Rate-Distortion Functions for Gaussian Markov Processes*

By BARRY J. BUNIN

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The rate-distortion function with a mean square error distortion criterion is investigated for a class of Gaussian Markov sources. It is found that for rates greater than a certain minimum, the rate-distortion function is equivalent to that of an independent letter source. This minimum rate was found to be less than n *bits per symbol, where* n *is the order of the Markov sequence. Comparisons between the rate-distortion function, and two quantizing systems are made.*

I. INTRODUCTION

Suppose in the communication system of Fig. 1, the source emits a sequence of continuous-valued random variables. The exact specification of such variates requires an infinite number of binary digits. Hence exact transmission would require a channel of infinite capacity. Since no physical channels possess infinite capacity, we see that exact transmission is impossible through this system.

However, if we are willing to accept some error in our specification of the source output, then finitely many binary digits are necessary. In the study of digital encoding systems, a useful quantity to know is the fewest number of binary digits necessary to represent an analog signal within a certain error. Such a quantity would give us a performance criterion with which to compare existing systems, and also tell us how much improvement is possible.

The quantity we seek is given by Shannon's rate-distortion function.[1,2] The rate-distortion function gives, for any bit rate, the minimum possible error achievable.

In this paper we study the rate-distortion functions for the important

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Fig. 1 — General communication system.

class of gaussian Markov sources. We measure our error by the mean square error criterion. Also, the performance of two quantizing systems, differential PCM and block quantizing, is compared to the rate-distortion bound.

II. DISCUSSION OF RESULTS

We have studied the rate-distortion functions of gaussian Markov sources with a mean square error criterion. We express our results in Fig. 2 by plotting signal-to-noise ratio in dB, versus bit rate R. The signal-to-noise ratio is given by

$$\mathrm{S/N} = 10 \log_{10} \frac{\sigma^2}{D} \tag{1}$$

where σ^2 is the variance of the source output, and D is the mean square error.

It was found that for rates R greater than a certain $R_{\min}$, the rate distortion function is given by

$$R = \tfrac{1}{2} \log_2 \frac{\sigma_m^2}{D} \qquad 0 \leqq D \leqq \sigma_m^2 \tag{2}$$

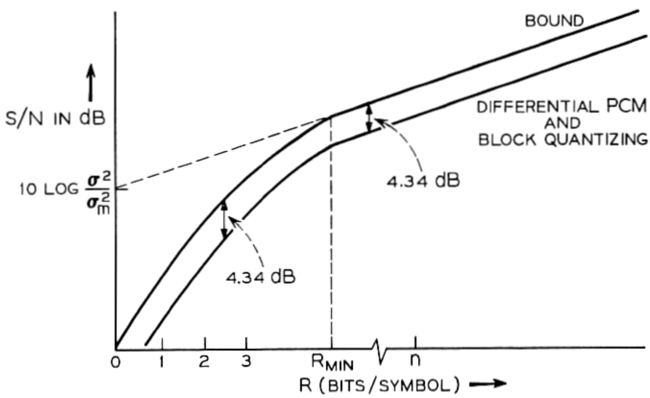

Fig. 2 — Rate-distortion bound of a Markov-n source compared with block quantizing system and differential PCM.

or

$$S/N = 6.02R + 10 \log_{10} \frac{\sigma^2}{\sigma_m^2} \tag{3}$$

where σ_m^2 is the minimum mean square prediction error one step ahead. The point $R_{\min}$ occurs in the interval $(0, n)$ where n is the order of the Markov process that the source emits. The exact location of $R_{\min}$ depends on the exact shape of the power spectral density of the process, as we shall see. At $R = R_{\min}$, the rate-distortion function has a discontinuity in the third derivative.

If the source were followed by the optimum prediction system of Fig. 3 then the output sequence produced would be uncorrelated with variance σ_m^2. Such a sequence has the rate-distortion function given by (2). Hence for rates greater than $R_{\min}$ the sequences at the input and output of the prediction system have equal rate-distortion functions. For rates less than $R_{\min}$ they do not.

A lower bound on the performance achievable by the block quantizing system of Fig. 4 was found. The result is also shown in Fig. 2, where it is seen that this system can be made to perform within 4.34 dB of the bound.

Also shown in Fig. 2 is the performance bound for a differential PCM system (see Fig. 5) as derived by O'Neal. This bound however, holds only for high bit rates.

III. RATE DISTORTION FUNCTIONS FOR MARKOV-N SOURCES

3.1 *Introduction*

Consider again the communication system of Fig. 1. The source emits the discrete time, stationary random process x_t, $t = 0, \pm 1, \pm 2, \cdots$. After N seconds, a column N vector X is obtained, and after encoding, transmission and decoding, the receiver obtains a replica $\hat{X}$ of X. The mean square error between the transmitted and received vectors is

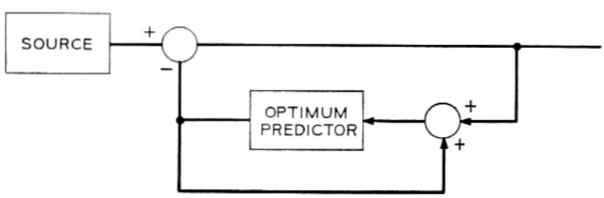

Fig. 3 — Predictive communication system.

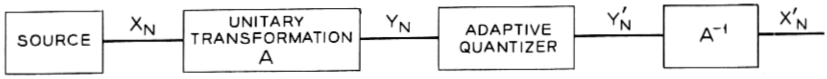

Fig. 4 — Block quantizer for correlated source.

defined by

$$D = \frac{1}{N} E(X - \hat{X})^T(X - \hat{X}) \tag{4}$$

where E denotes expectation and X^T is the transpose of X. It is reasonable to ask what the minimum bit rate is, at which we must transmit, so as to be able to achieve a mean square error less than some prescribed amount. The answer is given by Shannon's rate-distortion function which is defined as follows:[1,2]

$$R(D) = \lim_{N\to\infty} \min \frac{1}{N} \iint p(X_N)p(\hat{X}_N \mid X_N) \cdot \log_2 \frac{p(\hat{X}_N \mid X_N)}{p(\hat{X}_N)} \, dX_N \, d\hat{X}_N \tag{5}$$

where the minimization is taken over all $p(\hat{X}_N \mid X_N)$ satisfying

$$\langle D \rangle = \frac{1}{N} \iint (X_N - \hat{X}_N)^T(X_N - \hat{X}_N) \cdot p(X_N)p(\hat{X}_N \mid X_N) \, dX_N \, d\hat{X}_N \leqq D \tag{6}$$

and where

$p(X_N)$ = probability measure of the source vector X_N
$p(\hat{X}_N \mid X_N)$ = conditional probability measure of $\hat{X}_N$ given X_N
$p(\hat{X}_N)$ = probability measure induced on $\hat{X}_N$ by $p(X_N)$ and $p(\hat{X}_N \mid X_N)$.

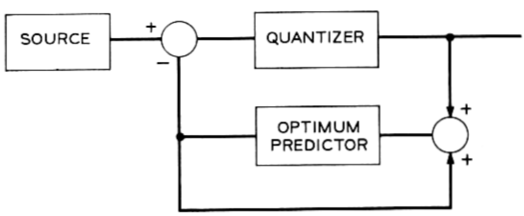

Fig. 5 — Differential pulse code modulation system.

(The subscript N is included to emphasize that we are dealing with an N-vector.)

Suppose the source emits a stationary gaussian time series with correlations $E(x_j x_k) = r_{j-k} = r_\tau$. Then the discrete time power spectral density is given by

$$f(\lambda) = \sum_{\tau=-\infty}^{\infty} r_\tau e^{i\tau\lambda} \qquad -\pi \leqq \lambda \leqq \pi \tag{7}$$

and the rate distortion function is given parametrically by[3] (see Fig. 6 for interpretation)

$$R(\phi) = \frac{1}{2}\int_A \log \frac{f(\lambda)}{\phi} \frac{d\lambda}{2\pi} \qquad 8(a)$$

$$D(\phi) = \int_A \phi \frac{d\lambda}{2\pi} + \int_{A'} f(\lambda) \frac{d\lambda}{2\pi} \qquad 8(b)$$

$$A = \{\lambda : f(\lambda) \geqq \phi\}$$

$$A' = \{\lambda : f(\lambda) < \phi\}$$

and

$$A \cup A' = (-\pi, \pi).$$

Hence, if we are given a distortion D, from (8b) we can find ϕ, and then from (8a) we can find the theoretically minimum rate R necessary to achieve a mean square error less than or equal to D. If $\{x_t\}$ consists

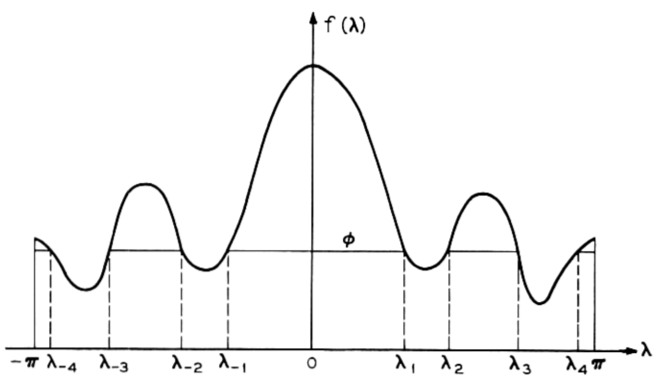

Fig. 6 — Graphical interpretation of equations 8a and b. The set $A = (-\pi, \lambda_{-4}) \cup (\lambda_{-3}, \lambda_{-2}) \cup (\lambda_{-1}, \lambda_1) \cup (\lambda_2, \lambda_3) \cup (\lambda_4, \pi)$. $A' = (\lambda_{-4}, \lambda_{-3}) \cup (\lambda_{-2}, \lambda_{-1}) \cup (\lambda_1, \lambda_2) \cup (\lambda_3, \lambda_4)$.

of independent Gaussian variates, with variance σ^2, then $f(\lambda) = \sigma^2$ and (8a) becomes

$$R(D) = \tfrac{1}{2} \log_2 \frac{\sigma^2}{D} \text{ bits/symbol.} \tag{9}$$

If we restrict the class of sources to be wide sense Markov of order n, then $f(\lambda)$ assumes the following form:

$$f(\lambda) = \frac{K}{\prod_{j=1}^{n} | e^{i\lambda} - a_j |^2} \tag{10}$$

with $0 < a_j < 1$, $a_j \neq a_k$ if $j \neq k$, and K is chosen to satisfy

$$\sigma^2 \equiv E\{x_\tau^2\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda)\, d\lambda. \tag{11}$$

In the remainder of this paper we consider some properties of the rate distortion function as given by (8a) and (8b) for processes with power spectral density (10).*

3.2 *The Markov-n Sequence*

In this section we present some results from prediction theory. For details and proofs see Refs. 6 and 7.

A process with power spectral density given in (10) is known as a Markov-n process.[7] Performing the indicated multiplication in (10) results in

$$f(\lambda) = \frac{K}{\prod_{j=1}^{n} | e^{i\lambda} - a_j |^2} = \frac{K}{| e^{in\lambda} + b_1 e^{i(n-1)\lambda} + \cdots + b_n |^2}. \tag{12}$$

A sequence with the spectrum (12) can be shown to satisfy the autoregressive relation

$$x_n + \sum_{i=1}^{n} b_i x_{n-i} = \epsilon_n \tag{13}$$

where $\{\epsilon_n\}$ is a sequence of uncorrelated random variables with variance K.

Writing (13) in the form

$$x_n = -\sum_{i=1}^{n} b_i x_{n-i} + \epsilon_n \tag{14}$$

* T. Berger, in a recent paper considers similar properties for the Weiner process[4].

it can be shown by the orthogonality principle (Ref. 8, Section VII-C) that the best linear predictor in the mean square sense, of x_n given the infinite past is just

$$\hat{x}_n = -\sum_{i=1}^{n} b_i x_{n-i} . \tag{15}$$

Hence for a Markov-n process the best prediction involves only the n previous samples.

The error is

$$e \equiv x_n - \hat{x}_n = \epsilon_n . \tag{16}$$

The minimum mean square error is thus

$$\sigma_m^2 \equiv E(\epsilon_n)^2 = K. \tag{17}$$

From (10) and (17)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 f(\lambda)\, d\lambda = \log_2 \sigma_m^2 - \frac{1}{2\pi}\sum_{j=1}^{n}\int_{-\pi}^{\pi} \log_2 |\, e^{i\lambda} - a_j\, |^2. \tag{18}$$

From Peirce's tables,[9] number 540, it can be shown that the integral is zero (recalling that $0 < a_j < 1$). We state our conclusion as a theorem.

Theorem 1: For a sequence with spectrum given in (10) *the minimum mean square error resulting from an optimal prediction one step ahead is* σ_m^2, *where*

$$\log_2 \sigma_m^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 f(\lambda)\, d\lambda. \tag{19}$$

Theorem 1 is a special case of the theorem proved in Ref. 6, page 183.

3.3 *Evaluation of* $R(D)$ *for* $D \leqq f(\pi)$

We next consider the particular form that equations (8a) and (8b) assume when $f(\lambda)$ is as given in (10).

Theorem 2: Given a process with

$$f(\lambda) = \frac{K}{\prod_{j=1}^{n} |\, e^{i\lambda} - a_j\, |^2}$$

for some integer n. *For mean square errors satisfying* $0 \leqq D \leqq f(\pi)$, R(D) *is given by*

$$R(D) = \tfrac{1}{2} \log_2 \frac{\sigma_m^2}{D} \text{ bits/symbol}. \tag{20}$$

Proof: From (8a) and (8b)

$$R(\phi) = \frac{1}{2} \int_A \log_2 \frac{f(\lambda)}{\phi} \frac{d\lambda}{2\pi}$$

$$D = \frac{1}{2\pi} \int_A \phi \, d\lambda + \int_{A'} f(\lambda) \, d\lambda.$$

The power spectral density $f(\lambda)$ is monotonically decreasing with a minimum at $\lambda = \pi$. Hence for ϕ in the range $0 \leqq \phi \leqq f(\pi)$, $A = (-\pi, \pi)$, $A' = \varnothing$, and

$$D = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi \, d\lambda = \phi. \tag{21}$$

It follows that

$$R(\phi) = R(D) = \frac{1}{2} \int_{-\pi}^{\pi} \log_2 f(\lambda) \frac{d\lambda}{2\pi} - \tfrac{1}{2} \log_2 D. \tag{22}$$

From Theorem 1 the first term is $\frac{1}{2} \log \sigma_m^2$ so $R(D) = \frac{1}{2} \log_2 \sigma_m^2/D$ which holds for $0 < D \leqq f(\pi)$. This is (20).

The rate-distortion function (20) is precisely the rate-distortion function of a process consisting of independent gaussian random variables with mean 0 and variance σ_m^2 [see (9)].

Figure 7 illustrates why the rate-distortion function depends on $f(\pi)$ in this way. The shape of the spectrum of D in (8b) is that which would be assumed by water if it were poured into a container shaped as $f(\lambda)$. As we pour in water, it distributes itself uniformly so long as its level is below $f(\pi)$. Hence D is independent of $f(\lambda)$ so long as $D < f(\pi)$. Once $D = f(\pi)$ the exact shape of $f(\lambda)$ comes into play.

Consider next the predictive communication system of Fig. 4. The source emits the gaussian process with power spectral density (10). The

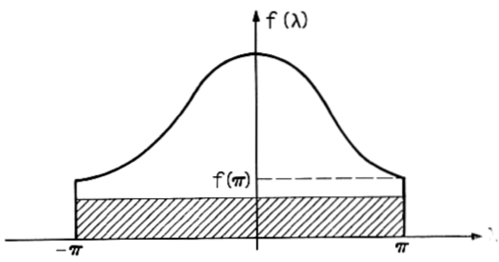

Fig. 7 — Typical Markov spectrum, illustrating water filling interpretation of the rate-distortion function.

optimum predictor makes a prediction of x_n based on $\{x_k\}_{k=0}^{n-1}$. This prediction is then subtracted from x_n and the error is transmitted. The transmitted sequence is thus the sequence $\{\epsilon_n\}$ [see (14)] which is a sequence of uncorrelated gaussian random variables with variance σ_m^2 . Its rate-distortion function is thus also given by (20), for D in the interval $0 < D \leqq \sigma_m^2$.

From (1)

$$\begin{aligned} S/N &= 10 \log_{10} \frac{\sigma^2}{D} \\ &= 10 \log_{10} \frac{\sigma^2 \sigma_m^2}{\sigma_m^2 D} \\ &= 3.01 \log_2 \frac{\sigma_m^2}{D} + 10 \log_{10} \frac{\sigma^2}{\sigma_m^2} \\ &= 6.02R + 10 \log_{10} \frac{\sigma^2}{\sigma_m^2} \end{aligned} \tag{23}$$

since R is given by (20). Hence S/N is a linear function of R over the range of R for which $0 \leqq D \leqq f(\pi)$. This range depends on n, the order of the Markov process, as given in theorem 3.

Theorem 3: *For an* n*th order gaussian Markov process, the rate-distortion function is given by*

$$R(D) = \tfrac{1}{2} \log_2 \frac{\sigma_m^2}{D} \text{ bits/symbol}$$

for rates $R \geqq R_{min}$. *The value of* R_{min} *depends on the exact shape of the power spectral density* $f(\lambda)$ *and assumes a value satisfying*

$$0 < R_{min} < n \quad \text{bits/symbol} \tag{24}$$

depending on the a_j*'s of* $f(\lambda)$ [*see* (10)].

Proof: From (10)

$$f(\lambda) = \frac{K}{\prod_{j=1}^{n} | e^{i\lambda} - a_j |^2}.$$

From this

$$f(\pi) = \frac{K}{\prod_{i=1}^{n} | 1 + a_i |^2}. \tag{25}$$

At $D = f(\pi)$

$$R_{\min} = R(f(\pi)) = \tfrac{1}{2} \log_2 \frac{\sigma_m^2}{f(\pi)} \text{ bits/symbol} \tag{26}$$

which from Theorem 1 is

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 f(\lambda)\, d\lambda - \log_2 f(\pi)\right]$$

$$= \frac{1}{2}\left[\log_2 K - \frac{1}{2\pi}\sum_{j=1}^{n}\int_{-\pi}^{\pi} \log_2 |\, e^{i\lambda} - a_j \,|^2\, d\lambda - \log_2 K + \sum_{j=1}^{n} \log_2 (1 + a_j)^2\right]. \tag{27}$$

As in (18) the integral is zero and

$$R_{\min} = \sum_{j=1}^{n} \log_2 (1 + a_j) \text{ bits/symbol.} \tag{28}$$

Since $|\, a_j \,| < 1$, $R_{\min} < n$ bits/symbol. Hence, $0 < R_{\min} < n$ bits/symbol, which is the desired result.

3.4 *Behavior of $R(D)$ at $D = f(\pi)$*

With $f(\lambda)$ as given in (10), the rate-distortion function is, from (20)

$$R(D) = \tfrac{1}{2} \log_2 \frac{\sigma_m^2}{D}$$

for $0 < D \leqq f(\pi)$, and from (8a) and (8b)

$$R(\lambda) = \frac{1}{2\pi}\int_0^{\lambda} \log_2 \frac{f(\gamma)}{f(\lambda)}\, d\gamma \tag{29a}$$

$$D(\lambda) = \frac{1}{\pi}\left[\int_0^{\lambda} f(\lambda)\, d\gamma + \int_{\lambda}^{\pi} f(\gamma)\, d\gamma\right] \tag{29b}$$

for $f(\pi) \leqq D \leqq \sigma^2$. Writing (8a) and (8b) in this form follows from the observation that for a monotonically decreasing power spectral density the set A equals the simply connected interval $(0, \lambda)$ and $\phi = f(\lambda)$, for the appropriate λ.

From (20)

$$\frac{d^n R}{dD^n} = (-1)^n \frac{(n-1)!}{2} D^{-n} \ln 2 \qquad 0 < D < f(\pi) \tag{30}$$

and from (29)

$$\frac{dR}{dD} = \frac{1}{2}\frac{1}{f(\lambda)} \ln 2 \tag{31}$$

$$\frac{d^2R}{dD^2} = \frac{\pi}{2}\frac{1}{\lambda f^2(\lambda)} \ln 2 \tag{32}$$

$$\frac{d^3R}{dD^3} = -\frac{\pi^2}{2}\frac{f^2(\lambda) + 2\lambda f(\lambda)f'(\lambda)}{\lambda^3 f^4(\lambda)f'(\lambda)} \ln 2 \tag{33}$$

for $f(\pi) < D < \sigma^2$, where $f'(\lambda) = df(\lambda)/d\lambda$

From (30), (31), and (32) we see that dR/dD and d^2R/dD^2 are continuous at $D = f(\pi)$. But from (33) we see that $d^3R/dD^3 \to -\infty$ as $D \to f(\pi)$ from above (since $f'(\pi) \to 0$), whereas d^3R/dD^3 is bounded as $D \to f(\pi)$ from below. Hence d^3R/dD^3 is discontinuous at $D = f(\pi)$.

IV. QUANTIZING CORRELATED SOURCES

4.1 *Introduction*

Consider a source that emits a sequence of independent gaussian random variables of mean 0, variance σ^2. It is desired to optimally quantize the source by using an M level quantizer. Max[10] has shown that by optimally choosing the quantizer input ranges and output levels, a mean square quantization error of

$$D_q = K(M)\frac{\sigma^2}{M^2} \tag{34}$$

can be achieved where $K(M)$ is a function of M. Further, it is shown numerically that $K(M) \leqq 2.72$, and that the inequality becomes an equality as $M \to \infty$. Hence for any M

$$D_q \leqq 2.72 \frac{\sigma^2}{M^2}. \tag{35}$$

For an M level quantizer the number of bits/symbol is $R = \log_2 M$, so that (35) can be written

$$D_q \leqq 2.72 \frac{\sigma^2}{2^{2R}}. \tag{36}$$

The rate-distortion function of the process is from (9)

$$R = \tfrac{1}{2}\log_2 \frac{\sigma^2}{D}$$

so that the minimum possible mean square error achievable with a fixed bit rate R is

$$D_{\min} \equiv \frac{\sigma^2}{2^{2R}}. \tag{37}$$

Hence Max's scheme can be made to achieve a mean square error satisfying

$$D_q \leqq 2.72\, D_{\min} \tag{38}$$

where $D_{\min}$ is the minimum mean square error as given by rate-distortion theory.

In this section we find a bound on a quantizing system studied by Huang and Schultheiss.[11] Our result is that (38) holds also for correlated sources, when $D_{\min}$ is as given by the appropriate rate-distortion function. For the case of Markov sources we plot this result in Fig. 2.

4.2 *Description of the System*

Referring to Fig. 4, the source emits correlated gaussian variates (not necessarily Markov), of mean 0 and with correlation matrix $\mathcal{R} = E(XX^T)$. The operator A accumulates source N-vectors X, and rotates them in such a way that

$$Y = AX \tag{39}$$

and

$$E(YY^T) = E(AXX^TA^T) = AE(XX^T)A^T = A\mathcal{R}A^T = J \tag{40}$$

where J is a diagonal matrix whose ith entry is λ_i, the ith eigenvalue of $\mathcal{R}$. Hence Y is an N-vector whose components are independent random variables with mean 0 and variance λ_i, and A is a unitary transformation.

The sequence of independent variates $\{y_i\}$ (the components of Y_N) are then quantized step by step.[10,11] The jth quantization can be optimized to produce a mean square error of

$$D_j = K(M_j)\lambda_j M_j^{-2} < 2.72\lambda_j M_j^{-2} \tag{41}$$

where M_j is the number of quantization levels used to quantize y_j. Denoting the output of the quantizer by the vector Y', the average mean square error is

$$\begin{aligned} D = \frac{1}{N} E(Y - Y')^T(Y - Y') &= \frac{1}{N} E(Y - Y')^T A^T A(Y - Y') \\ &= \frac{1}{N} E(X - X')^T(X - X') \end{aligned} \tag{42}$$

where we have used the fact that for a unitary transformation $AA^T = AA^{-1} = I$, the identity matrix. Hence the system mean square error equals the quantizer mean square error.

From (41) and (42)

$$D = \frac{1}{N} E(Y - Y')^T(Y - Y') = \frac{1}{N} E \sum_{i=1}^{N} (y_i - y_i')^2$$

$$\leqq \frac{1}{N} 2.72 \sum_{i=1}^{N} \lambda_i M_i^{-2} \equiv D_u . \quad (43)$$

4.3 *Optimization over the* M_j

We next tighten the upper bound by optimally choosing the M_j's subject to the following constraints.

(*i*) $M_j \geqq 1$ for every j. The quantizer must have at least one output level.

(*ii*) The bit rate is limited by the channel capacity, C bits per symbol. We can thus use $M = 2^C$ levels per symbol or M^N levels per vector. This implies the constraint

$$M^N = \prod_{i=1}^{N} M_i . \quad (44)$$

Hence we wish to minimize the right side of (43) subject to (44), while keeping in mind constraint (*i*).

With ν a Lagrange multiplier, we form

$$F = D_u + \nu M^N. \quad (45)$$

A differentiation with respect to M_k yields

$$\frac{\lambda_k}{M_k^2} = \mu \quad (46)$$

where μ is a constant. Using (44) to solve for the constant gives

$$M_k = M \left[\frac{\lambda_k}{\left(\prod_{i=1}^{N} \lambda_i \right)^{1/N}} \right]^{1/2} \quad (47)$$

and

$$D_u = \frac{2.72}{M^2} \left(\prod_{i=1}^{N} \lambda_i \right)^{1/N}. \quad (48)$$

However constraint (*i*) will only hold if in (47)

$$\lambda_k \geqq \frac{\left(\prod_{i=1}^{N} \lambda_i\right)^{1/N}}{M^2}. \tag{49}$$

for every k.

The right side of (49) can be written

$$\frac{\left(\prod_{i=1}^{N} \lambda_i\right)^{1/N}}{M^2} = \frac{2^{\frac{1}{N}\sum_{i=1}^{N} \log_2 \lambda_i}}{M^2} \tag{50}$$

$$\sim = \frac{2^{\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 f(\lambda)\, d\lambda\right\}}}{M^2} \tag{51}$$

$$= \frac{\sigma_m^2}{M^2} \tag{52}$$

where we have used the fact that the eigenvalues of $\mathfrak{R}$ approach the ordinates of $f(\lambda)$ equally spaced in $(-\pi, \pi)$ as $N \to \infty$ (see Ref. 6), and then applied the definition of a Riemann integral. Finally, we used (19). Hence the constraint (i) is met if

$$\lambda_k \geqq \frac{\sigma_m^2}{M^2} \tag{53}$$

for all k. Using (50), (51), and (52), (48) becomes

$$D_u = 2.72 \frac{\sigma_m^2}{M^2}. \tag{54}$$

In terms of signal to noise ratio we get

$$\begin{aligned} \text{S/N} &= 10 \log_{10} \frac{\sigma^2}{D} \geqq 10 \log_{10} \frac{\sigma^2}{D_u} \\ &= 10 \log_{10} \frac{\sigma^2}{\sigma_m^2} + 20 \log_{10} 2 \log_2 M - 4.34 \\ &= 10 \log_{10} \frac{\sigma^2}{\sigma_m^2} + 6.02R - 4.34 \end{aligned} \tag{55}$$

for

$$R > \tfrac{1}{2} \log_2 \frac{\sigma_m^2}{f(\pi)} \tag{56}$$

and where we used the relation

$$R = \log_2 M. \tag{57}$$

Suppose, however, that for some λ_k's (53) is not met. Specifically, arrange the eigenvalues such that $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \cdots \geqq \lambda_N$ and suppose that (47) yields

$$M_k \geqq 1 \qquad k = 1, 2, \cdots J \tag{58a}$$

$$M_k < 1 \qquad k = J + 1 \cdots N. \tag{58b}$$

Set those M_k in (58b) equal to one, and reoptimize over the M_k of (58a), the expression

$$D_J = 2.72 \sum_{k=1}^{J} \frac{\lambda_k}{M_k^2} \tag{59}$$

subject to the constraint

$$\prod_{k=1}^{J} M_k = M^N. \tag{60}$$

We would find that optimally

$$\frac{\lambda_k}{M_k^2} = \frac{\left(\prod_{i=1}^{J} \lambda_i\right)^{1/J}}{M^{2N/J}} = \gamma \qquad k = 1 \cdots J \tag{61}$$

where the right side of (61) is a constant. Without loss of generality, we can assume that all M_k obtained from (61) are greater than or equal to one. Otherwise we would set the infeasible M_k equal to one, and reoptimize. The procedure would return us to an equation similar to (61). As $N \to \infty$

$$\begin{aligned} D_q &\leqq 2.72 \frac{1}{N} \left(\sum_{i=1}^{J} \frac{\lambda_i}{M_i^2} + \sum_{i=J+1}^{N} \lambda_i \right) \\ &= 2.72 \frac{1}{N} \left(\sum_{i=1}^{J} \gamma + \sum_{i=J+1}^{N} \lambda_i \right) \\ \sim &= 2.72 \left[\frac{1}{2\pi} \int_A \gamma \, d\lambda + \frac{1}{2\pi} \int_{A'} f(\lambda) \, d\lambda \right] \end{aligned} \tag{62}$$

where A and A' are as given in (8) with ϕ replaced by γ.

Similarly

$$\gamma = \frac{\left(\prod_{i=1}^{J} \lambda_i\right)^{1/J}}{M^{2N/J}} \tag{63}$$

which, upon rearrangement, becomes

$$\begin{aligned} R \equiv \log_2 M &= \frac{1}{2N} \sum_{i=1}^{J} \log_2 \frac{\lambda_i}{\gamma} \\ \sim &= \frac{1}{4\pi} \int_A \log_2 \frac{f(\lambda)}{\gamma} \, d\lambda. \end{aligned} \tag{64}$$

By comparing (8a) and (8b) with (62) and (64) we see that (62) has the optimal spectrum for a rate given by (64). This implies that our procedure of setting infeasible M_k's equal to one does indeed lead to an optimum result.

Further, the terms in brackets in (62) is the minimum mean square error for a rate given by (64). Hence the quantization procedure has yielded

$$D_q \leqq 2.72 \, D_{\min}$$

which is (38).

This result is plotted in dB in Fig. 2, for the case of a Markov-n process.

There is an approximation involved in obtaining this result. The M_i obtained may not be integers. However, the large M_i will be little affected by rounding, and the looseness of the bound of (38) for small M_i counteracts the effects of rounding the small M_i. In fact, for very small M_i the bound is conservative, as we can see from Fig. 2. Clearly S/N should approach zero as R goes to zero. Hence our lower bound on S/N is loose in this range.

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