Some Theory and Applications of Periodically Coupled Waves

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Parallel-traveling waves can interact with complete power transfer even though they have different phase constants, provided that the coupling is periodic. This paper outlines some possible applications of this phenomenon, including mode transforming devices, frequency-selective filters in the microwave and laser wavelength regions, and parametric amplifiers or converters. This paper also gives some coupled-wave equations for interactions in a nonlinear medium and a generalization of the Tien conditions for parametric wave interaction.

I. INTRODUCTION

In a previous paper it was shown that two parallel-traveling coupled waves can interact with complete power interchange even though they have different phase constants.[1] This is accomplished by introducing a variation in coupling in the direction of wave propagation. The ideal coupling variation is a pure phase variation whose period exactly matches the beat period between the uncoupled waves, however, it was also shown in that paper that a simple periodic magnitude variation of the coupling can also yield complete power interchange between waves having different phase constants.

In this paper we outline some of the possible applications of periodic coupling. Complete power exchange between two modes of a single hollow metallic waveguide is illustrated. In two dielectric or hollow metallic waveguides, or in a combination of them, complete power exchange (or a desired fractional exchange) can be arranged. Frequency selective filters in the above structures can be obtained or broadband interactions can be chosen by suitable design. The periodic coupling phenomenon can be applied in lumped element parametric devices by modulating the pump waveform periodically; we give the resulting conditions that the signal frequency, idler frequency, pump frequency, and modulation frequency must fulfill.

Finally, in distributed parametric devices the periodic coupling principle can be used to advantage; spatial variation of the coupling gives a modified phase-matching relation that may render useful long lengths (with guided waves or unguided waves) of materials not useful with previous vectorial phase matching relations; time modulation of the pump introduces new frequency relations of possible use in modulators or frequency translators. The frequency range in which such applications may be useful extends from the laser region to the lowest frequency at which distributed coupled-wave interactions are convenient.

Section II presents some theory needed to understand the device illustrations. In Appendices A and B and in the discussion of parametric devices, we develop some coupled-wave equations to facilitate analysis of nonlinear circuits with generalized time- and space- dependent couplings. This paper is a survey of potential applications and is intended as a stimulus for further work. Complete design relations and experimental verification are not included.

II. GENERAL THEORY

We deal with devices or situations in which two waves of amplitude E_1 and E_2 are coupled according to

$$\frac{d}{dz} E_1(z) = -\gamma_1 E_1 + c_{21}(z) E_2 \tag{1}$$

$$\frac{d}{dz} E_2(z) = -\gamma_2 E_2 + c_{12}(z) E_1 \tag{2}$$

in which γ_1 and γ_2 are the complex propagation constants and c_{12} and c_{21} are coupling functions. In a previous paper we showed that the coupling distributions summarized in Table I lead to wave interactions virtually the same as those which are familiar for c_{21} and c_{12} independent of z, provided that transformations for coupling magnitude c_* and differential phase constant $\Delta\beta_*$ are appropriately defined. For $E_1 = 1.0$ and $E_2 = 0$ at $z = 0$ the solutions for equations (1) and (2) are

$$E_1(z) = \exp(-\gamma_1 z)[A \exp(r_1 z) + B \exp(r_2 z)] \tag{3}$$

$$E_2(z) = \frac{\exp(-\gamma_1 z)}{2\sqrt{\ }} [\exp(r_1 z) - \exp(r_2 z)] \tag{4}$$

TABLE I — VALUES OF c_* AND $\Delta\beta_*$ FOR VARIOUS PERIODIC COUPLING FUNCTIONS

Coupling Type	Coupling Definition	c_*	$\Delta\beta_*$
1	$c_{12} = c_{21} = jc$	c	$\Delta\beta = \beta_1 - \beta_2$
2	$c_{12} = jc \exp\left(-j\,\frac{2\pi z}{\lambda_m}\right)$ $c_{21} = jc \exp\left(j\,\frac{2\pi z}{\lambda_m}\right)$	c	$\Delta\beta - \frac{2\pi}{\lambda_m}$
3	$c_{12} = c_{21} = jc \sin\left(\frac{2\pi z}{\lambda_m}\right)$	$\frac{c}{2}$	$\Delta\beta - 2\pi/\lambda_m$
4	symmetrical square wave $c_{12} = c_{21} = jc$ for $n\lambda_m < z < \lambda_m(n + \frac{1}{2})$ $c_{12} = c_{21} = -jc$ for $(n + \frac{1}{2})\lambda_m < z < (n + 1)\lambda_m \quad n = 0, 1, 2, \cdots$	$\frac{2}{\pi} c$	$\Delta\beta - \frac{2\pi}{\lambda_m}\left[1 - \left(\frac{c\lambda_m}{\pi}\right)^2\right]^{\frac{1}{2}}$
5	raised square wave $c_{12} = c_{21} = j2c$ for $n\lambda_m < z < \lambda_m(n + \frac{1}{2})$ $c_{12} = c_{21} = 0$ for $(n + \frac{1}{2})\lambda_m < z < (n + 1)\lambda_m \; n = 0, 1, 2, \cdots$	$\frac{2}{\pi} c$ c	$\Delta\beta - \frac{2\pi}{\lambda_m}\left[1 - \left(\frac{c\lambda_m}{\pi}\right)^2\right]^{\frac{1}{2}}$ $\Delta\beta$

in which

$$A = \frac{1}{2} - \frac{1}{2}\,\frac{\left(\frac{\Delta\beta_*}{2c_*}\right) - i\left(\frac{\Delta\alpha}{2c_*}\right)}{\sqrt{}} \tag{5}$$

$$B = \frac{1}{2} + \frac{1}{2}\,\frac{\left(\frac{\Delta\beta_*}{2c_*}\right) - i\left(\frac{\Delta\alpha}{2c_*}\right)}{\sqrt{}} \tag{6}$$

$$r_{\substack{1\\2}} = \frac{\Delta\gamma_*}{2} \pm ic_*\sqrt{} \tag{7}$$

$$\sqrt{} = \left[1 + \left(\frac{\Delta\beta_*}{2c_*}\right)^2 - \left(\frac{\Delta\alpha}{2c_*}\right)^2 - i2\left(\frac{\Delta\beta_*}{2c_*}\right)\left(\frac{\Delta\alpha}{2c_*}\right)\right]^{\frac{1}{2}} \tag{8}$$

$$\begin{aligned} \Delta\gamma_* &= (\alpha_1 - \alpha_2) + i\Delta\beta_* \\ \gamma_1 &= \alpha_1 + i\beta_1 \\ \gamma_2 &= \alpha_2 + i\beta_2 \\ \gamma_1 - \gamma_2 &= \Delta\alpha + i\Delta\beta. \end{aligned} \tag{9}$$

In Table I we define the quantities c_* and $\Delta\beta_*$; λ_m is the wavelength of the coupling variation as defined in the second column of Table I.

In Table I, type 1 coupling is the familiar uniform coupling, independent of z. For negligible attenuation and for $\Delta\beta = 0$ the wave energy is exchanged cyclically between the two waves according to

$$E_1 = \cos (cz) \tag{10}$$

$$E_2 = i \sin (cz); \tag{11}$$

and for other values of $\Delta\gamma$ limited wave interactions occur. This has been described previously.[2]

In Table I, type 2 coupling corresponds to the exact transformations given for c_* and $\Delta\beta_*$; the other type couplings correspond to the approximate values given for c_* and $\Delta\beta_*$. For coupling types 1 and 2, equations (3) and (4) give exactly the coupled-wave amplitudes; for coupling types 3 and 4, equations (3) and (4) give the coupled wave amplitudes exactly at z equal to a multiple of $\lambda_m/2$, and may be in error by no more than about $0.2c\lambda_m/\pi$ at other values of z. The error may be slightly larger for coupling type 5, but is negligible for small $c\lambda_m$.

Figure 1 shows the initial buildup of the wave amplitude E_2 for coupling types 4 and 5. At $z = \lambda_m/2$ further extension of uniform coupling would result in added components to E_2 at such a phase as to diminish E_2. By reversing the sign of the type 4 coupling, the added components in the region $0.5\ \lambda_m < z < \lambda_m$ cause an increase in E_2. By reducing the magnitude of the type 5 coupling to zero at $\lambda_m = 0.5$, no components are added to E_2 in the region $0.5\ \lambda_m < z < \lambda_m$. At $z = \lambda_m$ the cycle repeats. In this way the amplitude variation in coupling versus z causes an average in-phase transfer of energy. The same behavior exists for an arbitrary amplitude variation of coupling $c(z)$; the fundamental Fourier component may be taken as the type 3 coupling and the resulting wave interaction calculated. The result is accurate provided that $c_p\lambda_m \ll 1$, where c_p is the peak of the coupling waveform.

III. FREQUENCY SENSITIVITY

In many coupled-wave devices the objective is to transfer all of the power from one wave to the other, and frequency sensitivity may be desirable (as in channel-selecting filters of a communication system) or may be undesirable. We show the magnitude of this frequency sensitivity.

Consider first two dielectric waveguides where most of the energy travels in the central dielectrics designated n_1 and n_2 (indexes of refraction) in Fig. 2. Periodic coupling is induced by the dielectric sheets labeled n_3, corresponding to type 5 coupling in Table I. Then, approximately

$$\Delta\beta = \frac{2\pi}{\lambda}(n_1 - n_2) \tag{12}$$

in which λ is free space wavelength. We assume the complete transfer condition, which is

$$c_* L = \frac{\pi}{2} \tag{13}$$

with L being the length of the coupling region. Also let

$$L = N\lambda_m \tag{14}$$

with

$$\lambda_m = \frac{2\pi}{\Delta\beta_o} \tag{15}$$

and $\Delta\beta_o$ defined as $\Delta\beta$ at the midband frequency $f = f_o$. Now $\Delta\beta_*$ as a

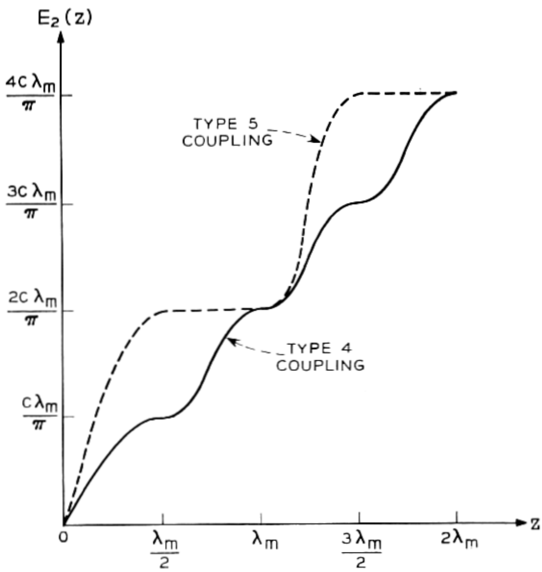

Fig. 1 — Transferred wave amplitude E_2 versus length of coupling region for type 4 and type 5 coupling (see Table I).

function of frequency is

$$\Delta\beta_*(f) = \Delta\beta(f) - \Delta\beta(f_o). \tag{16}$$

Expressing the frequency as a deviation from f_o

$$f = (1 + \delta)f_o , \tag{17}$$

we find

$$\Delta\beta_* = \frac{2\pi\ \delta}{\lambda_o}(n_1 - n_2) \tag{18}$$

with λ_o equal to λ at $f = f_o$. Using equations (18), (13), and (14) and assuming the typical case of negligible dependence of c_* on frequency, we find

$$\frac{\Delta\beta_*(f)}{c_*} = 4\ \delta N. \tag{19}$$

This ratio uniquely determines the frequency sensitivity of the wave interaction, according to

$$| E_2 | = \frac{1}{\left[\left(\frac{\Delta\beta_*}{2c_*}\right)^2 + 1\right]^{\frac{1}{2}}} \left| \sin \left\{ \left[\left(\frac{\Delta\beta_*}{2c_*}\right)^2 + 1\right]^{\frac{1}{2}} c_* L \right\} \right| \tag{20}$$

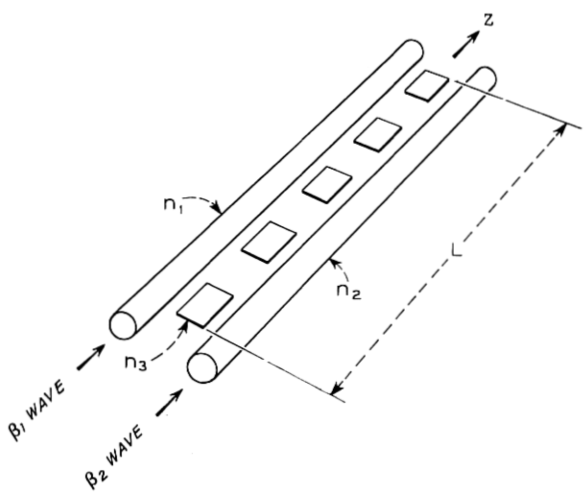

Fig. 2 — Dielectric waveguides (having indices of refraction n_1 and n_2) with periodic coupling.

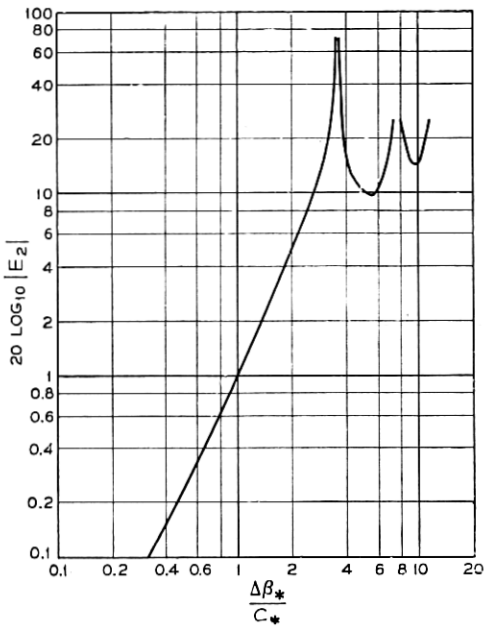

Fig. 3 — Transferred wave amplitude E_2 versus $\Delta\beta_*/c_*$, the frequency dependent parameters, for $c_*L = \pi/2$.

which follows from equation (4) with $\Delta\alpha = 0$. With complete transfer conditions $c_*L = \pi/2$ and with λ_m chosen to make $\Delta\beta_* = 0$ at $f = f_o$, equation (20) becomes unity at $f = f_o$ and falls off as $\Delta\beta_*(f)$ differs from zero, that is, as δ differs from zero in equation (17). Figure 3 shows E_2 versus $\Delta\beta_*/c_*$ for $c_*L = \pi/2$; values for this graph can be calculated from equation (20)†. Using these results and equation (19) we find the bandwidth properties of the periodically coupled wave interaction on dielectric waveguides. A few examples are listed in Table II. The first three rows illustrate broadband coupling; as long as N (the number of coupling periods in the total coupling length L) is five or less, very little variation from the complete transfer condition occurs. The fourth row illustrates that intentional frequency selectivity can be induced by using a large N; the 0.2 percent band at $N = 865$ yields $\Delta\beta_*/c_* = 3.46$, the location of the first null in Fig. 3. Structures analogous to Fig. 2 but actually fabricated in a solid sheet continuum are under consideration for laser beam circuitry. If a 20Å bandwidth to the first nulls is desired

† For $\Delta\beta_*/c_* < 2$, $20 \log_{10} | E_2 | \cong -1.1 \, |\Delta\beta_*/c_*|^2$ dB.

TABLE II — BANDWIDTH PROPERTIES OF PERIODICALLY COUPLED WAVES ON DIELECTRIC WAVEGUIDES

Percentage Bandwidth (200δ)	N	$20 \log \lvert E_2 \rvert$ Band Edge Loss (dB)
10	1	−0.04
10	3	−0.36
10	5	−1.1
0.2	865	$-\infty$

at 10,000Å midband and if $(n_1 - n_2) = 0.1$, we find $\lambda_m = 10\ \mu$m and the coupling length $L = 8.65$ mm. Frequency selectivity obtained in this way does not require low heat loss in the circuit; as long as the two waves have the same attenuation coefficient, loss does not limit the filter selectivity.

For waves in an infinite medium or in other types of waveguides, equation (20) remains valid but relations other than (19) must be found to describe the way $\Delta\beta_*/c_*$ varies with frequency. For waves in hollow metallic tubes the results are very similar to those for waves on dielectric rods. We show this with two illustrative examples as follows.

In any hollow metallic waveguide the phase constant of a mode is given by

$$\beta = \frac{2\pi}{\lambda}[1 - \mu^2]^{\frac{1}{2}} \tag{21}$$

where

λ = free space wavelength,
$\mu = f_c/f$,
f_c = cutoff frequency for the particular mode, and
f = operating frequency.

By defining $\lambda_o = \lambda$ at $f = f_o$

$$\mu_{10} = \mu \text{ for wave 1 at } f = f_o$$
$$\mu_1 = \mu \text{ for wave 1 at } f = f_o(1 + \delta);$$

and using similar definitions (not written out) for wave number 2, we find

$$\begin{aligned}\Delta\beta_*(f) &= \Delta\beta(f) - \Delta\beta(f_o)\\ &= \frac{2\pi}{\lambda_o}(1+\delta)[(1-\mu_1^2)^{\frac{1}{2}} - (1-\mu_2^2)^{\frac{1}{2}}]\\ &\quad - \frac{2\pi}{\lambda_o}[(1-\mu_{10}^2)^{\frac{1}{2}} - (1-\mu_{20}^2)^{\frac{1}{2}}].\end{aligned} \tag{22}$$

To develop a physical model, we take parameters typical of a 24,000 MHz $TE_{10}^{\square} - TE_{01}^{\circ}$ transducer similar to one described in connection with Fig. 42 of Ref. 2. We keep the same coupling length $L = 0.417$ m for complete transfer of power, corresponding to $c_* = 3.76\ \text{m}^{-1}$. We arbitrarily choose to explore the bandwidth when $\lambda_m = L/3 = 0.139\ m$. We keep the same rectangular guide width, 0.340 inches, which at $f_o = 24{,}000$ MHz gives $\mu_{10} = 0.723$. This determines that $\mu_{20} = 0.625$; there is a round guide diameter of 0.96 inches (optionally a particular μ_{20} larger than 0.723 could have been selected to give the same $|\Delta 2(f_o)|$ and λ_m). We can now calculate $\Delta\beta_*(f)/c_*$ from equation (22), neglecting variations in c_* for this estimate. For a 10 percent frequency band, that is, $\delta = 0.05$, we find $\Delta\beta_*/c_* = 1.01$ and the loss $20 \log_{10} E_2 = 1.1$ dB. The case, $N = L/\lambda_m = 3$, thus yields a result very similar to that obtained for dielectrically guided waves using equations (19) and (20) and shows broadband interaction capability for waves in guided tubes provided N is not too large. Sections V and VI discuss some factors which may motivate one to use periodic coupling instead of constant coupling.

Consider a second example in hollow metallic guides to illustrate intentional frequency selectivity. Assume we need a filter with center frequency $f = 50$ GHz and a 3 dB bandwidth of 1000 MHz. Then $\delta = 0.01$ and from equation (20) or Fig. 3 we find $(\Delta\beta_*/c_*) \cong 1.6$. We keep one wave at $\mu_{10} = 0.723$ as before and choose $\mu_{20} = 0.91$. We can calculate $\Delta\beta_*$ from these choices using equation (22) which yields $\Delta\beta_* = 8.95\ \text{m}^{-1}$ at $f = 1.01 f_o$. At this frequency we need $(\Delta\beta_*/c_*) = 1.6$, so c_* needs to be $5.58\ \text{m}^{-1}$ and complete transfer at f_o (that is, $c_* L = \pi/2$), requires $L = 0.28$ m. These are reasonable values physically; Section IV illustrates possible coupling and waveguide cross-sectional geometries. We now note that $N = L/\lambda_m$ for this case is 12.7. The same values of $\delta(0.01)$ and $N(12.7)$ for a dielectrically guided wave pair yield from equation (19) $\Delta\beta_*/c_* = 5.1$, indicating somewhat more selectivity in the dielectrically guided waves than in the hollow-tube guided waves, for the same number of coupling periods N.

IV. STRUCTURES FOR PASSIVE WAVE INTERACTIONS

We describe a few structures in which guided waves may be coupled periodically. The general diagram is given in Fig. 4. Most typically there is no input to wave 2 in this discussion although the transformations of Table I and equations (1) and (2) may be used to treat general inputs to the periodically coupled region. In some cases the two waves occupy the same space as discrete modes of a single structure. In other cases separate guiding structures for the two waves are provided.

In Ref. 1 a structure is described for hollow metallic waveguide

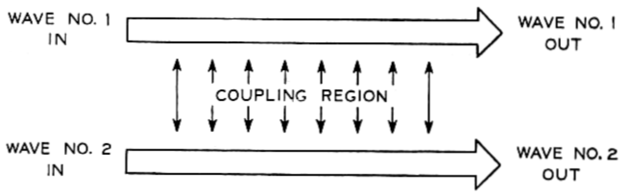

Fig. 4 — Two coupled waves; the dimension for the coupling region may be distance or time.

$TE_{10}^{\square} - TE_{01}^{\circ}$ coupling which closely approximates type 2 coupling and yields the simple transformation for $\Delta\beta_*$ of Table I without "harmonic" transformations for $\Delta\beta_*$. The harmonic transformations, discussed fully in Ref. 1, are characteristic of square-wave or sinusoidal coupling patterns and may yield appreciable wave interactions when $\Delta\beta_* \cong \Delta\beta p/\lambda_m$ with p an odd integer. The exponential type 2 coupling is thus a desirable one. However, because the harmonic interactions are weaker than the fundamental and may occur at greatly different frequencies, the square-wave and sinusoidal couplings are useful.

Figure 2 shows two dielectric waveguides periodically coupled with dielectric sheets yielding type 5 coupling of Table I. Its possible use as a frequency selective filter has already been referred to. Figure 5 shows the form it might take in laser circuitry where λ_m of 10 μm could be sought using photolithographic techniques; the substrate index n_s is to be less than n_1 and n_2.[3]

Figures 6 and 7 illustrate the way two modes of a single hollow metallic waveguide can be coupled periodically to achieve complete or partial

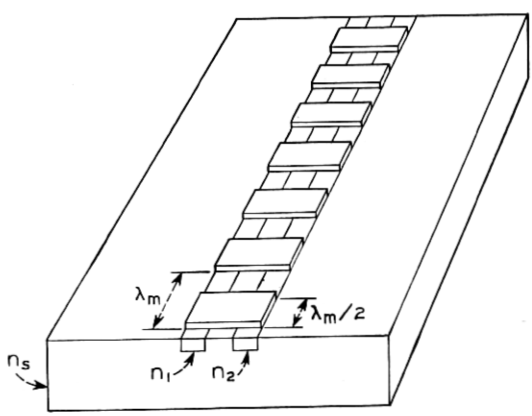

Fig. 5 — Periodically coupled dielectric waveguides.

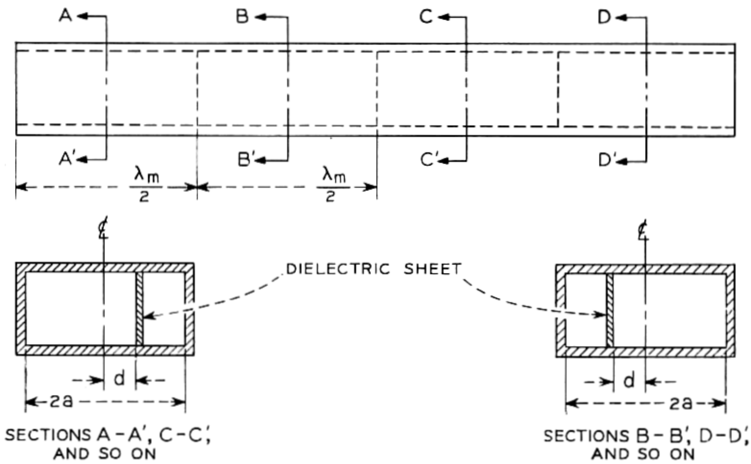

Fig. 6 — Periodic coupling structure for waves in a hollow, rectangular, metallic waveguide.

power interchange. In Fig. 6 the $TE_{10}^{\square}$ and $TE_{20}^{\square}$ modes are coupled by the dielectric sheet. The fields of these modes in a transverse plane are sketched in Fig. 8; a thin dielectric sheet introduces maximum coupling at a distance $d = 0.392a$, where the product of the two fields is a maximum. The coupling between the modes is reversed by moving the sheet to the opposite side of the guide centerline, as in section $B - B'$ of Fig. 6. A similar maximum coupling position can be found for the $TE_{11}^{\circ} - TE_{01}^{\circ}$ coupling, the fields for which are sketched in Fig. 9;

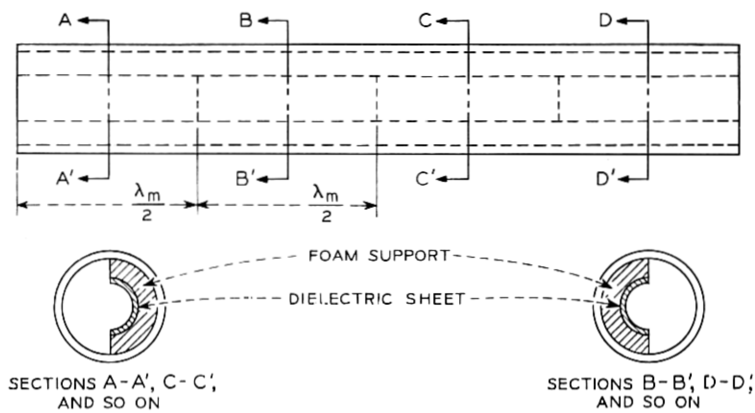

Fig. 7 — Periodic coupling structure for waves in a hollow, round, metallic waveguide.

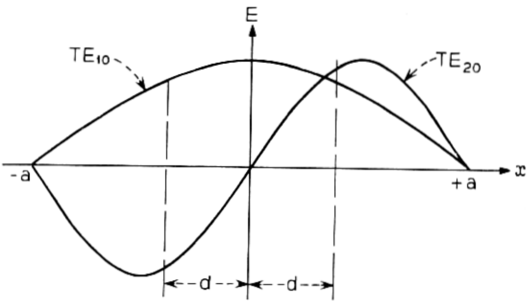

Fig. 8 — Transverse field distributions for $TE_{10}^{\square}$ and $TE_{20}^{\square}$.

Fig. 7 shows the structural form of coupler. In both Figs. 6 and 7 the length $\lambda_m/2$ is that at which the coupled modes develop π radians phase difference. This length is near that for π radians phase difference in an empty guide, which for Fig. 7 is approximately one diameter. (Specifically, in a $\frac{1}{3}$ inch-inside diameter guide at 54 GHz the half-beat wavelength for $TE_{11}^{\circ} - TE_{01}^{\circ}$ is about $\frac{1}{3}$ inch.) Structures of the type in Figs. 6 and 7 provide mode transformation without complicated and expensive shaping of the metallic walls.

Figures 10 and 11, which show the transverse cross sections of the guides, illustrate coupling between modes of different hollow metallic waveguides. Although $TE_{10}^{\square} - TE_{01}^{\circ}$ and $TE_{01}^{\circ} - TE_{01}^{\circ}$ couplings are indicated, any mode pair having common field components at the coupling aperture may be used. Figure 12 illustrates the type 5 coupling distribution, simulated by a series of discrete point couplings which should be spaced no more than about one-third guide wavelength. Either broadband power interchange or intentional frequency selectivity may be obtained.

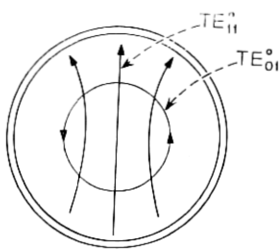

Fig. 9 — Transverse electric field lines for TE_{11}° and TE_{01}°.

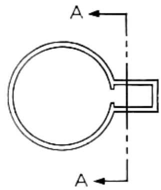

Fig. 10 — Transverse cross section for $TE_{10}{}^{\square} - TE_{01}{}^{\circ}$ coupling in hollow metallic waveguides.

V. LUMPED-ELEMENT PARAMETRIC DEVICES

Periodic coupling can be applied to lumped-element parametric devices; Figure 13 is a simplified version. The box labelled ω_1 is a filter presenting a short circuit at ω_1 and an open circuit at other frequencies; the filter box labelled ω_2 has similar characteristics

We assume a general time-varying capacitor

$$\mathcal{C}(t) = \mathcal{C}_0 + \mathcal{C}_p(t) \tag{23}$$

in which $\mathcal{C}_0$ is a constant. Appendix A shows that the normalized amplitudes representing the voltages and currents in the two resonant circuits can be described by the coupled-wave equations:

$$\frac{da_1}{dt} = j\omega_1 a_1 - \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_1 - a_1^*)}{\mathcal{C}_{11}} - \frac{(a_2 - a_2^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\} \tag{24}$$

$$\frac{da_1^*}{dt} = -j\omega_1 a_1^* + \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_1 - a_1^*)}{\mathcal{C}_{11}} - \frac{(a_2 - a_2^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\} \tag{25}$$

$$\frac{da_2}{dt} = j\omega_2 a_2 - \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_2 - a_2^*)}{\mathcal{C}_{22}} - \frac{(a_1 - a_1^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\} \tag{26}$$

$$\frac{da_2^*}{dt} = -j\omega_2 a_2^* + \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_2 - a_2^*)}{\mathcal{C}_{22}} - \frac{(a_1 - a_1^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\}. \tag{27}$$

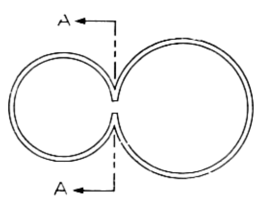

Fig. 11 — Transverse cross section for $TE_{01}{}^{\circ} - TE_{01}{}^{\circ}$ coupling in hollow metallic waveguides.

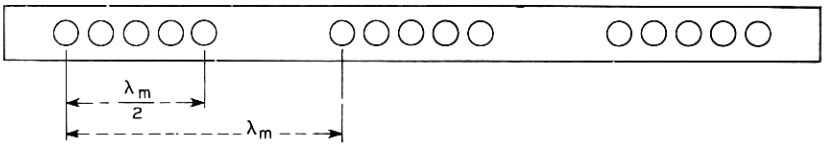

Fig. 12 — Section A-A′ for Figs. 10 and 11.

For $\mathcal{C}_p = 0$ the solutions to equations (24) through (27) are of the form

$$a_1 = A_1 \exp (j\omega_1 t) \tag{28}$$

$$a_1^* = A_1^* \exp (-j\omega_1 t) \tag{29}$$

$$a_2 = A_2 \exp (j\omega_2 t) \tag{30}$$

$$a_2^* = A_2^* \exp (-j\omega_2 t). \tag{31}$$

We now specify a periodically varying capacitance component

$$\mathcal{C}_p = \Delta\mathcal{C} \cos (\omega_p t + \varphi) \cos \omega_c t \tag{32}$$

and we proceed to determine the coupling coefficients in equations (24) through (27) and to deduce the frequency interrelations governing the parametric interaction.

In equation (24) only the frequencies of the term in $d(\ \ )/dt$ at ω_1 result in large coupled-wave interaction; similarly in equations (25) through (27) only frequencies near $-\omega_1$ are important. Moreover, in equation (24) the term in $(a_1 - a_1^*)$ is a reaction of circuit 1 upon itself, which for small coupling is negligible; we drop terms of that type. With these criteria for selection of important terms we find that putting equation (32) in equations (24) through (27) leads to the following as the only significant wave interaction

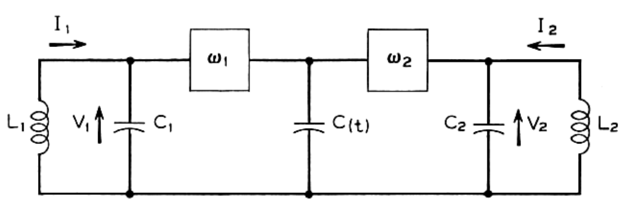

Fig. 13 — Lumped element parametric circuit.

$$\frac{da_1}{dt} = j\omega_1 a_1 - \frac{\Delta\mathcal{C}}{8[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}} \frac{d}{dt}$$

$$\cdot (a_2^* \{\exp [j(\omega_p t + \omega_c t + \varphi)] + \exp [j(\omega_p t - \omega_c t + \varphi)]\}) \tag{33}$$

$$\frac{da_2^*}{dt} = -j\omega_2 a_2^* - \frac{\Delta\mathcal{C}}{8[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}} \frac{d}{dt}$$

$$\cdot (a_1 \{\exp [j(\omega_c t - \omega_p t - \varphi)] + \exp [j(-\omega_c t - \omega_p y - \varphi)]\}). \tag{34}$$

Noting that $dA_2^*/dt \ll (\omega_p \pm \omega_c)$ in our loose coupling approximation [A_2^* defined as in equation (31)] and similarly for dA_1/dt, we find equations (33) and (34) reduce to

$$\frac{da_1}{dt} = j\omega_1 a_1 + c_{121} a_2^* \exp [j(\omega_p + \omega_c)t] + c_{122} a_2^* \exp [j(\omega_p - \omega_c)t] \tag{35}$$

$$c_{121} = \frac{-\Delta\mathcal{C} \exp (j\varphi)}{8[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}} j(\omega_p - \omega_2 + \omega_c) \tag{36}$$

$$c_{122} = \frac{-\Delta\mathcal{C} \exp (j\varphi)}{8[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}} j(\omega_p - \omega_2 - \omega_c) \tag{37}$$

$$\frac{da_2^*}{dt} = -j\omega_2 a_2^* + c_{212} a_1 \exp [j(-\omega_p + \omega_c)t] + c_{211} a_1 \exp [j(-\omega_p - \omega_c)t] \tag{38}$$

$$c_{211} = \frac{-\Delta\mathcal{C} \exp (-j\varphi)}{8[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}} j(-\omega_p - \omega_c + \omega_1) \tag{39}$$

$$c_{212} = \frac{-\Delta\mathcal{C} \exp (-j\varphi)}{8[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}} j(-\omega_p + \omega_c + \omega_1). \tag{40}$$

Note that

$$c_{212}^* = \frac{(\omega_p - \omega_c - \omega_1)}{(\omega_p - \omega_c - \omega_2)} c_{122} \tag{41}$$

$$c_{211}^* = \frac{(\omega_p + \omega_c - \omega_1)}{(\omega_p + \omega_c - \omega_2)} c_{121} . \tag{42}$$

Using relations (28) and (31) for a_1 and a_2^*, equations (35) and (38) reduce to

$$\frac{dA_1}{dt} = c_{121} A_2^* \exp [j(\omega_p + \omega_c - \omega_1 - \omega_2)t]$$

$$+ c_{122} A_2^* \exp [j(\omega_p - \omega_c - \omega_1 - \omega_2)t] \tag{43}$$

$$\frac{dA_2^*}{dt} = c_{211}A_1 \exp [j(-\omega_p - \omega_c + \omega_1 + \omega_2)t] + c_{212}A_1 \exp [j(-\omega_p + \omega_c + \omega_1 + \omega_2)t]. \tag{44}$$

For simple exponential buildup of A_1 and A_2^* there are two possible frequency relations; one is

$$\omega_1 + \omega_2 = \omega_p + \omega_c \tag{45}$$

which reduces equations (43) and (44) to

$$\frac{dA_1}{dt} = c_{121}A_2^* + c_{122}A_2^* \exp (-j2\omega_c t) \tag{46}$$

$$\frac{dA_2^*}{dt} = c_{211}A_1 + c_{212}A_1 \exp (j2\omega_c t). \tag{47}$$

Here the $c_{121} - c_{211}$ terms are important; the other terms give a small cyclical variation on the exponential buildup.

The other important frequency relation is

$$\omega_1 + \omega_2 = \omega_p - \omega_c \tag{48}$$

which reduces equations (43) and (44) to

$$\frac{dA_1}{dt} = c_{121}A_2^* \exp (j2\omega_c t) + c_{122}A_2^* \tag{49}$$

$$\frac{dA_2^*}{dt} = c_{211}A_1 \exp (-j2\omega_c t) + c_{212}A_1 . \tag{50}$$

Here the $c_{122} - c_{212}$ terms are important; the other terms give a small cyclical variation on the exponential buildup.

Thus the effect of periodically varying the coupling in the lumped parametric circuit is to modify the frequency-relation requirement to equations (45) and (48). The result is eminently reasonable and perhaps superficially obvious. We can see this as follows: equation (32) can be rewritten

$$\mathcal{C}_p = \frac{\Delta\mathcal{C}}{2} \{\cos [(\omega_p + \omega_c)t + \varphi] + \cos [(\omega_p - \omega_c)t + \varphi]\}.$$

Suppose we assume a $\mathcal{C}_p$ of

$$\mathcal{C}_p = \frac{\Delta\mathcal{C}}{2} \cos [(\omega_p + \omega_c)t + \varphi].$$

Then the previously known frequency condition for strong interaction is[4]

$$\omega_1 + \omega_2 = \omega_p + \omega_c .$$

If instead we have

$$\mathcal{C}_p = \frac{\Delta \mathcal{C}}{2} \cos [(\omega_p - \omega_c)t + \varphi].$$

then the frequency condition for strong interaction is

$$\omega_1 + \omega_2 = \omega_p - \omega_c .$$

If we then assume linear superposition (unjustified in the nonlinear process) we could expect relations (45) and (48) for $\mathcal{C}_p$ of equation (32). The above analysis and associated discussion indicate the restrictions which must be met to achieve the desired result.

The periodic coupling variation need not be cosinusoidal as in (32). Instead, square wave or even low duty cycle pulse modulation of $\mathcal{C}_p$ again leads to equations (45) and (48), although care must be exercised to assure that pulse modulation of the pump properly reproduces the signal content in a parametric amplifier.

VI. DISTRIBUTED PARAMETRIC WAVE INTERACTIONS

Coupling in distributed parametric wave interactions can be periodic in two ways: (*i*) with respect to time at a particular point, and (*ii*) with respect to distance in the direction of propagation at a particular instant of time. We derive the constraints on propagation constants and on frequencies which result from such periodicity and then indicate some physical structures in which these wave interactions may prove useful.

Figure 14 shows a simplified model of a distributed transmission medium. The distributed capacitance is nonlinear and is a function of time as well as of the position z in the direction of propagation. A number of waves of frequencies ω_1, ω_2, and ω_p may propagate. The distributed inductance L_n is independent of current magnitude but may have

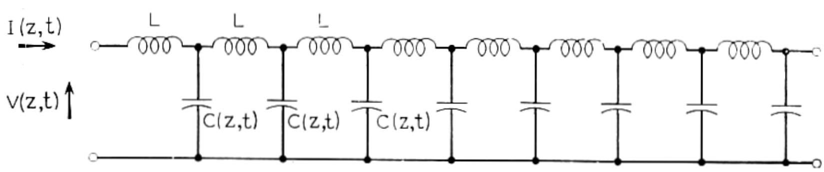

Fig. 14 — Distributed parametric circuit.

different values at different frequencies ω_n . In Appendix B the following coupled wave equations are derived for the normalized amplitudes of the traveling waves in Fig. 14

$$\frac{d}{dz} a_1 = -j\beta_1 a_1 - \frac{[z_{01}]^{\frac{1}{2}}}{2} \exp(-j\omega_1 t) \frac{\partial(V\mathcal{C}_p)}{\partial t}\Big|_{\omega_1} \tag{51}$$

$$\frac{da_1^*}{dz} = j\beta_1 a_1^* - \frac{[z_{01}]^{\frac{1}{2}}}{2} \exp(j\omega_1 t) \frac{\partial(V\mathcal{C}_p)}{\partial t}\Big|_{-\omega_1} \tag{52}$$

$$\frac{da_2}{dt} = -j\beta_2 a_2 - \frac{[z_{02}]^{\frac{1}{2}}}{2} \exp(-j\omega_2 t) \frac{\partial(V\mathcal{C}_p)}{\partial t}\Big|_{\omega_2} \tag{53}$$

$$\frac{da_2^*}{dt} = j\beta_2 a_2 - \frac{[z_{02}]^{\frac{1}{2}}}{2} \exp(j\omega_2 t) \frac{\partial(V\mathcal{C}_p)}{\partial t}\Big|_{-\omega_2} \tag{54}$$

in which we define, at frequency ω_n ,

$$\mathcal{C}(z, t) = \mathcal{C}_{on} + \mathcal{C}_p(z, t) \tag{55}$$

$$z_{on} = \left[\frac{L_n}{\mathcal{C}_{on}}\right]^{\frac{1}{2}} \tag{56}$$

$$\beta_n = \omega_n [L_n \mathcal{C}_{on}]^{\frac{1}{2}}. \tag{57}$$

The time and space varying portion of $\mathcal{C}(z, t)$ is all contained within $\mathcal{C}_p(z, t)$, and $\mathcal{C}_{on}$ is dependent only on frequency.

Equations (51) through (54) may be used to explore the effects of any periodic coupling behavior. Because the normalized amplitudes a_1 , a_1^* , a_2 , and a_2^* are dependent on z only (according to equations 142, 143, 130, and 131), only the terms of the partial derivatives of (51) through (54), which yield zero time dependence of the coupling coefficient, result in appreciable coupled-wave interaction. This condition produces the frequency interrelations for parametric interaction. Similarly, only the terms of the partial derivatives of equations (51) through (54), which ultimately yield constant coupling between the traveling waves at all z, cause appreciable wave interaction; this condition produces the interrelations between the propagation constants (the β_n) necessary for parametric interaction. We proceed to apply this technique.

6.1 *Traveling-Wave Pump with Spatial and Time Periodicity*

We specify a function for the nonlinear distributed capacitance (type 3 coupling of Table I)

$$\mathcal{C}_p(z, t) = \frac{\Delta\mathcal{C}}{2} \cos\beta_c z\{\cos[(\omega + \omega_c)t - \beta_+ z] + \cos[(\omega - \omega_c)t - \beta_- z]\} \tag{58}$$

in which β_+ is the phase constant at frequency $(\omega + \omega_c)$ and β_- is the phase constant at $(\omega - \omega_c)$. This corresponds to driving the nonlinear medium with traveling wave at a modulated pump frequency $\cos \omega_c t$ in which ω_c is the modulation; the $\cos \beta_c z$ factor represents a spatial periodic variation in the coupling. Structures which produce spatially periodic parametric interactions are described later in this section.

We use equation (58) in equations (51) through (54) and select the terms which are capable of yielding a zero time dependence to the coupling terms. This shows a_1 and a_2^* to be the waves with significant coupling and the selected terms are

$$\begin{aligned}\frac{da_1}{dz} &= -j\beta_1 a_1 + c_{121} a_2^* \exp\left[-j(\beta_+ + \beta_c) + j(\omega + \omega_c - \omega_1 - \omega_2)t\right] \\ &+ c_{121} a_2^* \exp\left[-j(\beta_+ - \beta_c) + j(\omega + \omega_c - \omega_1 - \omega_2)t\right] \\ &+ c_{122} a_2^* \exp\left[-j(\beta_- + \beta_c) + j(\omega - \omega_c - \omega_1 - \omega_2)t\right] \\ &+ c_{122} a_2^* \exp\left[-j(\beta_- - \beta_c) + j(\omega - \omega_c - \omega_1 - \omega_2)t\right]\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{da_2^*}{dz} &= j\beta_2 a_2^* + c_{211} a_1 \exp\left[j(\beta_+ + \beta_c) - j(\omega + \omega_c - \omega_1 - \omega_2)t\right] \\ &+ c_{211} a_1 \exp\left[j(\beta_+ - \beta_c) - j(\omega + \omega_c - \omega_1 - \omega_2)t\right] \\ &+ c_{212} a_1 \exp\left[j(\beta_- + \beta_c) - j(\omega - \omega_c - \omega_1 - \omega_2)t\right] \\ &+ c_{212} a_1 \exp\left[j(\beta_- - \beta_c) - j(\omega - \omega_c - \omega_1 - \omega_2)t\right]\end{aligned} \tag{60}$$

in which

$$c_{121} = \frac{-\Delta \mathcal{C}}{16}\, j(\omega + \omega_c - \omega_2)(z_{01} z_{02})^{\frac{1}{2}} \tag{61}$$

$$c_{122} = \frac{-\Delta \mathcal{C}}{16}\, j(\omega - \omega_c - \omega_2)(z_{01} z_{02})^{\frac{1}{2}} \tag{62}$$

$$c_{211} = \frac{\Delta \mathcal{C}}{16}\, j(\omega + \omega_c - \omega_1)(z_{01} z_{02})^{\frac{1}{2}} \tag{63}$$

$$c_{212} = \frac{\Delta \mathcal{C}}{16}\, j(\omega - \omega_c - \omega_1)(z_{01} z_{02})^{\frac{1}{2}}. \tag{64}$$

From equations (59) and (60) one sees that there are two frequency conditions which can yield large wave interactions. When

$$\omega + \omega_c = \omega_1 + \omega_2 \tag{65}$$

the c_{121} and c_{211} terms dominate and the other terms produce only minor fluctuations. Also, when

$$\omega - \omega_c = \omega_1 + \omega_2 \tag{66}$$

the c_{122} and c_{212} terms dominate. When equation (65) is valid, $(\omega + \omega_c - \omega_2) = \omega_1$ and the coupling coefficients reduce to

$$c_{121} = \frac{-\Delta\mathcal{C}}{16} j\left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}}\left(\frac{\beta_1\beta_2}{\mathcal{C}_{01}\mathcal{C}_{02}}\right)^{\frac{1}{2}} \tag{67}$$

$$c_{211} = \frac{\Delta\mathcal{C}}{16} j\left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}}\left(\frac{\beta_1\beta_2}{\mathcal{C}_{01}\mathcal{C}_{02}}\right)^{\frac{1}{2}}. \tag{68}$$

Note that

$$c_{121} = \frac{\omega_1}{\omega_2} c_{211}^* . \tag{69}$$

When equation (66) is valid, the coupling coefficients of importance are c_{122} which reduces to equation (67) and c_{212} which reduces to equation (68), so that again

$$c_{122} = \frac{\omega_1}{\omega_2} c_{212}^* . \tag{70}$$

To find the necessary constraints on the phase constants we note that in the absence of coupling (that is, $\Delta\mathcal{C} = 0$) the solutions to equations (59) and (60) are of the form

$$a_1 = A_1 \exp(-j\beta_1 z) \tag{71}$$

$$a_2^* = A_2^* \exp(j\beta_2 z). \tag{72}$$

When equation (65) is valid, use of equations (71) and (72) in equations (59) and (60) reduces them to

$$\begin{aligned}\frac{dA_1}{dz} &= c_{12}A_2^*\{\exp[-j(\beta_+ - \beta_c - \beta_1 - \beta_2)z] \\ &\quad + \exp[-j(\beta_+ + \beta_c - \beta_1 - \beta_2)z]\}\end{aligned} \tag{73}$$

$$\begin{aligned}\frac{dA_2^*}{dz} &= c_{21}A_1 \\ &\quad \cdot\{\exp[j(\beta_+ - \beta_c - \beta_1 - \beta_2)z] + \exp[j(\beta_+ + \beta_c - \beta_1 - \beta_2)z]\}.\end{aligned} \tag{74}$$

We can now observe two conditions, either of which permit significant parametric wave interaction:

$$\beta_+ - \beta_c = \beta_1 + \beta_2 \tag{75}$$

$$\beta_+ + \beta_c = \beta_1 + \beta_2 \, . \tag{76}$$

Repeating the above procedure for equation (66) being valid instead of equation (65) yields two more permissible conditions at which in-phase wave interaction occurs at all z:

$$\beta_- - \beta_c = \beta_1 + \beta_2 \tag{77}$$

$$\beta_- + \beta_c = \beta_1 + \beta_2 \, . \tag{78}$$

When one of equations (75) through (78) is valid along with the corresponding frequency condition, equations (59) and (60) reduce to

$$\frac{dA_1}{dz} = c_{12} A_2^* \tag{79}$$

$$\frac{dA_2^*}{dz} = c_{21} A_1 \, . \tag{80}$$

These equations are satisfied by exponentials of the form

$$\exp \left[\pm \, (c_{12}c_{21})^{\frac{1}{2}} z\right].$$

When $(c_{12}c_{21})^{\frac{1}{2}}$ is pure real, growing and decaying waves are present and equations (67) and (68) meet this requirement. The parallel propagation of signal ω_1, idler ω_2, and pump ω results in gain, as is well known. Other configurations of signal, pump, and coupling peridocity can result in pure imaginary values of $(c_{12}c_{21})^{\frac{1}{2}}$ in which case a periodic interchange of power between waves is indicated.

The above discussion pertains to type 3 coupling of Table I, the difference between the sin and cos being negligible. For square wave coupling the physical model is often simpler to construct; we briefly consider this situation. In Fig. 15 we assume a region "a" in which the coupling is constant but the normal phase matching relations are not met, that is,

$$\beta_1 + \beta_2 \neq \beta.$$

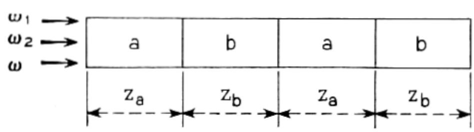

Fig. 15 — Model of a transmission medium with periodically varying properties.

Then the proper way to establish the periodic coupling is to make the length z_a such that the exponentials in equations (73) and (74) (with $\beta_c = 0$) become a half beat wavelength; for the specific case above there are two permissible choices,

$$(\beta_+ - \beta_1 - \beta_2)z_a = \pi \pm 2p\pi \tag{81}$$

$$(\beta_- - \beta_1 - \beta_2)z_a = \pi \pm 2p\pi \tag{82}$$

with the time modulation present; for cw pumping

$$(\beta - \beta_1 - \beta_2)z_a = \pi \pm 2p\pi \tag{83}$$

with p being any integer. Then, in the "b" region of Fig. 15, the coupling may be zero in which case we have type 5 coupling, or the coupling may be reversed compared with the "a" region, in which case we have type 4 coupling. In either case we require

$$(\beta - \beta_1 - \beta_2)z_b = \pi \pm 2p\pi. \tag{84}$$

The β's in the "a" and "b" regions need not be the same—the β's of equations (81) through (84) are to be those values characteristic of the waves' location. Earlier work has made use of some of these possibilities.[5,6]

Figure 15 shows square-wave coupling which, as discussed above, applies generally to passive wave interactions as well as to other parametric interactions. The conditions analogous to equations (81) through (84) follow from making the exponents in the appropriate coupled-wave equations, analogous to equations (73) and (74), equal to π or an odd multiple of π.

6.2 *CW Traveling-Wave Pump with Simultaneous Modulation of the Entire Medium*

A case related to that discussed in Section 6.1 is described by

$$\mathcal{C}_p(t) = \Delta\mathcal{C} \cos w_c t \cos (\omega t - \beta z). \tag{85}$$

Here the pump wave is a continuous wave and the entire array of variable capacitors is simultaneously modulated. This may occur when the modulating wave ω_c is brought into the nonlinear medium at right angle to z, or when ω_c is so small that the entire length of nonlinear medium is a lumped element in the ω_c circuit. Analysis similar to that in Section 6.1 shows that the frequency conditions are again given by equations (65) and (66), the coupling coefficients are twice those given by equations (67) and (68), and the phase constant condition is

$$\beta = \beta_1 + \beta_2 . \tag{86}$$

6.3 *Second-Harmonic Generation with Spatially Periodic Coupling*

The capacitance function for second-harmonic generation with spatially periodic coupling is

$$\mathcal{C}_p(t) = \Delta\mathcal{C} \cos \beta_c z \cos (\omega_1 t - \beta_1 z). \tag{87}$$

We look for coupling with $\omega_2 = 2\omega_1$ in equations (51) through (54) and find the interaction between a_1 and a_2 . The coupling coefficients are

$$c_{12} = j \frac{\Delta\mathcal{C}}{4} \left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}} \left(\frac{\beta_1\beta_2}{\mathcal{C}_{01}\mathcal{C}_{02}}\right)^{\frac{1}{2}} \tag{88}$$

$$c_{21} = j \frac{\Delta\mathcal{C}}{4} \left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}} \left(\frac{\beta_1\beta_2}{\mathcal{C}_{01}\mathcal{C}_{02}}\right)^{\frac{1}{2}} \tag{89}$$

and the phase-constant requirement is

$$\beta_2 = 2\beta_1 \pm \beta_c . \tag{90}$$

In this case $(c_{12}c_{21})^{1/2}$ is pure imaginary, so the wave solutions, varying as

$$\exp [(c_{12}c_{21})^{\frac{1}{2}}z] \pm \exp [-(c_{12}c_{21})^{\frac{1}{2}}z],$$

represent a cyclical interchange of power between a_1 and a_2. However the mathematical model represented by equation (87) is not valid when a_1 diminishes appreciably because it no longer is the principal field on the variable capacitors as called for in equation (87).

If square-wave coupling is used in the configuration of Fig. 15, the phase constant and length relations are

$$\begin{aligned} (2\beta_{1a} - \beta_{2a})z_a &= \pi \pm 2p\pi \\ (2\beta_{1b} - \beta_{2b})z_b &= \pi \pm 2p\pi \end{aligned} \tag{91}$$

with p being any integer including zero; the subscripts a or b on the β's denotes the region of Fig. 15 involved. As in the previous discussion of Fig. 15, a constant coupling in the "a" regions may be paired with either zero coupling or reversed coupling in the "b" regions to form types 4 or 5 coupling of Table I.

6.4 *Frequency Converter with Spatially Periodic Coupling*

Consider a medium driven nonlinear simultaneously at all z by a frequency ω_c according to

$$\mathcal{C}_p(z, t) = \Delta\mathcal{C} \cos \beta_c z \cos \omega_c t . \tag{92}$$

With waves of frequency ω_1 and ω_2 in the medium, from equations (51) through (54) we find that there is strong coupling between a_1 and a_2 at the frequency

$$\omega_c = \omega_1 - \omega_2 . \tag{93}$$

The phase constant condition is

$$\beta_1 - \beta_2 = \pm\beta_c \tag{94}$$

and the coupling coefficients are

$$c_{12} = j\frac{\Delta\mathcal{C}}{4}\left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}}\left(\frac{\beta_1\beta_2}{\mathcal{C}_{01}\mathcal{C}_{02}}\right)^{\frac{1}{2}} \tag{95}$$

$$c_{21} = j\frac{\Delta\mathcal{C}}{4}\left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}}\left(\frac{\beta_1\beta_2}{\mathcal{C}_{01}\mathcal{C}_{02}}\right)^{\frac{1}{2}}. \tag{96}$$

Since $(c_{12}c_{21})^{\frac{1}{2}}$ is pure imaginary there is a cyclical interchange of power between waves, and in this case the mathematical model is valid for complete interchange of power. If a wave at ω_1 is the input, the output will be solely a wave at ω_2 at a medium length z_t such that

$$|\,(c_{12}c_{21})^{\frac{1}{2}}\,|\,z_t = \frac{\pi}{2} \tag{97}$$

which yields

$$z_t = \frac{2\pi(\mathcal{C}_{01}\mathcal{C}_{02})^{\frac{1}{2}}}{\Delta\mathcal{C}(\beta_1\beta_2)^{\frac{1}{2}}}. \tag{98}$$

When square-wave coupling in a periodic structure of the form of Fig. 15 is used, the phase constant and length relations become

$$(\beta_{1a} - \beta_{2a})z_a = \pi \pm 2p\pi \tag{99}$$

$$(\beta_{1b} - \beta_{2b})z_b = \pi \pm 2p\pi \tag{100}$$

with p being any integer.

6.5 *Structural Forms of Periodic Parametric Devices*

We suggest here a few forms which periodic parametric devices might take. Figure 15 has already been referred to; it is apparent that the diagram is applicable to all of the preceding cases. The "b" region might simply be an index-matching oil without coupling effects. In other cases, it may be possible to achieve a reversal of the coupling.[6]

Figure 16 shows a centrosymmetric crystal such as (potassium tantalum niobate) with associated electrodes and potentials to achieve

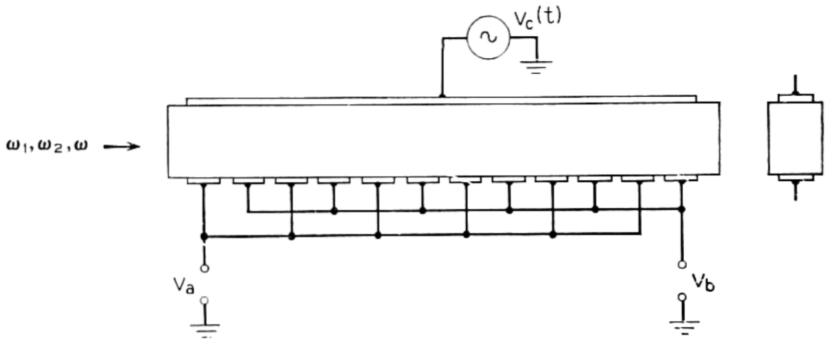

Fig. 16 — Model of a nonlinear crystal with wave coupling that is periodic both in time and space.

the periodic coupling. In such a crystal, a change in index of refraction is a parabolic function of the biasing field through the electro-optic effect. We have in mind laser wavelengths for the ω_1, ω_2, and ω waves. For V_a positive and V_b negative in Fig. 16, the slope of index versus RF field at frequency ω is positive in the "a" region and negative in the "b" region. Therefore, a spatial variation of coupling of the general form described in Section 6.1 is established; instead of the $\cos \beta_c z$ term in equation (58), a square-wave variation results from Fig. 16 with $V_c(t) = 0$ and dc biases of $V_a = +V$, $V_b = -V$.

With the addition of $V_c(t)$ in Fig. 16, a component $\cos \omega_c t$ as in equation (85) adds a simultaneous modulation of the medium, of the general form discussed in Section 6.2; to conform to equation (85) the voltages V_a and V_b should be made equal to zero. Second harmonic generation can be achieved using Fig. 16 with the ω wave omitted, $V_c(t) = 0$, $V_a = V$, and $V_b = -V$.

Frequency conversion of the type discussed in Section 6.4 might also be accomplished in the structure of Fig. 16. In this case the ω wave is omitted, the $\cos \omega_c t$ variation of equation (92) is produced by $V_c(t)$, and the biases V_a and V_b yield a square-wave spatial periodicity. Notice that V_b may be zero, approximating a type 5 coupling of Table I.

Figure 17 shows an alternate wave feeding arrangement for simultaneously modulating the entire nonlinear medium at a laser frequency rate. This could apply to Section 6.4 as well as to Section 6.2 with the addition of an ω wave parallel to the ω_1 and ω_2 waves.

In all cases, a guided wave may be used in the nonlinear medium by having a transverse index variation such as to produce a dielectric

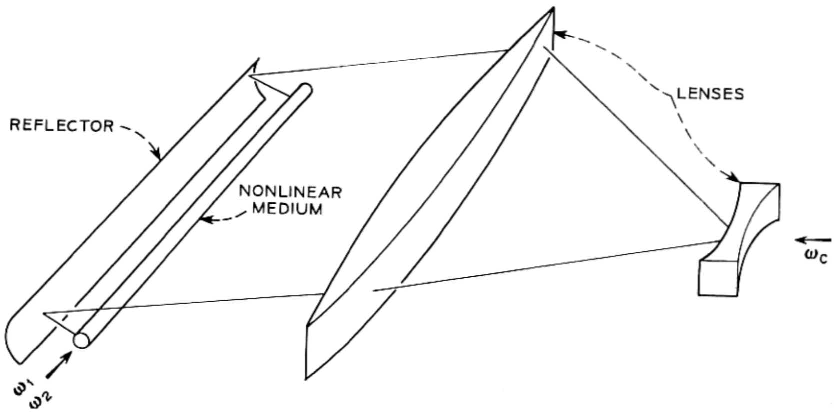

Fig. 17 — Parametric device with simultaneous modulation of the entire length of the nonlinear medium.

waveguide effect. This permits much longer regions of nonlinear interaction by holding the field within a small transverse area.

VII. CONCLUSION

We have outlined a wide variety of coupled-wave interactions in which a periodic variation in coupling may be used. The advantage in using periodic coupling rather then uniform coupling is frequently to achieve large power transfer between waves under conditions where uniform coupling will not do so—that is, where it is not possible for one reason or another to establish identical phase constants between the waves. Then by matching the periodicity of the coupling to the difference between the phase constants of the coupled waves, one can achieve nearly the same wave interactions as for matched phase constants and uniform coupling.

With frequency-selective filters, dispersion in the phase constants in combination with periodic coupling produces a desirable frequency-selective transfer of power. In the case of parmetric coupled-wave devices, periodic coupling requires a generalization of the Tien conditions which the frequencies and phase constants must meet.[7] These are outlined in Section VI.

VIII. ACKNOWLEDGMENT

The writer is indebted to D. H. Ring and Tingye Li for a careful reading of the manuscript.

APPENDIX A

Lumped Element Parametric Circuit

We now derive the coupled wave equations for the lumped circuit of Fig. 13 with a general time-varying capacitor†

$$\mathcal{C}(t) = \mathcal{C}_o + \mathcal{C}_p(t). \tag{101}$$

We define ω_1 and ω_2 by

$$\omega_1^2 L_1 \mathcal{C}_{11} = 1 \tag{102}$$

$$\omega_2^2 L_2 \mathcal{C}_{22} = 1 \tag{103}$$

$$\mathcal{C}_{11} = \mathcal{C}_1 + \mathcal{C}_o \tag{104}$$

$$\mathcal{C}_{22} = \mathcal{C}_2 + \mathcal{C}_o \ . \tag{105}$$

Then

$$\frac{dI_1}{dt} = -\frac{1}{L_1} V_1 \tag{106}$$

$$\frac{dI_2}{dt} = -\frac{1}{L_2} V_2 \ . \tag{107}$$

With the filter denoted by ω_1 in Fig. 13, a short circuit at ω_1 and an open circuit at other frequencies, and similarly for the filter ω_2

$$I_1 = \frac{d}{dt}\{[\mathcal{C}_1 + \mathcal{C}(t)V_1] - \mathcal{C}(t)V_2\} \tag{108}$$

$$I_2 = \frac{d}{dt}\{[\mathcal{C}_2 + \mathcal{C}(t)V_2] - \mathcal{C}(t)V_1\}. \tag{109}$$

Expanding equation (108)

$$I_1 = (\mathcal{C}_{11} + \mathcal{C}_p)\frac{dV_1}{dt} + V_1 \frac{d}{dt}(\mathcal{C}_{11} + \mathcal{C}_p) - V_2 \frac{d}{dt}(\mathcal{C}_o + \mathcal{C}_p) - (\mathcal{C}_o + \mathcal{C}_p)\frac{dV_2}{dt}. \tag{110}$$

Rearranging terms,

$$\frac{dV_1}{dt} = \frac{I_1}{\mathcal{C}_{11}} - \frac{d}{dt}\left(\frac{\mathcal{C}_p V_1}{\mathcal{C}_{11}}\right) + \frac{V_2}{\mathcal{C}_{11}}\frac{d\mathcal{C}_p}{dt} + \frac{(\mathcal{C}_o + \mathcal{C}_p)}{\mathcal{C}_{11}}\frac{dV_2}{dt}. \tag{111}$$

† We follow the terminology of W. H. Louisell.[4]

Similarly,

$$\frac{dV_2}{dt} = \frac{I_2}{\mathcal{C}_{22}} - \frac{d}{dt}\left(\frac{\mathcal{C}_p V_2}{\mathcal{C}_{22}}\right) + \frac{V_1}{\mathcal{C}_{22}}\frac{d\mathcal{C}_p}{dt} + \left(\frac{\mathcal{C}_o + \mathcal{C}_p}{\mathcal{C}_{22}}\right)\frac{dV_1}{dt}. \tag{112}$$

As a result of the action of the ω_1 and ω_2 filters, V_1 contains only the frequency ω_1, and V_2 contains only the frequency ω_2. Hence dV_2/dt cannot contribute to dV_1/dt, and may be dropped in equation (111). Similarly, the last term of equation (112) may be dropped.

Multiplying the remainder of equation (111) by $j\omega_1\mathcal{C}_{11}$, adding equation (116), and multiplying each side by $(L_1)^{\frac{1}{2}}/2$, gives

$$\frac{(L_1)^{\frac{1}{2}}}{2}\left(\frac{dI_1}{dt} + j\omega_1\mathcal{C}_{11}\frac{dV_1}{dt}\right)$$
$$= \frac{(L_1)^{\frac{1}{2}}}{2}\left[-\frac{V_1}{L_1} + j\omega_1 I_1 - j\omega_1\mathcal{C}_{11}\frac{d}{dt}\left(\frac{\mathcal{C}_p V_1}{\mathcal{C}_{11}}\right) + j\omega_1 V_2\frac{d\mathcal{C}_p}{dt}\right]. \tag{113}$$

Using the normalized amplitudes a_1, a_2, and their complex conjugates

$$a_1 = \frac{(L_1)^{\frac{1}{2}}}{2}(I_1 + j\omega_1\mathcal{C}_{11}V_1) \tag{114}$$

$$a_1^* = \frac{(L_1)^{\frac{1}{2}}}{2}(I_1^* - j\omega_1\mathcal{C}_{11}V_1^*) \tag{115}$$

$$a_2 = \frac{(L_2)^{\frac{1}{2}}}{2}(I_2 + j\omega_2\mathcal{C}_{22}V_2) \tag{116}$$

$$a_2^* = \frac{(L_2)^{\frac{1}{2}}}{2}(I_2^* - j\omega_2\mathcal{C}_{22}V_2^*), \tag{117}$$

one may verify that equation (113) becomes

$$\frac{da_1}{dt} = j\omega_1 a_1 - \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_1 - a_1^*)}{\mathcal{C}_{11}} - \frac{(a_2 - a_2^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\}. \tag{118}$$

Using similar methods one can derive the other coupled wave equations

$$\frac{da_1^*}{dt} = -j\omega_1 a_1^* + \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_1 - a_1^*)}{\mathcal{C}_{11}} - \frac{(a_2 - a_2^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\} \tag{119}$$

$$\frac{da_2}{dt} = j\omega_2 a_2 - \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_2 - a_2^*)}{\mathcal{C}_{22}} - \frac{(a_1 - a_1^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\} \tag{120}$$

$$\frac{da_2^*}{dt} = -j\omega_2 a_2^* + \frac{d}{dt}\left\{\frac{\mathcal{C}_p}{2}\left[\frac{(a_2 - a_2^*)}{\mathcal{C}_{22}} - \frac{(a_1 - a_1^*)}{[\mathcal{C}_{11}\mathcal{C}_{22}]^{\frac{1}{2}}}\right]\right\}. \tag{121}$$

APPENDIX B

Distributed Parametric Medium

We now derive the coupled-wave equations for the distributed transmission medium of Fig. 14 with the general time- and space-varying distributed capacitance.

$$\mathcal{C}(z, t) = \mathcal{C}_{on} + \mathcal{C}_p(z, t) \tag{122}$$

where $\mathcal{C}_{on}$ is a constant relevant at angular frequency ω_n. Similarly the distributed inductance may have different values L_n at the various ω_n. From circuit theory

$$\frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \tag{123}$$

$$\frac{\partial I}{\partial z} = -\frac{\partial (V\mathcal{C})}{\partial t}. \tag{124}$$

Noting $\partial\mathcal{C}/\partial t = \partial\mathcal{C}_p/\partial t$, equation (124) becomes

$$\frac{\partial I}{\partial z} = -\mathcal{C}_{on} \frac{\partial V}{\partial t} - \frac{\partial}{\partial t}(V\mathcal{C}_p). \tag{125}$$

We define

$$Z_{01} = \left(\frac{L_1}{\mathcal{C}_{01}}\right)^{\frac{1}{2}} \tag{126}$$

$$z_{02} = \left(\frac{L_2}{\mathcal{C}_{02}}\right)^{\frac{1}{2}} \tag{127}$$

$$\beta_1 = \omega_1(L_1\mathcal{C}_{01})^{\frac{1}{2}} \tag{128}$$

$$\beta_2 = \omega_2(L_2\mathcal{C}_{02})^{\frac{1}{2}}. \tag{129}$$

Consider the case of propagating two waves in the medium of Fig. 14, one at ω_1 and one at ω_2. Then define

$$V(z, t) = V_1(z) \exp(j\omega_1 t) + V_2(z) \exp(j\omega_2 t) + V_1^*(z) \exp(-j\omega_1 t) + V_2^*(z) \exp(-j\omega_2 t) \tag{130}$$

$$I(z, t) = I_1(z) \exp(j\omega_1 t) + I_2(z) \exp(j\omega_2 t) + I_1^*(z) \exp(-j\omega_1 t) + I_2^*(z) \exp(-j\omega_2 t) \tag{131}$$

where the V_n and I_n are dependent only on z and the* denotes the complex conjugate. Then Equation (123) becomes

$$\exp(j\omega_1 t)\frac{dV_1}{dz} + \exp(j\omega_2 t)\frac{dV_2}{dz} + \cdots = -j\omega_1 L_1 \exp(j\omega_1 t) I_1 - j\omega_2 L_2 \exp(j\omega_2 t) I_2 + \cdots . \tag{132}$$

Equating terms of equal frequency

$$\frac{dV_1}{dz} = -j\omega_1 L_1 I_1 \tag{133}$$

$$\frac{dV_2}{dz} = -j\omega_2 L_2 I_2 \tag{134}$$

$$\frac{dV_1^*}{dz} = j\omega_1 L_1 I_1^* \tag{135}$$

$$\frac{dV_2^*}{dz} = j\omega_2 L_2 I_2^* . \tag{136}$$

Using equations (130) and (131), equation (125) becomes

$$\exp(j\omega_1 t)\frac{dI_1}{dz} + \exp(j\omega_2 t)\frac{dI_2}{dz} + \cdots = -j\omega_1 \mathcal{C}_{01} V_1 \exp(j\omega_1 t) - j\omega_2 \mathcal{C}_{02} V_2 \exp(j\omega_2 t) + \cdots - \frac{\partial(V\mathcal{C}_p)}{\partial t}. \tag{137}$$

Equating terms of equal frequency yields

$$\frac{dI_1}{dz} = -j\omega_1 \mathcal{C}_{01} V_1 - \exp(-j\omega_1 t)\left.\frac{\partial(V\mathcal{C}_p)}{\partial t}\right|_{\omega_1} \tag{138}$$

$$\frac{dI_1^*}{dz} = j\omega_1 \mathcal{C}_{01} V_1^* - \exp(j\omega_1 t)\left.\frac{\partial(V\mathcal{C}_p)}{\partial t}\right|_{-\omega_1} \tag{139}$$

$$\frac{dI_2}{dz} = -j\omega_2 \mathcal{C}_{02} V_2 - \exp(-j\omega_2 t)\left.\frac{\partial(V\mathcal{C}_p)}{\partial t}\right|_{\omega_2} \tag{140}$$

$$\frac{dI_2^*}{dz} = j\omega_2 \mathcal{C}_{02} V_2^* - \exp(j\omega_2 t)\left.\frac{\partial(V\mathcal{C}_p)}{\partial t}\right|_{-\omega_2} . \tag{141}$$

The partial derivatives are to be evaluated in the vicinity of ω_1 for equation (138), $-\omega_1$ for equation (139), and so on. Considering only forward waves, we define a normalized wave amplitude

$$a_1(z) = \frac{V_1}{(z_{01})^{\frac{1}{2}}} = I_1(z_{01})^{\frac{1}{2}} \tag{142}$$

$$a_1(z) = \frac{1}{2(z_0)^{\frac{1}{2}}}\{V_1 + z_{01} I_1\}. \tag{143}$$

Using equations (143), (133), and (138),

$$\frac{da_1}{dz} = -\frac{1}{2(z_{01})^{\frac{1}{2}}} \left[j\omega_1 L_1 I_1 + z_{01} j\omega_1 \mathcal{C}_{01} V_1 + z_{01} \exp(-j\omega_1 t) \frac{\partial(V\mathcal{C}_p)}{\partial t} \right].$$

Using equations (143), (128), and (126),

$$-j\beta_1 a_1 = \frac{-j\omega_1(L_1\mathcal{C}_{01})^{\frac{1}{2}}}{2(z_{01})^{\frac{1}{2}}} (V_1 + z_{01} I_1) \tag{144}$$

$$= -\frac{1}{2(z_{01})^{\frac{1}{2}}} (j\omega_1 \mathcal{C}_{01} V_1 z_{01} + j\omega_1 L_1 I_1).$$

Hence

$$\frac{d}{dz} a_1 = -j\beta_1 a_1 - \frac{(z_{01})^{\frac{1}{2}}}{2} \exp(-j\omega_1 t) \frac{\partial(V\mathcal{C}_p)}{\partial t} \bigg|_{\omega_1}. \tag{145}$$

Using similar substitutions one can show that

$$\frac{da_1^*}{dz} = j\beta_1 a_1^* - \frac{(z_{01})^{\frac{1}{2}}}{2} \exp(j\omega_1 t) \frac{\partial(V\mathcal{C}_p)}{\partial t} \bigg|_{-\omega_1} \tag{146}$$

$$\frac{da_2}{dt} = -j\beta_2 a_2 - \frac{(z_{02})^{\frac{1}{2}}}{2} \exp(-j\omega_2 t) \frac{\partial(V\mathcal{C}_p)}{\partial t} \bigg|_{\omega_2} \tag{147}$$

$$\frac{da_2^*}{dt} = j\beta_2 a_2^* - \frac{(z_{02})^{\frac{1}{2}}}{2} \exp(j\omega_2 t) \frac{\partial(V\mathcal{C}_p)}{\partial t} \bigg|_{-\omega_2}. \tag{148}$$

With the mode amplitudes normalized as above, the square of the amplitudes represents the power carried by the mode.

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