Dielectric Guide with Curved Axis and Truncated Parabolic Index

By E. A. J. MARCATILI

(Manuscript received May 5, 1970)

We find the field configurations and the propagation constants of the guided modes in a dielectric waveguide with curved axis and rectangular cross-section. Outside the guide, the refractive index is uniform. Inside, the index profile in the radial direction (intersection of the meridional plane and the plane of curvature) follows a parabolic law with the maximum at the center of the guide; in the direction perpendicular to the plane of curvature the index is either uniform or parabolic, again with the maximum at the center of the guide. The guide with mixed profiles has been proposed as an easy-to-support, low-loss, ribbon-like guide for millimeter and optical waves while the other, with parabolic profile in both directions, is similar to the "SELFOC®" or "GRIN" image transmitting guides.

The axial field components are small compared to the transverse components and consequently the modes are almost of the TEM kind. Within the guide the field distribution along a quadratic profile is a parabolic cylinder function of order close to an integer, and is sinusoidal along the uniform profile. The field components outside of the guide decay almost with exponential law.

Inside the SELFOC-like guide, the field distribution of the fundamental mode is gaussian and except for the attenuation the characteristics of the beam are similar to those obtained for a guide in which the parabolic index profile is not truncated.

The attenuation constant α of any mode is very sensitive to the radius of curvature R. Doubling R reduces α by several orders of magnitude.

Fixing R and the difference of refractive index between the center of the guide and the edge of it, the attenuation constant α passes through a minimum for a guide width measured in the plane of curvature which is only a few beam-widths.

Radiation loss for the fundamental gaussian mode is negligibly small if the distance between the center of the beam and the edge of the guide is two or more half beam-widths.

Guides with rectangular index profile in the plane of curvature have less radiation loss than similar guides with truncated parabolic profile.

I. INTRODUCTION

A dielectric guide in which the refractive index decreases with parabolic law away from its axis acts as a lens-like medium.[1,2] The transmission through it is known even if the axis is not straight[3] and if the parabolic decrease is different in two orthogonal directions[4] (astigmatic guide).

Though extremely useful in many respects the parabolic medium is not realizable since it has ever-decreasing refractive index away from the axis and this in turn produces an untenable physical result. Thus though we know that in any realizable dielectric guide with curved axis, radiation losses are inevitable,[5] the modes in the parabolic medium with curved axis can have no radiation loss since the refractive index tending towards infinity far away from the axis prevents it.

A more realistic model is achieved by truncating the parabolic index distribution. We begin, in Section II, studying the two dimensional guide, Fig. 1a, in which the index profile, Fig. 1b, varies as a truncated parabolic function along the x axis and is independent of y while outside of the guide the index is uniform.

Later, this guide is modified in such a way that along y, the index profile is either rectangular, Fig. 2a, or another truncated parabolic function, Fig. 2b.

The first of these guides has the index distribution of the dielectric thin-film guide proposed in Ref. 6 as a low-loss, easy-to-support ribbon-like guide for millimeter and optical waves. It has also the configuration of a possible guide for integrated optics.[7] This guide, with curved axis has been analyzed in Ref. 8 ignoring radiation due to curvature. In Section II, both the phase and attenuation coefficients of the guided modes are evaluated and compared to those in a similar guide with rectangular index profiles along both x and y.

The results obtained for the guide with truncated parabolic profiles along x and y, Fig. 2b, are applicable, at least in order of magnitude, to "SELFOC"[9] or "GRIN"[10] fibers, and tubular gas lenses[11] with curved axes.

Finally conclusions are drawn in Section III, while all the mathematics are given in the Appendix.

II. MODES IN THE CURVED GUIDE

Consider the two-dimensional curved guide in Fig. 1a. The parabolic refractive index within the guide is independent of y and equal to

$$n_i = n\left[1 - \Delta\left(1 + \frac{2x}{a}\right)^2\right], \tag{1}$$

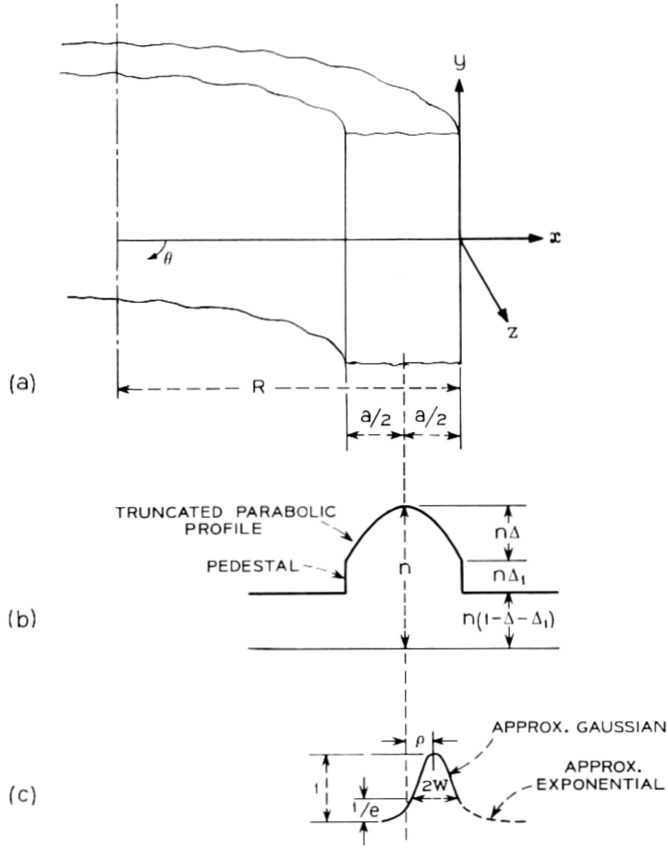

Fig. 1—(a) Two-dimensional truncated parabolic guide; (b) Refractive index profile; (c) Electric field distribution of the fundamental mode.

where a is the width of the guide, n, the refractive index in the center of it and $n(1 - \Delta)$, the refractive index at the edges. Outside the guide, the index is

$$n_o = n(1 - \Delta - \Delta_1). \tag{2}$$

We make the following assumptions:

$$\begin{aligned} \Delta &\ll 1 \\ \Delta_1 &\ll 1 \end{aligned} \tag{3}$$

and

$$\frac{\lambda}{a\sqrt{\Delta}} \ll 1 - \frac{a}{4\Delta R} \tag{4}$$

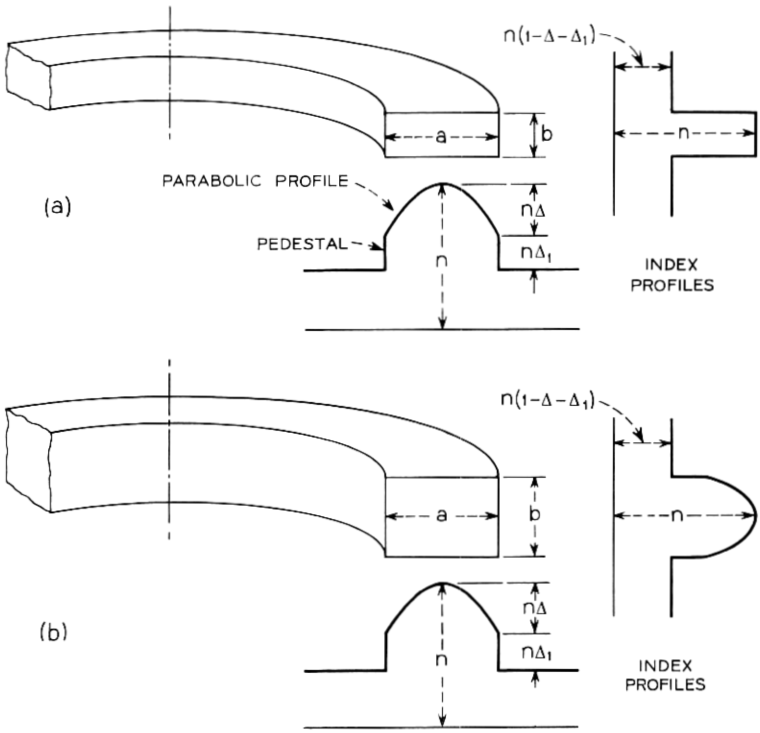

Fig. 2—(a) Inhomogeneous dielectric thin film guide; (b) "SELFOC®" or "GRIN" guides with rectangular cross-section.

where λ is the free-space wavelength and R the radius of curvature of the guide. The physical significance of inequality (3) is that the guided modes will have phase velocities quite comparable to that of a plane wave in a uniform medium of refractive index n. The inequality (4) insures that the amplitude of the field components at the edge of the guide are small compared to their maxima within the guide. In other words, most of the electromagnetic field is well confined within the guide, Fig. 1c, and consequently the loss per wavelength is small compared to unity. Considering only guided modes with field configurations independent of y, we can group them in two families: TE and TM. The field components of any mode of the first family are E_y, H_x and H_z while those of the second are H_y, E_x and E_z. In each family the transverse components are far larger than the axial components and consequently both families are essentially of the TEM kind.

The transverse components E_y, H_x, H_y and E_x of both families have the same functional dependence within and without the guide. Therefore we will talk from now on of the E field meaning either one of those four components.

Within the guide, and subject to the conditions (3) and (4), the E field distribution for the pth mode is essentially

$$E = \exp\left[-\left(\frac{x + \frac{a}{2} - \rho}{w}\right)^2\right] \mathrm{He}_p\left(2\,\frac{x + \frac{a}{2} - \rho}{w}\right) \exp\left[i(k_z z - \omega t)\right] \tag{5}$$

in which the first two factors describe the field distribution along x, and the last gives the propagating wave dependence along the curvilinear z axis. Similarly to the field distribution in the lens-like medium ($a = \infty$), the first factor is a gaussian with its maximum located at a distance

$$\rho = \frac{a^2}{8\Delta R} \tag{6}$$

from the center of the guide. The normalizing $1/e$ half-width is

$$w = \sqrt{\frac{a\lambda}{\pi n\sqrt{8\Delta}}}. \tag{7}$$

The second factor in equation (5) is a Hermite polynomial of order p which is also centered at $x = -(a/2) + \rho$ and the argument is normalized to $w/2$. Strictly speaking the expression (5) should have, instead of the Hermite polynomial, a Hermite function of order close to p. Interested readers can find the details in the Appendix.

For the fundamental mode $p = 0$ the Hermite polynomial is unity and the transverse field distribution is the well-known gaussian.

The propagation constant $k_z = \beta + i\alpha$ in equation (5) is complex and the phase and attenuation constants calculated in equations (36) and (37) are

$$\beta = \beta_\infty\left[1 + \frac{a^2}{16\Delta R^2} - \frac{2}{wkn}\,\frac{1 - M}{K(1 + M)}\right] \tag{8}$$

and

$$\alpha = \frac{\left[\frac{a}{w}(1 - d)\right]^{2p+1}}{2\sqrt{\pi\Delta}\, dRp!} \exp\left\{-\frac{\mathcal{R}}{3}\left[(1 - d)^2 + \frac{\Delta_1}{\Delta} - \left(p + \frac{1}{2}\right)\left(\frac{2w}{a}\right)^2\right]^{\frac{3}{2}} - \frac{a^2}{2w^2}(1 - d)^2\right\} \tag{9}$$

in which

$$\beta_\infty = kn\sqrt{1 - \left(\frac{2}{wkn}\right)^2\left(p + \frac{1}{2}\right)}, \tag{10}$$

$$d = \frac{2\rho}{a} = \frac{1}{\mathcal{R}}\left(\frac{a}{w}\right)^2, \tag{11}$$

$$\mathcal{R} = \frac{4\pi n}{\lambda}(2\Delta)^{\frac{3}{2}}R, \tag{12}$$

and the values of M and K can be found in equations (38) and (39). Let us discuss the physical meaning of some of these formulas.

The phase constant β given in equation (8) is the product of the phase constant β_∞(10) of the lens-like medium with straight axis ($R = a = \infty$), multiplied by a bracket essentially equal to one; the two small terms contained therein take into account the curvature of the axis and the truncation of the parabolic profile.

More interesting is the attenuation constant (9). The value $\sqrt{2\Delta}\ R\alpha$ which is the normalized attenuation per radian has been plotted in Fig. 3 for the fundamental mode $p = 0$ and $\Delta_1 = 0$. The abscissa is the square of the guide width a normalized to the beam-width $2w$ or its equivalent $(\pi na/\lambda)\sqrt{\Delta/2}$ which is the guide width normalized to the free wavelength. The parameter used for the solid curves is the normalized radius of curvature $\mathcal{R}$(12). For a given radius of curvature the loss per radian is highly sensitive to the width of the guide and passes through a minimum at width

$$\frac{a}{2w} = \left(\frac{\mathcal{R}}{8}\right)^{\frac{1}{2}}.$$

For a wide range of values of $\mathcal{R}$, say 10 to 1000, that minimum loss occurs when the guide width is only a few beam-widths.

The dotted lines are curves of constant d, that is constant ratio $2\rho/a$ between the beam displacement from the guide axis ρ and the guide half-width $a/2$. It is easy to understand the downward trend of these curves for large abscissas. Consider a guide with fixed geometry and decrease the wavelength λ of operation. The beam remains at the same distance ρ from the guide axis but it becomes narrower and consequently the field at the edge of the guide and the radiation loss decrease. It is surprising that the minimum radiation loss of the solid curves occurs when the beam displacement is a small part of the gidue width (d of the order of 0.1).

Why do the solid lines have a minimum? For very narrow guides

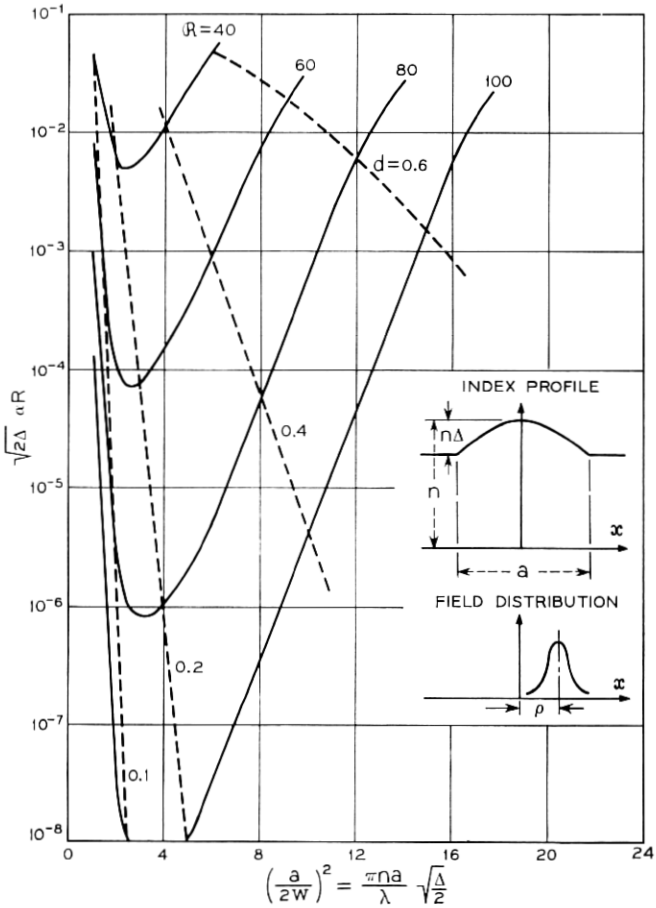

Fig. 3—Radiation loss in curved guides with truncated parabolic index profile.

$$w = \sqrt{\frac{a\lambda}{\pi n\sqrt{8\Delta}}}\ ; \qquad d = \frac{2\rho}{a}\ ; \qquad \rho = \frac{a^2}{8\Delta R}\ .$$

($a/2w \ll 1$), most of the electromagnetic field travels outside of the guide and any curvature of the axis introduces substantial radiation losses to this loosely guided beam. On the other hand, for very wide guides ($a/2w \gg 1$), any curvature of the axis displaces the beam close to one edge of the guide (d close to unity) and once again substantial losses occur. There must be a minimum in between.

It is interesting to compare the losses in these guides of truncated parabolic index profile with guides of identical width but with rectangular index profile of height $n\Delta$. In Fig. 4, the solid curves are a repetition of some of those in Fig. 3, while the dotted ones have been reproduced from Ref. 12. The abscissa is again $(a/2w)^2$ which is identical to $(\pi/4)a/A$ in which

$$A = \frac{\lambda}{n\sqrt{8\Delta}}$$

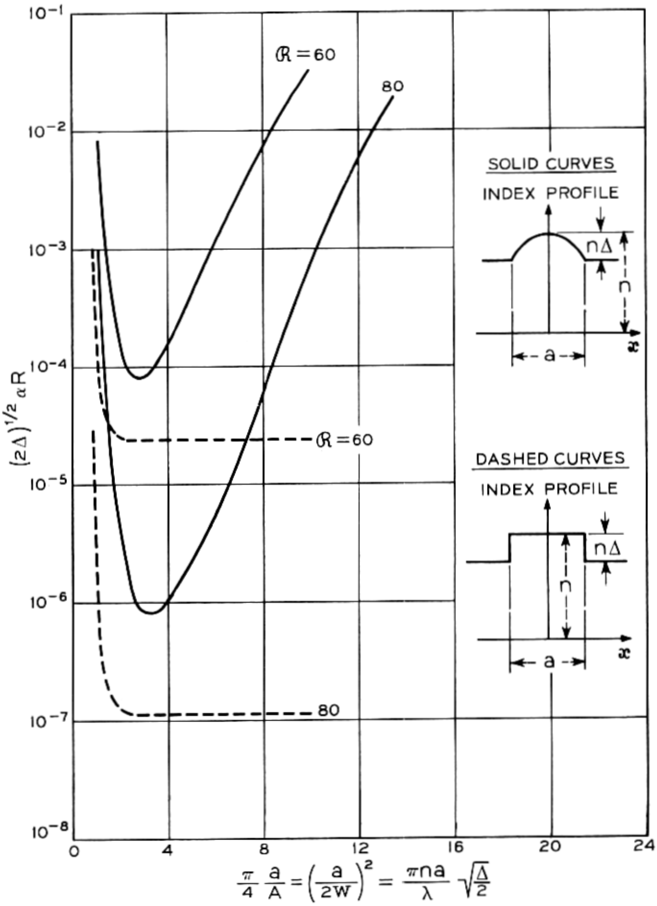

Fig. 4—Radiation loss in curved guides with truncated parabolic index profile (solid curves) and with rectangular index profile (dashed curves).

$$A = \frac{\lambda}{n\sqrt{8\Delta}}; \quad w = \sqrt{\frac{aA}{\pi}}; \quad \mathcal{R} = \frac{4\pi n R}{\lambda}(2\Delta)^{\frac{3}{2}}.$$

is a dimension such that for $a < A$, the guide with rectangular index profile supports a single mode and for $a > A$, the guide is multimode.

For the same radius of curvature, guide width, and same Δ on axis, the guide with truncated parabolic profile has more loss than the guide with rectangular profile. The difference is very marked for large abscissas, but this result should not be surprising because in the case of curved guides with truncated parabolic profile the beam travels close to one edge of the guide where there is little difference of refractive index between the inside and outside, while in the case of rectangular profile, though most of the power travels also close to one edge of the guide the full difference of refractive index $n\Delta$ is there to help in the guidance.

In Fig. 5 we have plotted again the attenuation per radian as a function of $(a/2w)^2$, but this time we use as parameter, the value of

$$h = \frac{\frac{a}{2} - \rho}{w}$$

which is the number of beam half-widths between the center of the beam and the external edge of the guide. The curves have asymptotes (dashed lines) parallel to both coordinates.

For $h \geqq 2$, $\Delta = 0.01$, the attenuation per radian αR turns out to be smaller than 0.003, which is very small for most purposes.

If the truncated parabolic profile is on a pedestal ($\Delta_1 \neq 0$), the losses are even smaller than those depicted in Fig. 4. The influence of Δ_1 in the attenuation constant (9), appears in the bracket of the exponent. The other two terms are in general small compared to unity. Therefore even a modest value of Δ_1, say $\Delta_1 = \Delta$, is enough to reduce the losses depicted in Figs. 3 and 5 by several orders of magnitude.

What happens when $p \neq 0$. From equation (9) we find as expected that for a given guide the radiation loss increases fast with the order p of the mode. The highest order mode that travels only slightly influenced by the guide width is characterized by

$$p_{\max} = \left[\frac{\frac{a}{2} - \rho}{w}\right]^2 - \tfrac{1}{2} = h^2 - \tfrac{1}{2}.$$

Naturally $p_{\max}$ is independent of Δ_1, and when the beam center is close to a beam half-width from the edge, $p_{\max} = 0$.

It is shown in the Appendix that if the refractive index profile along y, Fig. 1a, is not uniform but has either rectangular or truncated parabolic shape, Figs. 2a and 2b, the guides have different phase constants

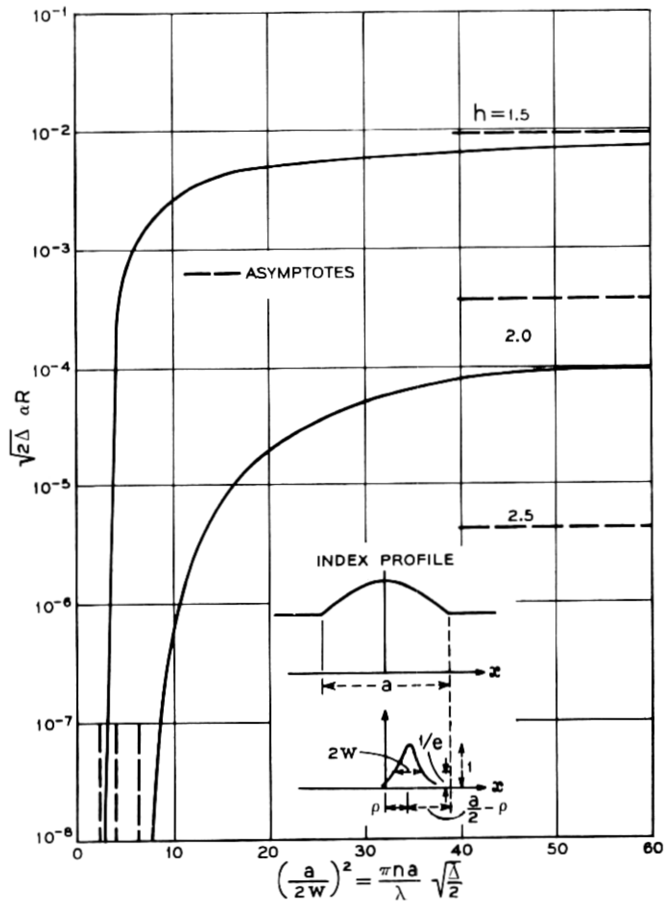

Fig. 5—Radiation loss in curved guides with truncated parabolic index profile.

$$h = \frac{(a/2) - \rho}{w}; \qquad w = \sqrt{\frac{a\lambda}{\pi n\sqrt{8\Delta}}}; \qquad \rho = \frac{a^2}{8\Delta R}.$$

than equation (8) but practically the same attenuation constant (9) provided that most of the electromagnetic field travels within the guide. Therefore everything said about attenuation in this section applies to the three guides.

For the following examples we will only use Figs. 3, 4 and 5 since all the important results and formulas are there.

2.1 *Example A*

For a guide such that

$$n = 1.5,$$
$$\Delta = 0.01,$$
$$\Delta_1 = 0,$$
$$a = 0.1 \text{ mm},$$
$$\lambda = 1\mu,$$

what is the radius of curvature R for which the loss per radian is of the order of 10^{-3}?

We calculate the abscissa and ordinate of Fig. 5 to be

$$\left(\frac{a}{2w}\right)^2 = \frac{\pi n a}{\lambda}\sqrt{\frac{\Delta}{2}} = 33$$

and

$$\sqrt{2\Delta}\alpha R = 1.4 \cdot 10^{-4}.$$

The parameter h obtained from Fig. 5 is approximately 2 and we derive

$$R = \frac{a^2}{8\Delta\rho} = 3.9 \text{ mm}.$$

A very small radius indeed.

2.2 *Example B*

For integrated optics a guide with truncated-parabolic profile may have the following characteristics

$$n = 1.5,$$
$$\Delta = 0.01,$$
$$\Delta_1 = 0,$$
$$a = 10\mu,$$
$$\lambda = 0.5\mu,$$
$$R = 0.6 \text{ mm}.$$

What is the loss per radian?

From Fig. 3 or 4 we get the abscissa and parameter

$$\left(\frac{a}{2w}\right)^2 = 6.7,$$

$$\mathcal{R} = \frac{4\pi nR}{\lambda}(2\Delta)^{\frac{3}{2}} \cong 60.$$

Consequently the loss per radian results

$$\alpha R = 0.018.$$

If instead of parabolic the index had been rectangular, from Fig. 4 we deduce that the loss per radian would have been 0.00018, two orders of magnitude smaller.

III. CONCLUSIONS

For losses small enough, the field configurations and phase constants of the modes in dielectric guides, Figs. 2a and 2b, with curved axis and parabolic index profile on a pedestal, are quite comparable to those in a similar guide in which the parabolic profile is extended to infinity.

The attenuation constant of a mode is very sensitive (exponential dependence) to the radius of curvature, size of the pedestal and order of the mode. The higher the order of the mode and the smaller the size of the pedestal the larger the loss.

Quantitative results about the attenuation constant for the fundamental gaussian mode in a guide without pedestal are given in Figs. 3, 4 and 5 and in typical examples at the end at the preceding section. We find in these figures the loss per radian αR as a function of the guide width a, using as parameter the radius of curvature R, or the ratio between beam displacement ρ and guide width or the ratio between the beam distance from the edge of the guide, $a/2 - \rho$ and the beam width w. The main conclusions are:

(*i*) Doubling R reduces the attenuation constant α several orders of magnitude.

(*ii*) For any R, there is a guide width that minimizes the loss per radian. That dimension is only a few beam-widths.

(*iii*) For comparable characteristics, guides with rectangular profiles have lower attenuation than those with truncated-parabolic profile. Therefore if the transmission of images is not important, such as in the case of the ribbon-like guide of Ref. 6

and guides for integrated optics, rectangular index profiles are more attractive than parabolic profiles.

(*iv*) The attenuation per 90° bend is smaller than 10^{-3} in a guide such that the distance between beam center and the external edge of the guide is larger than a couple of half beam-widths, that is, if

$$\frac{\frac{a}{2} - \rho}{2w} > 1 \cdot$$

APPENDIX

Modes in Curved Guides
With Truncated-Parabolic Index Profile

We start studying the two-dimensional curved guide depicted in Fig. 1a in cylindrical coordinates. Later we will introduce a variation of the index profile along y.

The parabolic refractive index distribution within the guide is

$$n_i = n\left[1 - \Delta\left(1 + 2\frac{r - R}{a}\right)^2\right] \tag{13}$$

where a is the width of the guide, n the refractive index in the center and $n(1 - \Delta)$ the refractive index at the edges. The refractive index outside the guide is

$$n_o = n(1 - \Delta - \Delta_1). \tag{14}$$

Assuming that the electromagnetic field does not vary along y and that the only component along that direction is H_y, all the field components either inside or outside the guide are[13]

$$\left.\begin{aligned} H_y &= H \\ E_r &= \frac{H}{\omega \epsilon_0 n_{\substack{i\\o}} r} \\ E_\theta &= \frac{i}{\omega \epsilon_0 n_{\substack{i\\o}}^2} \frac{\partial H}{\partial r} \end{aligned}\right\} \exp\left[i(\nu\theta - \omega t)\right] \tag{15}$$

where ω is the angular frequency, ϵ_0 the refractive index of free space, and the indices i and o refer to the inside and outside of the guide.

The resulting wave equation for both media is

$$\frac{d^2H}{dr^2} + \frac{1}{r}\frac{dH}{dr} + \left(k^2 n_{\substack{i\\o}}^2 - \frac{\nu^2}{r^2}\right)H = 0 \tag{16}$$

in which $k = 2\pi/\lambda$ and λ is the free space wavelength. Within the guide n_i is given by equation (13) and the wave equation can be reduced to

$$\frac{d^2H}{d\xi^2} + [\eta + \tfrac{1}{2} - \tfrac{1}{4}(\xi + \xi_0)^2]H = 0 \tag{17}$$

by making the following substitutions

$$\xi = \frac{2(r - R)}{w}, \qquad \xi_0 = \frac{a}{w}(1 - d), \tag{18}$$

$$\nu = k_z R, \tag{19}$$

$$\eta = \frac{k^2 n_1^2 - k_z^2}{4} w^2 - \frac{a^2 d}{2w^2}\left(1 - \frac{d}{2}\right) - \frac{1}{2}, \tag{20}$$

in which

$$w = \sqrt{\frac{aA}{\pi}} = \sqrt{\frac{a\lambda}{\pi n \sqrt{8\Delta}}}, \tag{21}$$

$$d = \frac{a^2}{w^2 \mathcal{R}} = \frac{a}{4\Delta R} = \frac{2\rho}{a}, \tag{22}$$

$$\mathcal{R} = \frac{4\pi n}{\lambda}(2\Delta)^{\frac{1}{2}} R, \tag{23}$$

and

$$A = \frac{\lambda}{n\sqrt{8\Delta}}. \tag{24}$$

Furthermore, equation (17) has been derived making the following simplifying assumptions

$$\Delta \ll 1, \qquad \Delta_1 \ll 1, \qquad \frac{\lambda}{a\sqrt{\Delta}} \ll 1 - \frac{a}{4\Delta R}. \tag{25}$$

The physical significance of w, d, A and the inequalities are given in the text.

The solution of equation (17) is[14]

$$H_i = D_\eta(\xi + \xi_0) = \exp\left[-\left(\frac{\xi + \xi_0}{2}\right)^2\right] \mathrm{He}_\eta(\xi + \xi_0) \tag{26}$$

where $D_\eta(\xi + \xi_0)$ is the parabolic cylinder function of order η and $\mathrm{He}_\eta\,(\xi + \xi_0)$ is the Hermite function of order η. Only if $a \to \infty$, η becomes an integer, the Hermite function is reduced to a polynomial and H_i becomes the well-known solution of the parabolic lens-like medium extending to infinity.[3]

Outside of the guide, that is for $r > R$, the refractive index n_o is uniform, equation (14), and the solution of the wave equation (16) is[13] the Hankel function of order ν and argument kn_or. That is

$$H_o = H_\nu^{(1)}(kn_or). \tag{27}$$

To match fields at the boundary $r = R$, the radial admittance H_ν/E_θ inside and outside the guide must be identical. With the help of equations (15), (26) and (27), we obtain the characteristic equation

$$k\frac{w}{2}\frac{nD_\eta(\xi_0)}{D_\eta'(\xi_0)} = \frac{H_\nu^{(1)}(kn_oR)}{H_\nu^{(1)\prime}(kn_0R)} \tag{28}$$

in which the derivatives are taken with respect to the arguments of the functions.

We should have another boundary equation for the other side of the guide, $r = R - a$, but we are interested in guides with radius of curvature R small enough to push the field away from the center of the guide, and consequently the field at the interface $r = R - a$ is negligibly small.

To solve explicitly the boundary or characteristic equation (28) for k_z, we need asymptotic expansions of the functions involved. From the inequalities in equation (25), it can be deduced that

$$|\,\xi_0\,| \gg 1 \quad \text{and} \quad |\,\xi_0\,| \gg |\,\eta\,|. \tag{29}$$

The asymptotic expansion for $D_\eta(\xi_0)$ is then[14]

$$D_\eta(\xi_0) \cong \xi_0^\eta \exp\left(-\frac{\xi_0^2}{4} - i\pi\eta\right) + \frac{\sqrt{2\pi}}{\Gamma(-\eta)}\,\xi_0^{-\eta-1}\exp\left(\frac{\xi_0^2}{4}\right) \tag{30}$$

where $\Gamma(-\eta)$ is the gaussian function of argument $(-\eta)$.

The asymptotic expansion for the Hankel function results from observing that as a consequence of equation (25)

$$\begin{aligned} kn_oR &\gg 1, \\ k_zR &\gg 1, \\ \frac{kn_0}{k_z} &\simeq 1 \end{aligned} \tag{31}$$

and

$$(k_z^2 - k^2 n_o^2)^{\frac{3}{2}} \frac{R}{k_z^2} \gg 1.$$

Therefore we can replace the Hankel function by Watson's approximation.[14] This approximation involves Bessel functions of order one-third and large arguments. Keeping the first term of their asymptotic expansions, the Hankel function results

$$H_\nu^{(1)}(kn_o R) = \sqrt{\frac{2}{\pi R (k_z^2 - k^2 n_o^2)^{\frac{1}{2}}}} \left\{ -i \exp\left[\frac{R}{3k_z^2}(k_z^2 - k^2 n_o^2)^{\frac{3}{2}}\right] + \frac{1}{2} \exp\left[-\frac{R}{3k_z^2}(k_z^2 - k^2 n_o^2)^{\frac{3}{2}}\right]\right\}. \tag{32}$$

Substituting equations (30) and (32) in equation (28) we obtain a simplified version of the characteristic equation

$$\frac{1 + \dfrac{\sqrt{2\pi}}{\Gamma(-\eta)} \xi_0^{-2\eta - 1} \exp\left(\dfrac{\xi_0^2}{2} + i\pi\eta\right)}{1 - \dfrac{\sqrt{2\pi}}{\Gamma(-\eta)} \xi_0^{-2\eta - 1} \exp\left(\dfrac{\xi_0^2}{2} + i\pi\eta\right)} = \xi_0 \frac{1 + i \exp\left[-\dfrac{2}{3}\dfrac{R}{k_z^2}(k_z^2 - k^2 n_o^2)^{\frac{3}{2}}\right]}{(k_z^2 - k^2 n_o^2)^{\frac{1}{2}} w}. \tag{33}$$

To solve this equation for k_z we rewrite it as

$$\Gamma(-\eta) = F(\eta) \tag{34}$$

and notice that $F(\eta)$ is a large quantity. Therefore the gamma function is also large and hence η must be near a pole, which makes η close to an integer p. Then we can replace the gamma function by the first term of the Laurent series $(-1)^p/p!(p - \eta)$, and equation (34) becomes

$$\eta = p - \frac{(-1)^p}{p!\, F(p)}. \tag{35}$$

Substituting η by the value given in equation (20) we derive the explicit value of k_z. This propagation constant is complex, $k_z = \beta + i\alpha$, and the real and imaginary parts are the phase and attenuation constants of the pth mode:

$$\begin{aligned} \beta &= \operatorname{Re} k_z \\ &= kn\left\{1 - \frac{2}{(wkn)^2}\left[p + \frac{1}{2} + \frac{1}{2}\mathcal{R}d^2\left(1 - \frac{d}{2}\right) + \frac{1 - M}{K(1 + M)}\right]\right\} \end{aligned} \tag{36}$$

$$\alpha = \operatorname{Im} k_z = \frac{\exp\left[-\frac{\mathcal{R}}{3}\left(\frac{1-d}{M}\right)^3\right]}{dKR\sqrt{2\Delta}} \frac{1+2M-M^2}{(1+M)^2} \tag{37}$$

where

$$M = \left[1 + \frac{\frac{\Delta_1}{\Delta} - \left(p+\frac{1}{2}\right)\frac{4}{\mathcal{R}d}}{(1-d)^2}\right]^{-\frac{1}{2}} \tag{38}$$

$$K = \sqrt{2\pi}\, p! \frac{\exp\left[\frac{\mathcal{R}d}{2}(1-d)^2\right]}{[\sqrt{\mathcal{R}d}\,(1-d)]^{2p+1}}. \tag{39}$$

In equation (37), M affects the value of α mostly via the exponential and not via the fraction

$$\frac{1+2M-M^2}{(1+M)^2}$$

which for all practical purposes can be replaced by 1. Consequently the normalized loss per radian $\sqrt{2\Delta}R\alpha$ results

$$L = \sqrt{2\Delta}\, R\alpha = \frac{\exp\left[-\frac{\mathcal{R}}{3}\left(\frac{1-d}{M}\right)^3\right]}{dK}. \tag{40}$$

Now we turn to guides in which the refractive index is a function of y, Figs. 2a and 2b.

Let us start with the ribbon-like structure of Fig. 2a and assume as in Ref. 6 that

$$\Delta_1 \gg \Delta. \tag{41}$$

Provided that most of the electromagnetic field travels within the ribbon, the attenuation per radian is still given by equation (40), but the phase constant is a slight modification of equation (36). From Ref. 12 is deduced

$$\beta_1 = \beta - \frac{kn}{2}\left[\frac{\pi(q+1)}{b}\right]^2 \begin{cases} \left(1 + \frac{2(1-\Delta_1)^2 A_1}{\pi b}\right)^{-2} & \text{for field polarized along } y, \\ \left(1 + \frac{2}{\pi}\frac{A_1}{b}\right)^{-2} & \text{for field polarized along } x, \end{cases} \tag{42}$$

where $q + 1$ indicates the number of maxima of electric field within the guide along y and

$$A_1 = \frac{\lambda}{n\sqrt{8\Delta_1}}. \tag{43}$$

Consider another guide, Fig. 2b, with rectangular cross-section and truncated parabolic index profile along both the x and y directions

$$n_i = n\left[1 - \Delta\left(1 + 2\frac{r - R}{a}\right)^2 - \Delta\left(\frac{2y}{b}\right)^2\right]. \tag{44}$$

Provided that most of the electromagnetic field is within the guide cross-section, the loss per radian is still given by equation (40), but the phase constant becomes[4]

$$\beta_2 = \beta - \frac{2}{w_2^2 kn}\left\{q + \frac{1}{2} + 2\frac{1 - \left(1 + \frac{\Delta_1}{\Delta}\right)^{-\frac{1}{2}}}{\sqrt{2\pi}\; q!\left(\frac{b}{w_2}\right)^{2q+1}\left[1 + \left(1 + \frac{\Delta_1}{\Delta}\right)^{-\frac{1}{2}}\right]}\right\} \tag{45}$$

where $q + 1$ is the number of maxima of the electric field along y and

$$w_2 = \sqrt{\frac{b\lambda}{\pi n\sqrt{8\Delta}}}. \tag{46}$$

If

$$p = q = 0$$

and

$$a = b$$

the guide has square cross-section and equations (40) and (45) yield a first approximation of the phase and attenuation constants in a curved SELFOC[9] guide.

REFERENCES

1. Tonks, L., "Filamentary Standing-Wave Pattern in Solid State Maser," J. Appl. Phys., *33*, No. 6 (June 1962), pp. 1980–1986.
2. Kogelnik, H., "On the Propagation of Gaussian Beams of Light Through Lens-Like Media Including Those with a Loss or Gain Variation," Appl. Opt., *4*, No. 2 (December 1965), pp. 1562–1569.
3. Tien, P. K., Gordon, J. P., and Whinnery, J. R., "Focusing of a Light Beam of Gaussian Field Distribution in Continuous and Periodic Lens-Like Media," Proc. IEEE, *53*, No. 2 (February 1965), pp. 129–136.
4. Marcatili, E. A. J., "Modes in a Sequence of Thick Astigmatic Lens-Like Focusers," B.S.T.J., *43*, No. 6 (November 1964), pp. 2887–2903.
5. Marcatili, E. A. J., and Miller, S. E., "Improved Relations Describing Direc-

tional Control in Electromagnetic Wave Guidance," B.S.T.J., *48*, No. 7 (September 1969), pp. 2161–2188.

6. Kumagai, N., Kurazono, S., Sawa, S., and Yoshikawa, N., "Surface Waveguide Consisting of Inhomogeneous Dielectric Thin Film," Elec. and Commun. in Japan, *51-B*, No. 3 (March 1968), pp. 50–56.
7. Miller, S. E., "Integrated Optics: An Introduction," B.S.T.J., *48*, No. 7 (September 1969), pp. 2059–2070.
8. Sawa, S., and Kumagai, N., "Surface Wave Along a Circular *H*-Band of an Inhomogeneous Dielectric Thin Film," Elec. and Commun. in Japan, *52-B*, No. 3 (March 1969), pp. 44–50.
9. Uchida, T., Furukawa, M., Kitano, I., Kaizuki, K., and Matsumura, H., "A Light-Focusing Fiber Guide," 1969 IEEE Conference on Laser Eng. and Appl., Washington, D. C.
10. Rawson, E. G., Herriott, D. R., and McKenna, J., "Refractive Index Distributions in Cylindrical, Graded Index Glass Rods (GRIN Rods) Used as Image Relays," Appl. Opt., *9*, No. 3 (March 1970), p. 753–759.
11. Marcuse, D., and Miller, S. E., "Analysis of a Tubular Gas Lens," B.S.T.J., *43*, No. 4, Part 2 (July 1964), pp. 1759–1782.
12. Marcatili, E. A. J., "Bends in Optical Dielectric Guides," B.S.T.J., *48*, No. 7 (September 1969), pp. 2103–2132.
13. Stratton, J. A., *Electromagnetic Theory*, New York: McGraw-Hill, 1941, pp. 360–361.
14. Magnus, W., Oberhettinger, F., and Soni, R. P., *Formulas and Theorems for the Special Functions of Mathematical Physics*, Chelsea Publishing Co., New York: Springer-Verlag, 1966, p. 144 (on page 1660); p. 332 (on page 1659).