# B.S.T.J. BRIEFS

## On the Computation of the Far Field of Open Cassegrain Antennas

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(Manuscript received February 2, 1970)

The far field radiation patterns of open cassegrain antennas have been previously computed by a double integration[1]. Recently one integration has been eliminated by using a modified stationary phase approximation[2]. The rather remarkable accuracy thus obtained motivated an investigation to estimate the error in this approximation.

In this note we present an alternate formulation for the computation of the far field and provide an additional estimate for the error in the modified stationary phase approximation. It is also shown that for certain illuminations, there exists a closed form solution to the azimuthal integration. For illuminations derived from a linear combination of $TE^0_{1n}$ and $TM^0_{1n}$ waveguide modes, the integrals can be expressed in terms of convergent series of Bessel functions.

Based on the aperture field method, the far field electric $\bar{E}_f$ of an open cassegrain antenna is related to the reflected field at the aperture, $\bar{E}_r$, by the following double integral[1]

$$\bar{E}_f = j\,\frac{\exp(-jkR_a)}{R_a}\int_0^{\theta_m}\int_0^{2\pi}\bar{E}_r(\theta,\phi)\cdot\exp\left[jk\sin\theta_a(x_p\cos\phi_a + y_p\sin\phi_a)\right]r^2\sin\theta\,d\theta\,d\phi \tag{1}$$

where $R_a$, $\theta_a$, $\phi_a$ are the far field spherical observation coordinates; $\lambda$ is the free space wavelength and $k = 2\pi/\lambda$ is the propagation constant; and $\theta_m$ is the illumination angle. Referring to Fig. 1, the rectangular $x_p$ and $y_p$ coordinates are related to the spherical $\theta$, $\phi$ coordinates by

$$x_p = r(\cos\theta_0\sin\theta\cos\phi + \sin\theta_0\cos\theta), \tag{2}$$

$$y_p = r\sin\theta\sin\phi, \tag{3}$$

$$r = 2f/(a - b\cos\phi), \tag{4}$$

where

$$a = 1 + \cos\theta\cos\theta_0, \tag{5}$$

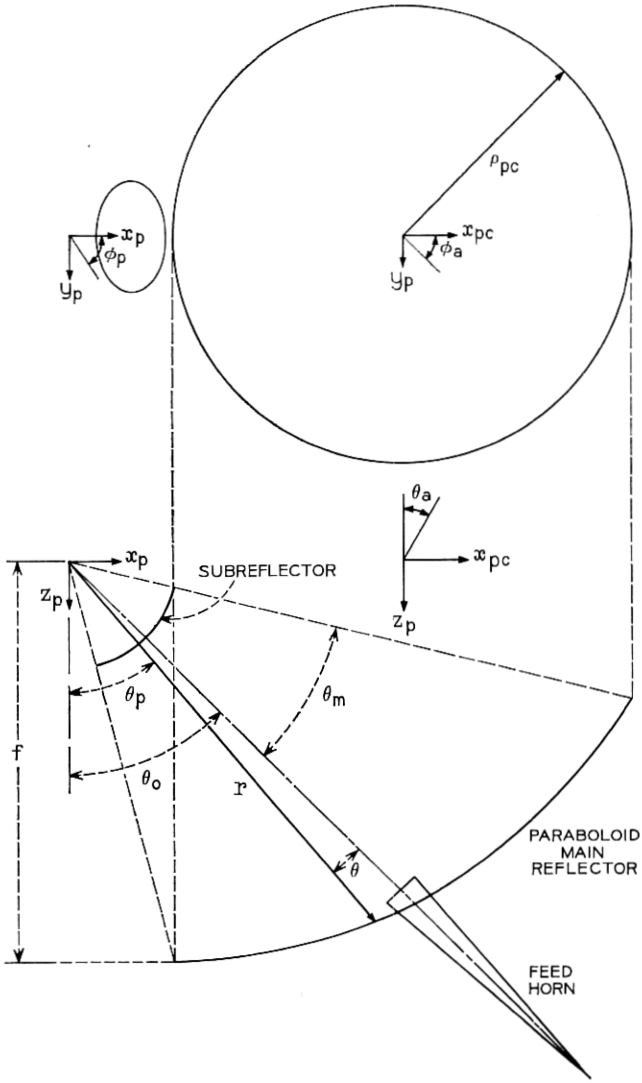


Fig. 1—Open cassegrain antenna.

$$b = \sin\theta \sin\theta_0 , \tag{6}$$

and

$$a^2 - b^2 = c^2, \tag{7}$$

$$c = \cos\theta + \cos\theta_0 . \tag{8}$$

$\theta_0$ is the offset angle, and $f$ is the focal length of the paraboloid.

For combined $TE_{1n}^0 - TM_{1n}^0$ mode excitation and for $y$ polarization of the horn feed the reflected electric field at the aperture, $(\bar{E}_{ry})$, can be written

$$(1 + \cos\theta_0)(\bar{E}_{ry}) = 1_{xp} \sin\phi \left[ cE_c(0) \cos\phi + (b + a\cos\phi)E_c\left(\frac{\pi}{2}\right)\right] - 1_{yp}\left[(a\cos\phi - b)E_c(0)\cos\phi + cE_c\left(\frac{\pi}{2}\right)\sin^2\phi\right] \tag{9}$$

where $E_c(0)$ and $E_c(\pi/c)$ are the fields radiated by the subreflector in the principal planes $\phi = 0$ and $\phi = \pi/2$ at a distance $2f/(1 + \cos\theta_0)$ from the focal point.

The $\phi$ integration in equation (1) was recently approximated by a modified stationary phase approximation[2]. An examination of this approximation reveals that the resulting phase factors and the arguments of the Bessel functions have a simple geometric interpretation. Specifically the phase factors are directly related to the location in the $x_p$ , $y_p$ plane of the centers of the $\theta$ = constant circles (see Fig. 2 in Ref. 2) and the arguments of the Bessel functions are related to the radii of these circles.

This suggests, for the $\phi$ integration, the following coordinate transformation

$$x_p = r_c \cos\phi_1 + x_c , \tag{10}$$

$$y_p = r_c \sin\phi_1 \tag{11}$$

with

$$r_c = \frac{2f\sin\theta}{c} , \tag{12}$$

$$x_c = \frac{2f\sin\theta_0}{c}. \tag{13}$$

This transformation gives in the $x_p$ , $y_p$ plane, for the $\theta$ = constant coordinates the same set of circles as obtained from equations (2) and

(3), and for the $\phi_1$ = constant coordinates a set of hyperbolas with the equation

$$x_p^2 - y_p^2 - 2x_p y_p \operatorname{ctn} \phi_1 + 4f(x_p - y_p \operatorname{ctn} \phi_1) \operatorname{ctn} \theta_0 = (2f)^2. \quad (14)$$

The sets of curves which correspond to the transformation (10) and (11) are shown in Fig. 2.

In the $\theta$, $\phi_1$ coordinate system the rectangular $x$, $y$ components of the far field have a much simpler representation. In particular for $y$ polarization these components are

$$(\bar{E}_{fy}) = j \frac{\exp(-jkR_a)}{\lambda R_a} \int_0^{\theta_m} (\bar{E}_y) \sin \theta \, d\theta \quad (15)$$

where the subscript $y$ is to designate the $y$ polarization, and the $x$, $y$ components of $(\bar{E}_y)$ are

$$(E_y)_y = -\frac{4f^2}{(1 + \cos \theta_0)c} \exp(jx_c \sin \theta_a \cos \phi_a) \cdot \int_0^{2\pi} \left[ E_c(0)(a \cos \phi_1 + b) \cos \phi_1 + E_c\left(\frac{\pi}{2}\right) c \sin^2 \phi_1 \right] \cdot \frac{\exp[jkr_c \sin \theta_a \cos(\phi_1 - \phi_a)]}{a + b \cos \phi_1} \, d\phi_1 \quad (16)$$

and

$$(E_y)_x = \frac{4f^2 \exp(jkx_c \sin \theta_a \cos \phi_a)}{(1 + \cos \theta_0)c} \cdot \int_0^{2\pi} \left[ E_c(0)(a \cos \phi_1 + b) - E_c\left(\frac{\pi}{2}\right) c \cos \phi_1 \right] \sin \phi_1 \cdot \frac{\exp[jkr_c \sin \theta_a \cos(\phi_1 - \phi_a)]}{a + b \cos \phi_1} \, d\phi_1 . \quad (17)$$

In deriving equations (15), (16) and (17), the Jacobian, $J$, of the transformation (10) and (11) was used, which is

$$J = \frac{(2f)^2}{c^3} (a + b \cos \phi_1) \sin \theta. \quad (18)$$

It is readily shown that by using a two term expansion of

$$[1/(a + b \cos \phi_1)] \approx (1/a)(1 - b/a \cos \phi_1),$$

the integrals (16) and (17) give the same values as those obtained by the modified stationary phase approximation but without the term

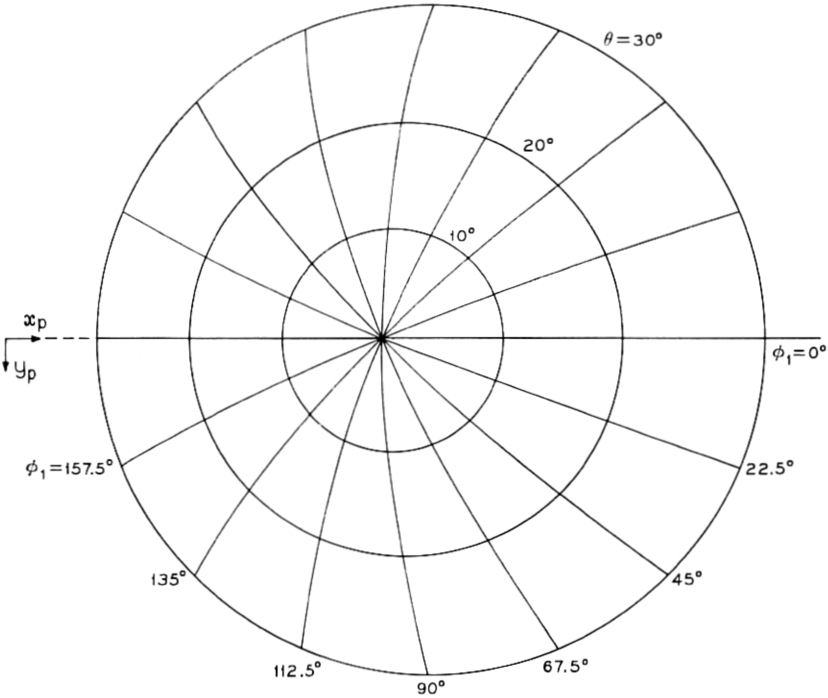


Fig. 2—Transformation of $(\theta, \phi_1)$ coordinates in the $(x_p, y_p)$ plane.

$(\sin \theta \sin \theta_0 \cos \phi_a)^2$ in the denominator. This shows that the modified stationary phase approximation is at least of order $(b/a)^2$, for all observation angles and with the additional property that for large angles off axis it is of order $[(b/a)^2/2kf \sin \theta_a]$.

There are two cases for which the integration of either equations (16) or (17) can be performed in closed form.

$$(i) \qquad aE_c(0) = cE_c\left(\frac{\pi}{2}\right)$$

where $a$ and $c$ are given by equations (5) and (8).

Such illuminations can be approximately realized by combined $TE_{11}^0 - TM_{11}^0$ mode excitations. For this case the integration of equation (16) yields

$$(E_y)_y = -8\pi^2 f^2 \frac{\exp(jkx_c \sin \theta_a \cos \phi_a)}{(1 + \cos \theta_0)c} E_c(0) J_0(x) \qquad (19)$$

where $J_0$ is a Bessel function of order zero with the argument

$$x = kr_c \sin \theta_a . \tag{20}$$

The evaluation of the corresponding expression based on the modified stationary phase approximation yields

$$(E_v)_v = -\frac{8\pi^2 f^2 \exp(jx_c \sin \theta_a \cos \phi_a)}{(1 + \cos \theta_0)c} \cdot E_c(0)\left[J_0(x) + \frac{b^2}{a^2 - b^2 \cos^2 \phi_a} \frac{J_1(x)}{x} \cos 2\phi_a\right]. \tag{21}$$

The error in the approximation is the second term in equation (21). That term is zero in the planes $\phi_a = \pi/4$ and $\phi_a = 3/4\pi$. In other planes, the error is of order $(b/a)^2/x$, and is in agreement with the error estimate given above.

The dependence of the second term in equation (21) on $\theta$ is shown in Fig. 3 for $\theta_0 = 55°$, $f = 80\lambda$ and $\theta_a = 2.5°$. These parameters are the same as those used in the computations in Ref. 2, Fig. 5. A comparison of the two figures shows good agreement, as would be expected, since the condition for the illumination was approximately satisfied.

*(ii)* $$cE_c(0) = aE\left(\frac{\pi}{2}\right).$$

These illuminations may also be approximately realized. For this condition the cross-polarized component has a closed form solution given by

$$(E_v)_x = \frac{j8\pi^2 f^2 \exp(jkx_c \sin \theta_a \cos \phi_a)}{(1 + \cos \theta_0)c} E_c(0)\left(\frac{b}{a}\right) J_1(x) \sin \phi_a . \tag{22}$$

An evaluation of the corresponding expression based on the stationary phase approximation gives*

$$(E_v)_x = j\frac{8\pi^2 f^2 \exp(jkx_c \sin \theta_a \cos \phi_a)}{(1 + \cos \theta_0)c} \cdot E_c(0)\left(\frac{b}{a}\right)\left[J_1(x) \sin \phi_a + \frac{b^2}{a^2 - b^2 \cos^2 \phi_a} \frac{J_2(x)}{x} \sin 3\phi_a\right]. \tag{23}$$

The error in the modified approximation is the second term and is of the same order as the error term in equation (21).

For illuminations derived from a linear combination of $TE^0_{1n}$ and $TM^0_{1n}$ waveguide modes, the integrals (15) and (16) can be expressed in convergent series of Bessel function, by using the following Fourier

* Equation (21) in Ref. 2 has a misprint and should be divided by 2.

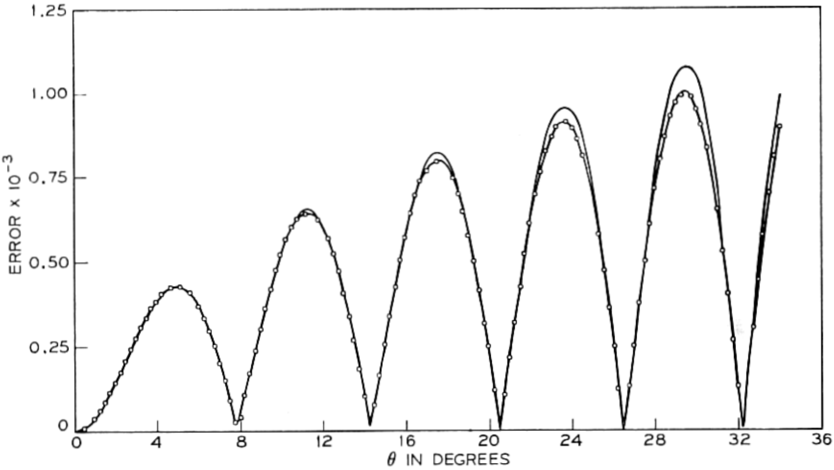


Fig. 3—Error introduced by the modified stationary phase approximation. Angle off axis, $\theta_a = 2.5$ degrees. ——— $\phi_a = 0$ (H plane); ○○○ $\phi_a = \pi/2$ (E plane).

series expansions

$$\exp\,[jx\cos(\phi - \phi_a)] = J_0(x) + 2\sum_{n=1}^{\infty} j^n J_n(x)\cos n(\phi - \phi_a) \tag{24}$$

and

$$\frac{1}{a + b\cos\phi} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos n\phi \tag{25}$$

with

$$c_n = \frac{2}{\pi}\int_0^{\pi} \frac{\cos n\phi}{a + b\cos\phi}\,d\phi. \tag{26}$$

The integral (26) is tabulated[3] and by using equation (7) gives

$$c_n = \frac{2}{c}(-1)^n\left(\frac{a-c}{a+c}\right)^{n/2}. \tag{27}$$

With these expansions, after some mathematical manipulations the following expressions are obtained for the integrals

$$\begin{aligned}(E_y)_y = -\frac{8\pi f^2 \exp\,(jx_c \sin\theta_a\cos\phi_a)}{(1+\cos\theta_0)(a+c)c}\Bigg\{c\left[E_c(0) + E_c\left(\frac{\pi}{2}\right)\right]J_0(x) \\ + \left[aE_c(0) - cE_c\left(\frac{\pi}{2}\right)\right]\Bigg[j\left(\frac{a-c}{a+c}\right)^{\frac{1}{2}} J_1(x)\cos\phi_a \\ - \frac{2c}{a+c}\sum_{n=0}^{\infty}(-j)^n\left(\frac{a-c}{a+c}\right)^{n/2} J_{n+2}(x)\cos(n+2)\phi_a\Bigg]\Bigg\}\end{aligned} \tag{28}$$

and

$$(E_y)_x = \frac{8\pi f^2 \exp(jx_c \sin\theta_a \cos\phi_a)}{(1 + \cos\theta_0)(a + c)^2 c} \left\{ jb\left[(a + 2c)E_c(0) + cE_c\left(\frac{\pi}{c}\right)\right] \cdot J_1(x)\sin\phi_a + 2c\left[cE_c(0) - aE_c\left(\frac{\pi}{2}\right)\right] \cdot \sum_{n=0}^{\infty} (-j)^n \left(\frac{a - c}{a + c}\right)^{n/2} J_{n+2}(x)\sin(n + 2)\phi_a \right\}. \tag{29}$$

From equations (5) and (7)

$$\left(\frac{a - c}{a + c}\right)^{\frac{1}{2}} = \tan\theta/2 \tan\theta_0/2. \tag{30}$$

For most antenna designs, equation (30) is much smaller than one, the series in equations (28) and (30) are, therefore, rapidly converging, and can in principle be evaluated to any desired accuracy.

It is noted that in the principal planes, these series can be related to Lommel functions of two variables[3].

The stimulating discussions with N. Amitay and E. R. Nagelberg are gratefully acknowledged by the author.

REFERENCES

1. Cook, J. S., Elam, E. M., and Zucker, H., "The Open Cassegrain Antenna," B.S.T.J., *44*, No. 7 (September 1965), pp. 1255–13001.
2. Ierley, W. H., and Zucker, H., "A Stationary Phase Method for the Computation of the Far Field of Open Cassegrain Antennas," B.S.T.J., *49*, No. 3 (March 1970), pp. 431–454.
3. Gradshteyn, I. S., and Ryzhik, I. M., *Tables of Integrals Series and Products*, New York: Academic Press, 1965, pp. 366, 987.

# Holographic Thin Film Couplers

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(Manuscript received June 17, 1970)

Recently P. K. Tien and his co-workers have described a prism coupler as a convenient means to feed light into a single mode of a guiding thin optical film.[1] Distributed couplers of this kind are of great interest for integrated optical devices. In this brief we describe thin