qquad \ell = 1, 2, 3, \cdots . \tag{118}$$

From equations (31), (117), and (118) we can now prove that

$$\sum_{\ell=1}^{K} \int_{(2\ell-1)\pi}^{2\ell\pi} \exp\left[-Q_R(x, y_s)\right] dx$$

$$\leqq \sum_{\ell=1}^{K} \pi \exp\left[-N^2 \frac{\sinh y_s}{y_s} \frac{1}{1 + \dfrac{y_s^2}{\pi^2(2\ell-1)^2}}\right]$$

$$= \pi \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right] \sum_{\ell=1}^{K} \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{(2\ell-1)^2 + \dfrac{y_s^2}{\pi^2}}\right]. \tag{119}$$

Further it can be shown that*

$$\sum_{\ell=1}^{K} \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{(2\ell-1)^2 + y_s^2/\pi^2}\right]$$

$$< \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{1 + y_s^2/\pi^2}\right]$$

$$+ (K-1) \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{9 + y_s^2/\pi^2}\right]. \tag{120}$$

* The upper bound derived in equation (120) can be improved in various ways. Since this makes only a minor contribution to the total integral we shall be satisfied with this simple bound.

From equations (119) and (120) we have

$$\sum_{\ell=1}^{K} \int_{(2\ell-1)\pi}^{2\ell\pi} \exp\left[-Q_R(x, y_s)\right] dx$$
$$< \pi \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left\{\exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{1+(y_s/\pi)^2}\right]\right.$$
$$\left. + (K-1)\exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{9+(y_s/\pi)^2}\right]\right\}. \tag{121}$$

For $2\ell\pi + \pi/2 \leqq x \leqq (2\ell+1)\pi$, $\ell \geqq 1$, we can show that

$$1 - \frac{y_s}{\tanh y_s}\frac{\sin x}{x} + y_s^2 \frac{1-\cos x}{x^2}$$
$$\geqq 1 - \frac{y_s}{\tanh y_s}\frac{2}{(4\ell+1)\pi}\sin x + \frac{y_s^2}{(2\ell+1)^2\pi^2}(1-\cos x)$$
$$\equiv J_\ell(x). \tag{122}$$

In this range of $x$

$$\cos x \leqq 0, \tag{123}$$

$$\sin x \geqq 0, \tag{124}$$

and

$$\frac{\partial J_\ell}{\partial x} = -\frac{y_s}{\tanh y_s}\frac{2}{(4\ell+1)\pi}\cos x + \frac{y_s^2}{(2\ell+1)^2\pi^2}\sin x \geqq 0. \tag{125}$$

We, therefore, have

$$1 - \frac{y_s}{\tanh y_s}\frac{\sin x}{x} + y_s^2\frac{1-\cos x}{x^2}$$
$$\geqq J_\ell(x) \geqq J_\ell\left(2\ell\pi + \frac{\pi}{2}\right) \equiv V_\ell(y_s), \tag{126}$$

where

$$V_\ell(y_s) = 1 - \frac{y_s}{\tanh y_s}\frac{2}{(4\ell+1)\pi} + \frac{y_s^2}{(2\ell+1)^2\pi^2}. \tag{127}$$

It can be shown that

$$V_\ell(0) = 1 - \frac{2}{(4\ell+1)\pi}, \tag{128}$$

that $V_\ell(y_s)$ reaches its minimum at

$$y_s \approx \left[1 + \frac{4\ell^2}{4\ell + 1}\right]\pi, \tag{129}$$

and

$$[V_\ell(y_s)]_{\min} \approx 1 - \frac{1}{4}\left[1 + \frac{1}{4\ell + 1}\right]^2 > 0, \qquad \ell \geqq 1. \tag{130}$$

Now for $2\ell\pi \leqq x \leqq 2\ell\pi + \pi/2$, it can also be shown that

$$1 - \frac{y_s}{\tanh y_s}\frac{\sin x}{x} + y_s^2\frac{1 - \cos x}{x^2} \geqq 1 - \frac{y_s}{\tanh y_s}\frac{\sin x}{2\ell\pi} + \frac{4y_s^2}{(4\ell + 1)^2\pi^2}(1 - \cos x) \equiv L_\ell(x). \tag{131}$$

One can prove that $L_\ell(x)$ reaches its minimum at

$$x = 2\ell\pi + \tan^{-1}\left[\frac{(4\ell + 1)^2}{8\ell}\frac{\pi}{y_s \tanh y_s}\right], \tag{132}$$

and

$$[L_\ell(x)]_{\min} \equiv U_\ell(y_s) = 1 + \frac{4y_s^2}{(4\ell + 1)^2\pi^2} - \left[\frac{16y_s^4}{(4\ell + 1)^4\pi^4} + \frac{y_s^2}{4\ell^2\pi^2 \tanh^2 y_s}\right]^{\frac{1}{2}} \tag{133}$$

Next we can show that

$$U_\ell(0) = 1 - \frac{1}{2\ell\pi} < 1 - \frac{2}{(4\ell + 1)\pi} = V_\ell(0), \tag{134}$$

that $U_\ell(y_s)$ is a monotonically decreasing function of $y_s$ (see Fig. 9), and

$$\lim_{y_s \to \infty} U_\ell(y_s) = 1 - \frac{(4\ell + 1)^2}{32\ell^2} > 0, \qquad \ell \geqq 1. \tag{135}$$

It can also be proved by numerical methods (see Fig. 9) that

$$U_\ell(y_s) \geqq U_1(y_s),\ y_s \geqq 0, \qquad \ell \geqq 1, \tag{136}$$

and

$$V_\ell(y_s) > U_1(y_s),\ y_s \geqq 0, \qquad \ell \geqq 1. \tag{137}$$

We therefore conclude that

$$1 - \frac{y_s}{\tanh y_s} \frac{\sin x}{x} + y_s^2 \frac{1 - \cos x}{x^2} \geqq U_1(y_s) = 1 + \frac{4y_s^2}{25\pi^2} - \left[\frac{16y_s^4}{625\pi^4} + \frac{y_s^2}{4\pi^2 \tanh^2 y_s}\right]^{\frac{1}{2}}, \qquad \ell \geqq 1, \qquad y_s \geqq 0,$$
$$2\ell\pi \leqq x \leqq (2\ell + 1)\pi. \tag{138}$$

Equations (31), (117), and (138) show that

$$Q_R(x, y_s) \geqq N^2 \frac{\sinh y_s}{y_s} U_1(y_s) \frac{(2\ell)^2\pi^2}{(2\ell)^2\pi^2 + y_s^2}, \qquad 2\ell\pi \leqq x \leqq (2\ell + 1)\pi. \tag{139}$$

From equation (139) we can now write

$$\sum_{\ell=1}^{K} \int_{2\ell\pi}^{(2\ell+1)\pi} \exp\left[-Q_R(x, y_s)\right] dx \leqq \sum_{\ell=1}^{K} \pi \exp\left[-N^2 \frac{\sinh y_s}{y_s} U_1(y_s) \frac{(2\ell)^2\pi^2}{(2\ell)^2\pi^2 + y_s^2}\right]$$
$$= \pi \exp\left[-N^2 \frac{\sinh y_s}{y_s} U_1(y_s)\right] \cdot \sum_{\ell=1}^{K} \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{(2\ell)^2 + (y_s/\pi)^2}\right]. \tag{140}$$

It can be shown that

$$\sum_{\ell=1}^{K} \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{(2\ell)^2 + (y_s/\pi)^2}\right] < \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{4 + (y_s/\pi)^2}\right] + (K - 1) \exp\left[\frac{N^2 y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{16 + (y_s/\pi)^2}\right]. \tag{141}$$

Equations (140) and (141) yield

$$\sum_{\ell=1}^{K} \int_{2\ell\pi}^{(2\ell+1)\pi} \exp\left[-Q_R(x, y_s)\right] dx < \pi \exp\left[-N^2 \frac{\sinh y_s}{y_s} U_1(y_s)\right] \cdot \left\{\exp\left[N^2 \frac{y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{4 + (y_s/\pi)^2}\right] + (K - 1) \exp\left[N^2 \frac{y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{16 + (y_s/\pi)^2}\right]\right\}. \tag{142}$$

From equations (113), (114), (121), and (142) we can write

$$| I_3 | < \left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}} E_3 \, , \tag{143}$$

where

$$\begin{aligned} E_3 = \left(\frac{N^2 A_2}{2\pi}\right)^{\frac{1}{2}} \Bigg\{ & 2\pi \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right] \\ & \cdot \left[\exp\left(\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{1 + (y_s/\pi)^2}\right)\right. \\ & \left. + (K - 1) \exp\left(\frac{N^2 y_s \sinh y_s}{\pi^2} \frac{1}{9 + (y_s/\pi)^2}\right)\right] \\ & + 2\pi \exp\left[-N^2 \frac{\sinh y_s}{y_s} U_1(y_s)\right] \\ & \cdot \left\{\exp\left[N^2 \frac{y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{4 + (y_s/\pi)^2}\right]\right. \\ & \left. + (K - 1) \exp\left[N^2 \frac{y_s \sinh y_s}{\pi^2} U_1(y_s) \frac{1}{16 + (y_s/\pi)^2}\right]\right\} \\ & + 2(x_2 - \pi) \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[1 + N^2 \cosh y_s \frac{\ln\left(\frac{x_2}{\pi}\right)}{x_2 - \pi}\right]\Bigg\} \cdot \end{aligned} \tag{144}$$

Finally, from equation (111) we have

$$| I_4 | \leqq 2 \int_{x_2}^{\infty} | G(x, y_s) | \, dx. \tag{145}$$

Now from equations (27)–(28)

$$\begin{aligned} | G(x, y_s) | = {} & \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right] \\ & \cdot \left| \exp\left[N^2 \frac{\sin (x + jy_s)}{x + jy_s}\right] - \left[1 + N^2 \frac{\sin (x + jy_s)}{x + jy_s}\right] \right| . \end{aligned} \tag{146}$$

If $z$ is a complex variable, it can be shown that

$$| \exp (z) - 1 - z | \leqq \frac{| z |^2}{2} \exp | z |. \tag{147}$$

From equations (112), (145)–(147), we can write

$$| I_4 | \leqq \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right] N^4 \cosh^2 y_s$$

$$\cdot \int_{x_2}^{\infty} \frac{1}{x^2 + y_s^2} \exp\left[\frac{N^2 \cosh y_s}{(x^2 + y_s^2)^{\frac{1}{2}}}\right] dx$$

$$= \frac{N^4 \cosh^2 y_s}{y_s} \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]$$

$$\cdot \int_0^{N^2 \cosh y_s/(x_2^2+y_s^2)^{\frac{1}{2}}} \frac{\exp(t)}{\left(\dfrac{N^4 \cosh^2 y_s}{y_s^2} - t^2\right)^{\frac{1}{2}}} dt$$

$$= \frac{N^4 \cosh^2 y_s}{y_s} \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]$$

$$\cdot \int_0^{\sin^{-1} y_s/(x_2^2+y_s^2)^{\frac{1}{2}}} \exp\left[\frac{N^2 \cosh y_s}{y_s} \sin\theta\right] d\theta. \qquad (148)$$

Since

$$0 \leqq \sin\theta \leqq \theta, \qquad 0 \leqq \theta \leqq \sin^{-1} \frac{y_s}{(x_2^2 + y_2^2)^{\frac{1}{2}}} < \frac{\pi}{2}, \qquad (149)$$

we can show from equation (148) that

$$| I_4 | < \frac{N^4 \cosh^2 y_s}{y_s} \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]$$

$$\cdot \int_0^{\sin^{-1} y_s/(x_2^2+y_s^2)^{\frac{1}{2}}} \exp\left[\frac{N^2 \cosh y_s}{y_s} \theta\right] d\theta$$

$$= N^2 \cosh y_s \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]$$

$$\cdot \left\{\exp\left[N^2 \frac{\cosh y_s}{y_s} \sin^{-1} \frac{y_s}{(x_2^2 + y_s^2)^{\frac{1}{2}}}\right] - 1\right\}. \qquad (150)$$

Now we have

$$0 \leqq \sin^{-1} \sigma \leqq \frac{\pi}{2} \sigma, \qquad 0 \leqq \sigma \leqq 1. \qquad (151)$$

From equations (149)–(151), we can write

$$| I_4 | < N^2 \cosh y_s \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]$$

$$\cdot \left\{\exp\left[\frac{\pi}{2} N^2 \frac{\cosh y_s}{(100N^4 \cosh^2 y_s + y_s^2)^{\frac{1}{2}}}\right] - 1\right\} < N^2 \cosh y_s$$

$$\cdot \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[\exp\left(\frac{\pi}{20}\right) - 1\right] = \left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}} E_4 , \qquad (152)$$

where

$$E_4 = \left(\frac{N^2 A_2}{2\pi}\right)^{\frac{1}{2}} N^2 \cosh y_s \exp\left[-N^2 \frac{\sinh y_s}{y_s}\right]\left[\exp\left(\frac{\pi}{20}\right) - 1\right]. \qquad (153)$$

From equations (27), (49), (69), (71), (84), (86), (102), (109), (143), and (152) we can write the following bounds for Re $I$:

$$\left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}} \{1 - E_1' - E_2 - E_3 - E_4\}$$
$$< \operatorname{Re} I < \left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}} [1 + E_1 + E_2 + E_3 + E_4]. \qquad (154)$$

It has been shown that

$$A_2 = \frac{\sinh y_s}{y_s} - \frac{2}{y_s} \frac{f}{N^2 W}. \qquad (155)$$

## V. UPPER AND LOWER BOUNDS TO $S_V(f)$

We have shown in the previous section that

$$S_V(f) = \exp(-N^2)\left\{\delta(f) + \frac{N^2}{2W}[u_{-1}(f + W) - u_{-1}(f - W)]\right\}$$
$$+ \frac{1}{2\pi W} \exp\left\{-2N^2\left[\cosh^2 \frac{y_s}{2} - \frac{\sinh y_s}{y_s}\right]\right\}\mu \qquad (156)$$

where

$$\left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}} \{1 - E_1' - E_2 - E_3 - E_4\}$$
$$< \mu < \left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}} [1 + E_1 + E_2 + E_3 + E_4], \qquad (157)$$

$$\frac{\cosh y_s}{y_s} - \frac{\sinh y_s}{y_s^2} = \frac{f}{N^2 W}, \qquad (25)$$

$$A_2 = \frac{\sinh y_s}{y_s} - \frac{2}{y_s} \frac{f}{N^2 W}. \qquad (155)$$

| *Parameter* | *Equation* |
|---|---|
| $E_1$ | 70 |
| $E_1'$ | 85 |
| $E_2$ | 103 or 104 |
| $E_3$ | 144 |
| $E_4$ | 153 |

For $N^2 = 10$ and 25 we plot, in Figs. 10–16, $E_1$, $E_1'$, $E_2$, $E_3$, and $E_4$. Notice that $E_1$, $E_1'$, $E_2$, $E_3$, and $E_4$ appearing in these bounds are all very small compared to unity so long as the modulation index is moderately high, and that $E_1$, $E_3$, and $E_4$ are monotonically decreasing functions of $f$ and $N^2$. Also notice that $E_1'$ and $E_2$ may first increase (see Figs. 11, 12) with $y_s$ (or $f$), reach their maxima and then decrease with $y_s$.* It can be shown that these maxima are all very small compared to unity for all $N^2$ which are even moderately high.

For all $f$, we can then write

$$\left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}}(1 - C) < \mu < \left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}}(1 + D), \tag{158}$$

where

$$C = E_1' + E_2 + E_3 + E_4 , \tag{159}$$

and

$$D = E_1 + E_2 + E_3 + E_4 . \tag{160}$$

From Figs. 10–16 and expressions for $C$ and $D$, we can show that $C$ and $D$ are both small (<2%) compared to unity for $N^2 > 25$ and for all $f$. Hence we deduce that

$$\mu \approx \left(\frac{2\pi}{N^2 A_2}\right)^{\frac{1}{2}}, \tag{161}$$

and that the fractional error in this approximation is very much less than unity (<2%).

For $N^2 = 10$ and 25 the spectral density $S_V(f)$ and the fractional error $C$ and $D$ obtained from equations (158)–(161) are plotted in Figs. 17–20. From these figures notice that $C$ and $D$ are less than 10 percent for $N^2 > 10$,† and that

$$C < 2\%, \quad \text{for} \quad N^2 \geqq 25, \tag{162}$$

$$D < 2\%, \quad \text{for} \quad N^2 \geqq 25, \tag{163}$$

proving the assertion made earlier in this paper.

For $N^2 = 6$ the spectral density obtained from equations (158), and (161) is given in Fig. 21; the percentage error between this spectral

* One of the terms in $E_1'$ is independent of $f$ and $N^2$.

† By modifying the contour of integration we also have been able to show that $C$ and $D$ are less than 8% for $N^2 \geqq 10$. Since this modified contour leads to unnecessary complications, we have not given that modified contour analysis in this paper.

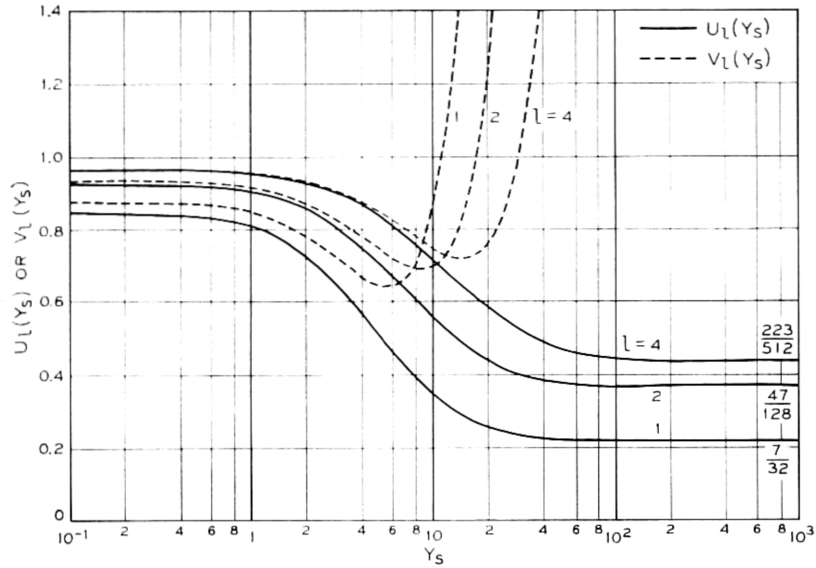


Fig. 9 — Functions $U_\ell(y_s)$ and $V_\ell(y_s)$. It can be observed that $V_\ell(y_s) > U_1(y_s) > 0$, $\ell \geqq 1$.

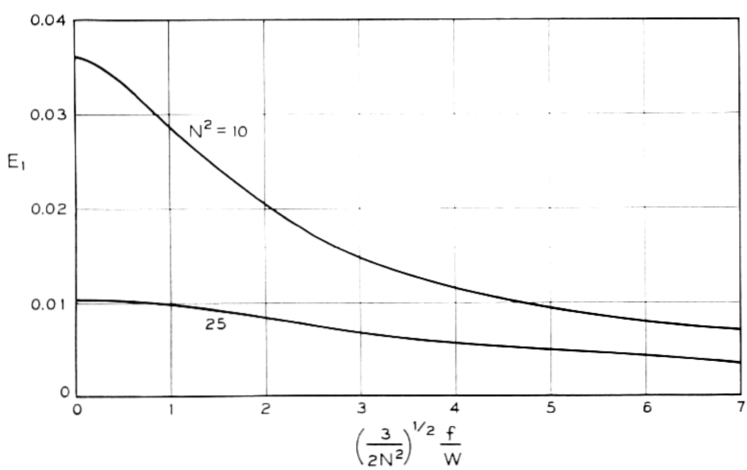


Fig. 10 — Parameter $E_1$ as a function of $\left(\frac{3}{2N^2}\right)^{\frac{1}{2}} \frac{f}{W}$.

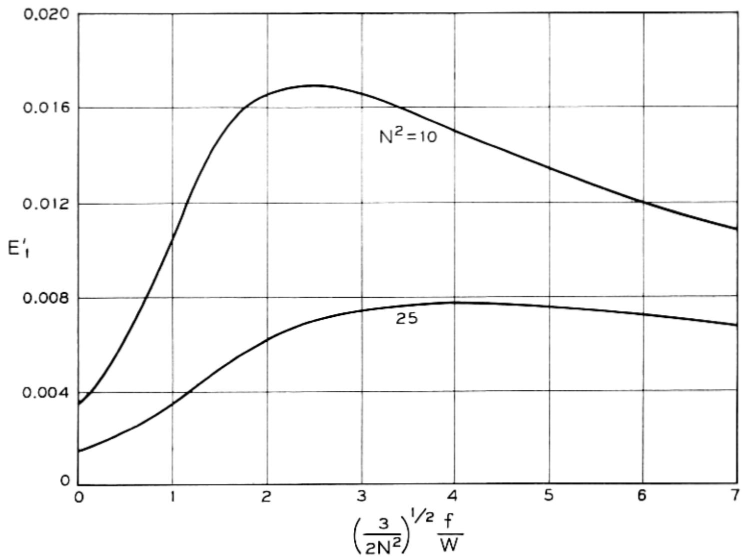


Fig. 11 — Parameter $E_1'$ as a function of $\left(\frac{3}{2N^2}\right)^{\frac{1}{2}} \frac{f}{W}$.

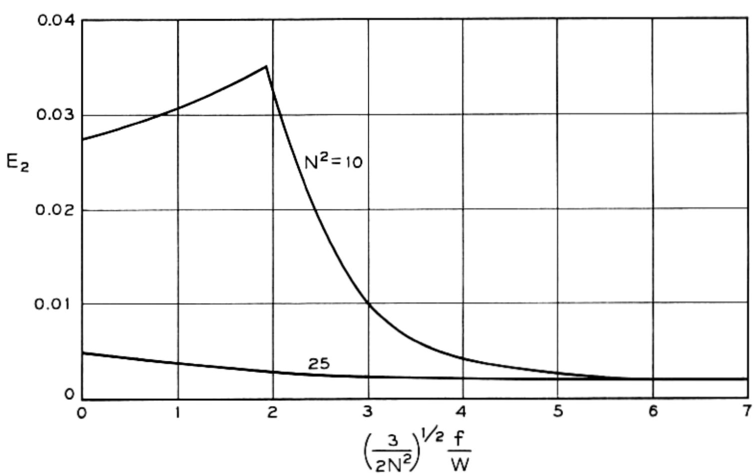


Fig. 12 — Parameter $E_2$ as a function of $\left(\frac{3}{2N^2}\right)^{\frac{1}{2}} \frac{f}{W}$.

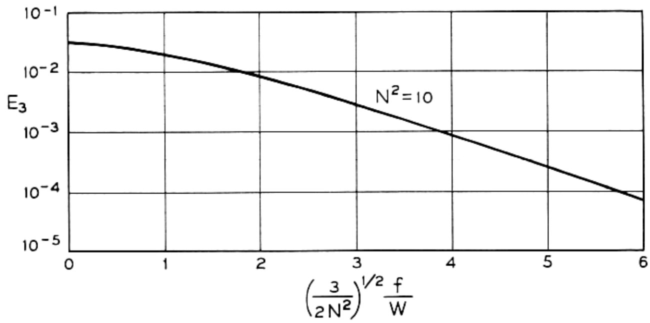


Fig. 13 — Parameter $E_3$ as a function of $\left(\frac{3}{2N^2}\right)^{\frac{1}{2}} \frac{f}{W}$, with $N^2 = 10$.

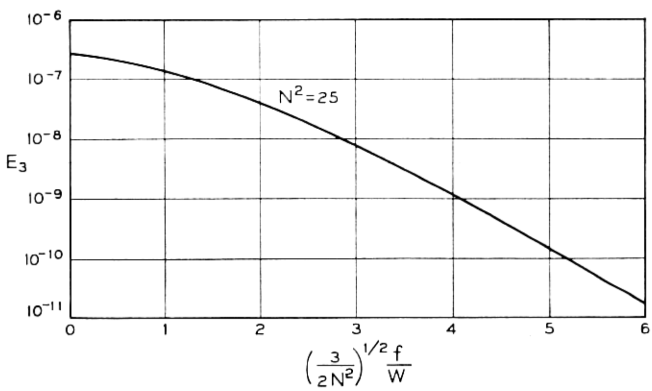


Fig. 14 — Parameter $E_3$ as a function of $\left(\frac{3}{2N^2}\right)^{\frac{1}{2}} \frac{f}{W}$, with $N^2 = 25$.

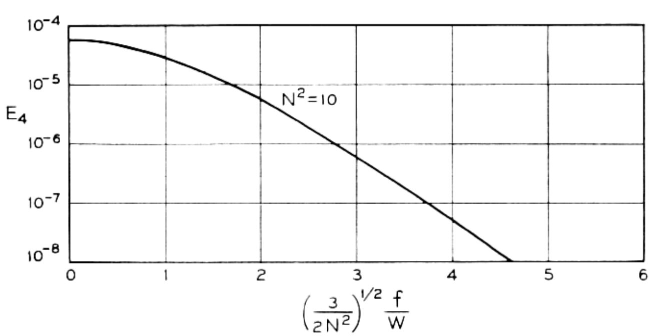


Fig. 15 — Parameter $E_4$ as a function of $\left(\frac{3}{2N^2}\right)^{\frac{1}{2}} \frac{f}{W}$, with $N^2 = 10$.

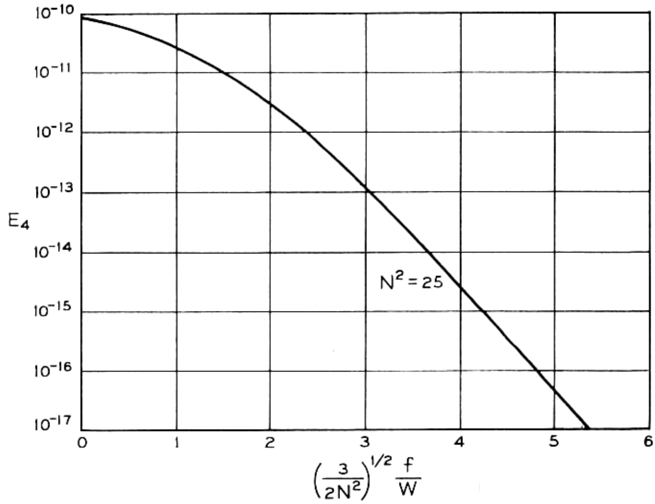


Fig. 16 — Parameter $E_4$ as a function of $\left(\frac{3}{2N^2}\right)^{\frac{1}{2}} \frac{f}{W}$, with $N^2 = 25$.

density and that obtained from equation (19) has been plotted in Fig. 22 (for a set of values of $f/W$). The scatter diagram in Fig. 22 indicates that the spectral densities obtained from the two methods agree very closely and that the saddle-point approximation error is not related in a simple way to the truncation error (it does not seem possible to draw a smooth curve through the points shown in Fig. 22).

For all practical purposes, including interference calculations, estimation of the spectrum to such an accuracy is almost always sufficient. It can therefore be said that the saddle-point approximation given by equations (25), (155), (158), and (161) is a good approximation to $S_V(f)$ as long as the modulation index is even moderately high ($N^2 > 10$). The spectrum can be estimated by this method for all values of $f$ even when it is millions of decibels smaller than the continuous part of the spectrum at $f = 0$.

Now compare the spectrum obtained from the quasistatic approximation* to that obtained from saddle-point approximation. For this purpose, the spectra obtained from equation (21) for $N^2 = 10$ and 25 are plotted in Figs. 17, and 19. We see that the spectra obtained from the quasistatic approximation agree very closely with those obtained from the saddle-point approximation for low frequencies, but that the quasi-

* See equation (21).

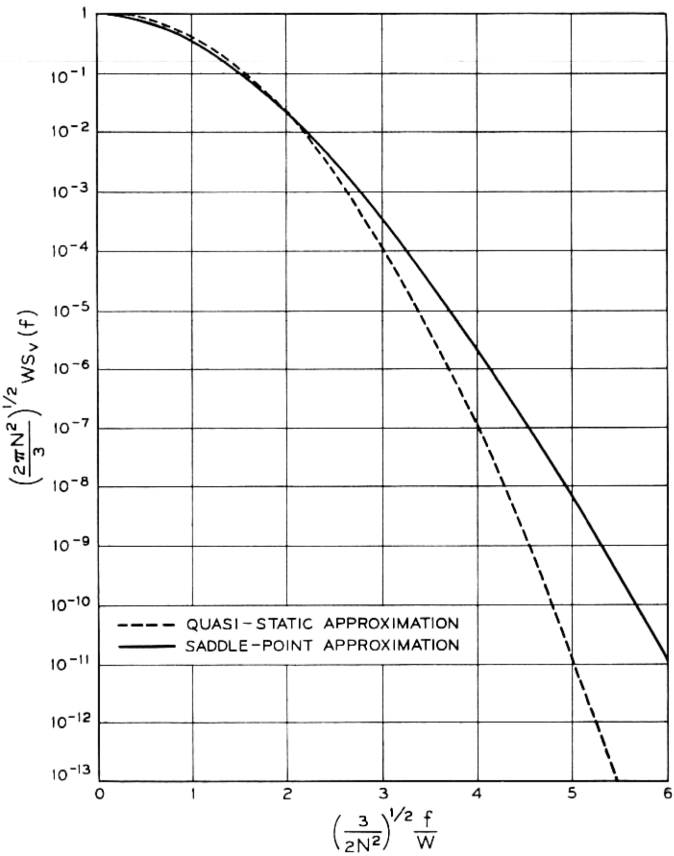


Fig. 17 — Spectral density of an angle-modulated wave, with gaussian phase modulation with a rectangular spectrum. $N = (10)^{1/2} \approx 3.162$ radians, rms phase deviation.

static approximation to $S_V(f)$ is too small for large $f$.* In fact for $N^2 = 10$ the quasistatic approximation is 30 dB too small for $f/W \approx 13.5$. We have therefore shown that the quasistatic approximation to the spectrum cannot be used in any interference calculations or in any calculations where the behavior of the spectrum on the tails is of importance.† The saddle-point approximation can be used at all frequencies as long as $N^2$ is moderately high.

* For small $f$ (or small $y_s$) it can easily be shown that the saddle-point approximation reduces to the quasistatic approximation.

† The higher the rms phase deviation, the further out will the low-frequency (quasistatic) approximation be valid,

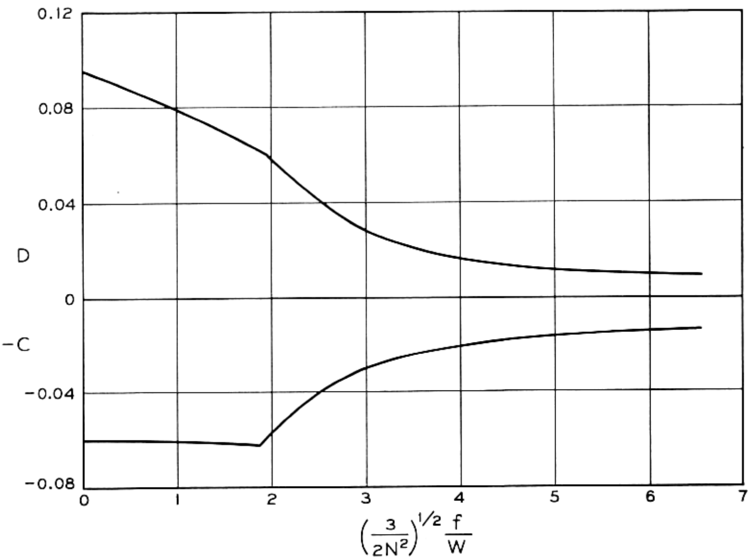


Fig. 18 — Bounds on fractional error in saddle-point approximation to the spectrum. $N^2 = 10$.

## VI. RESULTS AND CONCLUSIONS

A simple method (called the saddle-point method) has been presented in this paper to estimate the spectrum of a sinusoidal carrier phase modulated by gaussian noise having a rectangular power spectrum.

This method gives upper and lower bounds to the spectrum and shows that these bounds are very close for all $f$ and for all moderately high phase deviations. We also show that the fractional error in the saddle-point approximation is less than 2 percent for $N^2 \geqq 25$ and for all $f$.

The calculation of the spectrum by the saddle-point method is rather simple. For a given value of $f$, $N^2$, and $W$, we calculate $y_s$ from equation (25) and $A_2$ from equation (155). The spectrum $S_V(f)$ is then calculated from equations (156) and (161).

We have also shown in this paper that the quasistatic approximation to $S_V(f)$ is only good at low frequencies, and that for large $f$ the results obtained from that approximation are too small.

APPENDIX

It can be shown (see Ref. 7, p. 114) that

$$S_V(f) = \frac{1}{2\pi W} \int_{-\infty}^{\infty} \exp\left[ -\frac{N^2}{2W} \int_{-W}^{W} (1 - \cos 2\pi\mu\tau)\, d\mu \right] \cdot \exp\left[-j2\pi f\tau\right] d\tau, \qquad (164)$$

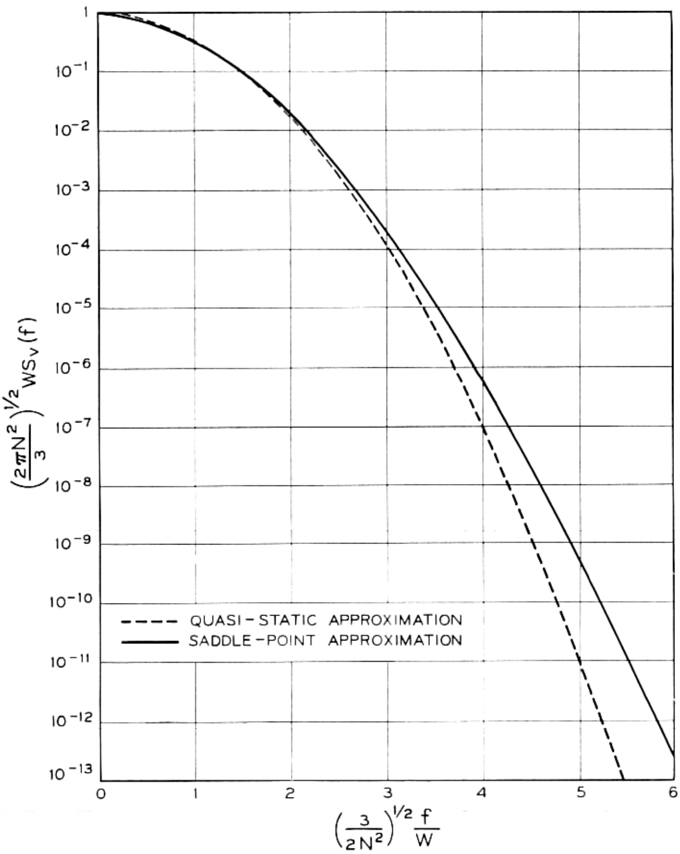


Fig. 19 — Spectral density of an angle-modulated wave, with gaussian phase modulation with a rectangular spectrum. $N = 5$ radians, rms phase deviation.

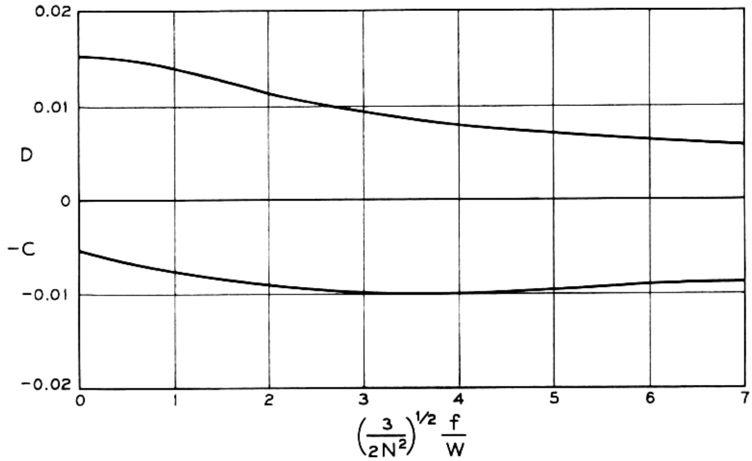


Fig. 20 — Bounds on fractional error in saddle-point approximation to the spectrum. $N^2 = 25$.

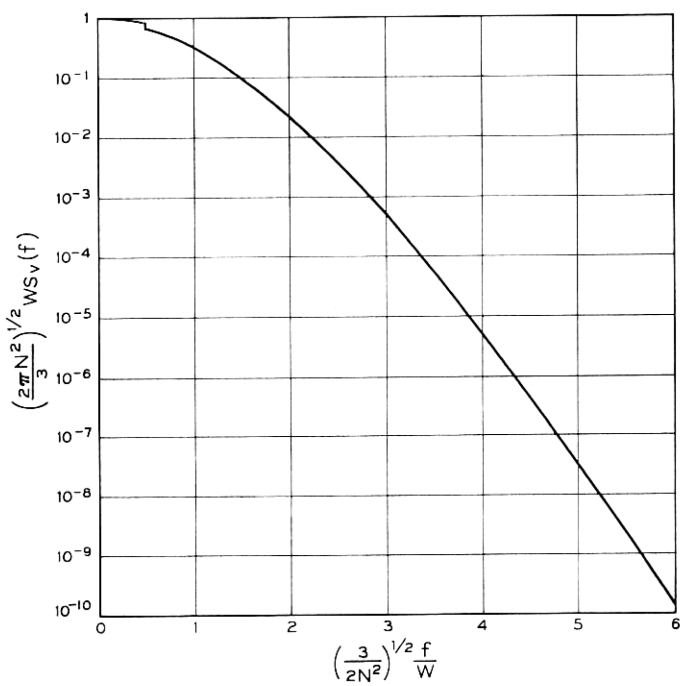


Fig. 21 — Spectral density of an angle-modulated wave, with gaussian phase modulation with a rectangular spectrum. $N = (6)^{1/2} \approx 2.449$ radians, rms phase deviation.

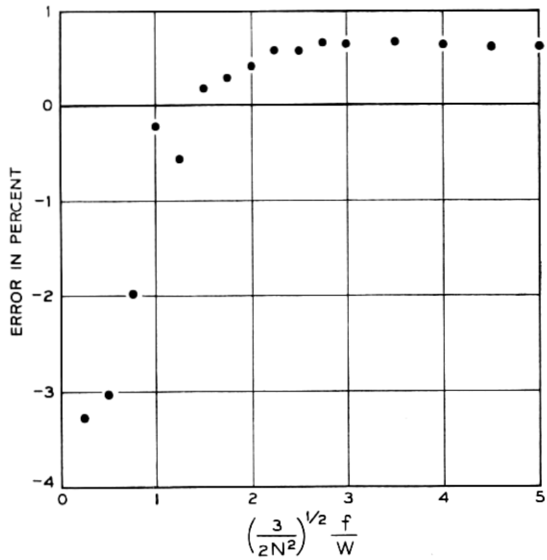


Fig. 22 — Percentage error between the spectral densities obtained from saddle-point approximation and that obtained from equation (19). It does not seem possible to draw a smooth curve through the points shown in this figure. We have, therefore, shown the error as a scatter diagram.

or

$$S(f) = \int_{-\infty}^{\infty} \left\{ \exp\left[ -\frac{N^2}{2W} \int_{-W}^{W} \left(1 - \cos\frac{\mu}{W} p\right) d\mu \right] - \exp\left[-N^2\right] \left[ 1 + N^2 \frac{\sin p}{p} \right] \right\} e^{j\lambda p}\, dp. \tag{165}$$

From equation (165), it can be shown that

$$I = \exp\left[ -N^2 \frac{\sinh y_s}{y_s} \right] \cdot \int_{-\infty}^{\infty} \left\{ \exp\left[ \frac{N^2}{2W} \int_{-W}^{W} \cos\frac{\mu}{W}(x + jy_s)\, d\mu \right] - \left[ 1 + N^2 \frac{\sin(x + jy_s)}{x + jy_s} \right] \right\} e^{j\lambda x}\, dx, \tag{166}$$

and

$$Q_R(x, y_s) = N^2 \frac{\sinh y_s}{y_s} - \operatorname{Re} \frac{N^2}{2W} \int_{-W}^{W} \cos\frac{\mu}{W}(x + jy_s)\, d\mu, \tag{167}$$

or

$$Q_R(x, y_s) = N^2 \frac{\sinh y_s}{y_s} - \frac{N^2}{W} \int_0^W \cos \frac{\mu}{W} x \cosh \frac{\mu}{W} y_s \, d\mu. \tag{168}$$

Equation (168) yields

$$\frac{\partial Q_R(x, y_s)}{\partial x} = \frac{N^2}{W} \int_0^W \frac{\mu}{W} \sin \frac{\mu}{W} x \cosh \frac{\mu}{W} y_s \, d\mu. \tag{169}$$

For $0 \leqq \mu x/W \leqq \pi$,

$$\sin \frac{\mu}{W} x \geqq 0. \tag{170}$$

For $y_s \geqq 0$, and $0 \leqq \mu \leqq W$,

$$0 \leqq x \leqq \pi, \tag{171}$$

$$\frac{\mu}{W} \sin \frac{\mu}{W} x \cosh \frac{\mu}{W} y_s \geqq 0, \tag{172}$$

and from (169),

$$\frac{\partial Q_R(x, y_s)}{\partial x} \geqq 0, \qquad 0 \leqq x \leqq \pi. \tag{173}$$

From equation (173) we then conclude that $Q_R(x, y_s)$ is a monotonically increasing function of $x$ for $0 \leqq x \leqq \pi$.

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