# The Capacity of the Gaussian Channel with Feedback

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*In this paper we provide a rigorous proof that feedback cannot increase the capacity of the channel with additive colored gaussian noise by more than a factor of two. We also give a tighter bound showing that any increase in capacity is less than the normalized correlation between the signal and noise. It is further shown that gaussian signals and linear feedback processing will achieve capacity.*

*The practical implications are that (i) feedback should be used to simplify encoding and decoding since there is little to be gained in the way of increased capacity and (ii) the various proposed schemes which use linear feedback are doing the correct thing.*

## I. INTRODUCTION

When Shannon first showed that feedback could not increase the capacity of a memoryless channel, he mentioned that the capacity could be increased when the channel had memory.[1] One example of such a channel is the additive colored gaussian noise channel with an average power limitation on the transmitted signal. We prove here that the capacity of this channel is never more than twice the capacity without feedback and as the noise becomes white the capacity approaches the forward capacity. The limiting case has been attributed to Shannon for years and has only recently been rigorously proven.[2]

We derive an exact expression for the mutual information between the input and output of the channel. The application of different bounds to this expression produces twice the forward capacity with the weakest bound, or the forward capacity plus the normalized correlation of the signal and noise with a slightly stronger bound. It is shown that a gaussian signal maximizes the information, and consequently the optimum feedback technique is linear.

Our results are based on the model shown in Fig. 1. The added noise

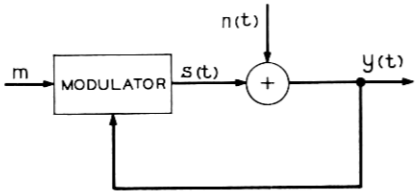


Fig. 1—Channel with noiseless feedback.

spectrum is normalized to 1 at infinite frequency, is bounded, and has an integrable logarithm. This allows us to represent the noise as in Fig. 2. The noise now consists of a white component plus a filtered version of the white noise. The imposed restrictions are for mathematical purposes only and are of no practical significance.

*Theorem 1: The mutual information between the input and output of a channel with additive gaussian noise with spectral density $N(\omega)$ and arbitrary causal feedback processing, as shown in Fig. 1, is given by:*

$$I(m;\, Y_T) = \frac{1}{2}\int_0^T E^2[s(t) + z(t) \mid m,\, Y_t]\, dt - \frac{1}{2}\int_0^T E^2[s(t) + z(t) \mid Y_t]\, dt \qquad (1)$$

*where $Y_t$ is $y(\tau)$, $0 \leqq \tau < t$ and the expectations are conditioned on $Y_t$ or $Y_t$ and $m$. $z(t)$ is a linear causal functional of white noise with the properties that:*

$$z(t) = \int_0^t h(t-\tau)\, dw(\tau) + \int_0^\infty h(t+\tau)\, dv(\tau)$$
$$|\, 1 + H(\omega)\, |^2 = N(\omega). \qquad (2)$$

*The two functions $w(t)$ and $v(t)$ are independent Wiener processes. The reason for introducing the second term is to make $n(t) = z(t) + \dot{w}(t)$ a stationary process.*

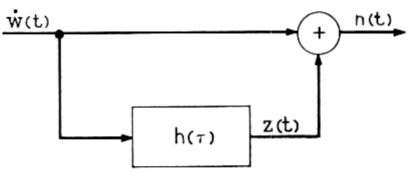


Fig. 2—Model of nonwhite noise.

*Proof:* We first observe that $\dot{w}(t) + z(t)$ is equivalent to noise with spectral density $N(\omega)$. A causal filter, $h(\tau)$, will exist whenever $N(\omega)$ represents the square magnitude of a causal filter

$$| G(\omega) |^2 = N(\omega)$$

$$H(\omega) = G(\omega) - 1.$$

The logarithm of $G(\omega)$ is

$$\tfrac{1}{2} \ln N(\omega) + iB(\omega)$$

where $B(\omega)$ is the phase characteristic of $G(\omega)$. The conditions of causality, no lower half plane poles, will be met when $B(\omega)$ is one half the Hilbert transform of $\ln N(\omega)$. The conditions on $N(\omega)$ insure that $\ln N(\omega)$ has a Hilbert transform.

Now to prove formula (1) we use a theorem due to Kadota, Zaki and Ziv[2], which we state without proof:

*Theorem A: The mutual information between the input parameter m and the output processes $Y_T$ of a finite power system disturbed by additive white gaussian noise is*

$$I(m; Y_T) = \tfrac{1}{2}E \int_0^T \phi^2(t, m, Y_t)\, dt - \tfrac{1}{2}E \int_0^T E^2[\phi(t, m, Y_t)/Y_t]\, dt,$$

*where $\phi(t, m, Y_t)$ is the causal modulating function.*

This result is applied to the non-white noise problem by considering $z(t)$ to be part of the signal. The inclusion is only useful when one is calculating the mutual information; it is not to be included in the calculation of transmitter power. Theorem A cannot be applied directly since the signal, $\phi$, which is taken as $s(t) + z(t)$ is not completely determined by $m$ and $Y_t$, but is also a function of the process $v(t)$. To find $I(m; Y_T)$ we use the decomposition,

$$I(m, V; Y_T) = I(m; Y_T) + I(V; Y_T \mid m), \tag{3}$$

where $V$ is the process $v(\tau)$.

From Theorem A we have,

$$I(m, V; Y_T) = \tfrac{1}{2}E \int_0^T [s(t) + z(t)]^2\, dt - \tfrac{1}{2}E \int_0^T E^2[s(t) + z(t) \mid Y_t]\, dt \tag{4}$$

and

$$I(V; Y_T \mid m) = \tfrac{1}{2}E \int_0^T [s(t) + z(t)]^2 \, dt - \tfrac{1}{2}E \int_0^T E^2[s(t) + z(t) \mid Y_t , m] \, dt,$$

which together with equation (3) proves Theorem 1. $s(t) + z(t)$ has finite energy because $s(t)$ must have finite energy and $z(t)$ will have finite energy whenever the channel has finite capacity without feedback, as we shall see when we evaluate $E[z^2(t)]$. With this basic result we can derive several interesting corollaries concerning the information.

*Corollary 1:* *(Pinsker)* Under the conditions of Theorem 1,*

$$\frac{I(m; Y_T)}{T} \leqq 2C$$

*where C is the capacity at the channel without feedback.*

First we observe by equation (3) that

$$I(m; Y_T) \leqq I(m, V; Y_T)$$

which is given by equation (4). Furthermore the second term in equation (4) is negative and can be ignored, thus

$$I(m; Y_T) \leqq \tfrac{1}{2}E \int_0^T (s + z)^2 \, dt. \tag{5}$$

$I(m; Y_T)$ can be further bounded by

$$I(m; Y_T) \leqq E \int_0^T s^2 \, dt + E \int_0^T z^2 \, dt \tag{6}$$

since $(s + z)^2 \leqq 2s^2 + 2z^2$.

The next step is to calculate the variance of $z$, since this enters directly into $I(m; Y_T)$.

$$E \int_0^T z^2(t) \, dt = TE(z^2),$$

$$E(z^2) = \frac{1}{2\pi} \int_{\infty}^{\infty} | H(\omega) |^2 \, d\omega = \frac{1}{2\pi} \int_{\infty}^{\infty} | G(\omega) - 1 |^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \exp\left[\frac{1}{2} \ln N(\omega) + \frac{i}{2} \widehat{\ln N(\omega)}\right]^{\dagger} - 1 \right|^2 d\omega$$

* The factor of 2 has been mentioned earlier by Pinsker but no proof has yet been published.

† Indicates the Hilbert transform.

$$= \frac{-1}{2\pi} \int_{-\infty}^{\infty} [1 - N(\omega)]\, d\omega$$

$$- \frac{\text{Re}}{\pi} \int_{-\infty}^{\infty} \left\{ \exp \left[ \frac{1}{2} \ln N(\omega) + \frac{i}{2} \widecheck{\ln N(\omega)} \right] - 1 \right\} d\omega.$$

This latter integral, as chance would have it, is almost identical in structure to an integral which arises in evaluating the spectral density of a single sideband FM wave (at the carrier frequency) which is modulated by a gaussian signal. The quantity $1/2 \ln N(\omega)$ here plays the role of the autocorrelation function of the gaussian signal, and although for our problem $1/2 \ln N(\omega)$ is not in general an autocorrelation function, the integral may be discussed *via* the technique used in the FM problem (see Mazo and Salz)[3].

Define:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2} \ln N(\omega) + \frac{i}{2} \widecheck{\ln N(\omega)} \right] e^{i\omega t} = f(t)$$

then

$$\frac{d}{d\omega} [G(\omega) - 1] = G(\omega) \frac{d}{d\omega} F(\omega) = [G(\omega) - 1] \frac{d}{d\omega} F(\omega) + \frac{d}{d\omega} F(\omega).$$

In the time domain this becomes

$$-ith(t) = -itf(t) - i \int_0^t \tau f(\tau) h(t - \tau)\, d\tau$$

because both $h(\tau)$ and $f(\tau)$ are zero for negative $\tau$. Both $f(\tau)$ and $h(\tau)$ are finite for small $\tau$ and thus

$$h(\tau = 0) = f(\tau = 0).$$

The integral we are interested in is $2 \text{ Re } h(\tau = 0)$ which is equal to

$$2 \text{ Re } f(\tau = 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln N(\omega)\, d\omega.$$

Thus far we have shown that

$$E \int_0^T s^2\, dt + E \int_0^T z^2\, dt$$

$$= E \int_0^T s^2\, dt - \frac{T}{2\pi} \int_{-\infty}^{\infty} [1 - N(\omega)]\, d\omega - \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln N(\omega)\, d\omega. \qquad (7)$$

One more trick is needed to prove the corollary. We have, up to this point, considered only normalized channels which had $N(\infty) = 1$.

This is valid because normalization cannot affect the ratio between capacity without feedback to that with feedback. Some channels cannot be normalized in this manner, i.e., $N(\infty) = \infty$ or $N(\infty) = 0$. The latter case has infinite capacity and thus the corollary applies. The former presents no problems due to the following lemma.

*Lemma: Consider the channel without feedback. By the water pouring argument*[4] *we know that the signal energy which achieves capacity obeys:*

$$S(\omega) = \begin{cases} K - N(\omega), & N(\omega) \leqq K; \\ 0, & \text{otherwise}. \end{cases}$$

*If we define a new noise* $N^0(\omega)$

$$N^0(\omega) = \begin{cases} N(\omega), & N(\omega) \leqq K; \\ K, & N(\omega) > K. \end{cases}$$

This new channel has the same capacity without feedback and a larger capacity with feedback.

*Proof:* The expression for capacity without feedback is the same for $N(\omega)$ and $N^0(\omega)$. The capacity with feedback can only be increased since $N^0(\omega) \leqq N(\omega)$ for all $\omega$. For if the capacity with $N(\omega)$ were larger, one could add a noise with spectrum $N(\omega) - N^0(\omega)$ at the receiver and do just as well as if the noise were $N(\omega)$.

We now normalize the noise, $N^0(\omega)$, in order to apply equation (6), which makes $K = 1$. The capacity without feedback is:

$$C = \frac{1}{4\pi} \int_{-\infty}^{\infty} \ln \frac{1}{N^0(\omega)}\, d\omega,$$
$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - N^0(\omega)]\, d\omega. \tag{8}$$

With feedback from equations (6), (7) and (8)

$$I(m;\, Y_T) \leqq E \int_0^T s^2\, dt - TP + 2TC$$

or

$$\frac{I(m;\, Y_T)}{T} \leqq 2C.$$

A tighter bound can be obtained by returning to equation (5) and writing:

$$I(m;\, Y_T) \leqq \frac{1}{2}\left[E \int_0^t s^2\, dt + E \int_0^t z^2\, dt\right] + E \int_0^t sz\, dt,$$

which by the preceding argument is equal to

$$C + E \int_0^t sn^0 \, dt.$$

The correlation $Esz^0$ is equal to $Esn^0$ because $n^0$ and $z^0$ only differ by a white component. Thus the capacity can be increased only by the correlation of the signal with the noise. The noise $n^0$ is not the original noise, however the difference occurs only at frequencies not used for signaling without feedback. As $N(w)$ becomes white, the energy in $z^0$ decreases and consequently $Esz^0$ must go to zero.

More insight into the problem is supplied by the following theorem.

*Theorem 2: Capacity can be attained with a gaussian signal* $s(t)$.

*Proof:* First we observe that

$$E[s(t) + z(t) \mid m, Y_t] = s(t, m, Y_t) + E[z(t) \mid W_t].$$

This is true because $s(t)$ is known given $m$ and $Y_t$, and $z(t)$ is dependent on $W_t$ which can be calculated given $Y_t$ and $s(t)$. $E[z(t) \mid W_t]$ is a linear functional of $\dot{w}$ because $\dot{w}$ is gaussian.

$$E[z(t) \mid W_t] = \int_0^t K(t, \tau) \, d\omega(\tau).$$

The first term in equation (1) depends only on the correlation properties of $s(t, m, Y_t)$ and $w(\tau)$ and therefore we can use a gaussian $s$ of the appropriate correlation. For the second term we use the property that a least-squares linear estimate has no more energy than the more general least square estimate.

$$Ex^2 = E\hat{x}^2 + E(x - \hat{x})^2 = E\bar{\hat{x}}^2 + E(x - \bar{\hat{x}})^2$$

where $\bar{\hat{x}}$ is the least-square linear estimate of $x$ and $\hat{x}$ is the least-square estimate. Since

$$E(x - \hat{x})^2 \leqq E(x - \bar{\hat{x}})^2,$$

$$E\hat{x}^2 \geqq E\bar{\hat{x}}^2.$$

Therefore, since $E[s(t) + z(t) \mid Y_t]$ is the least-squares estimate of $s(t) + z(t)$ given $Y_t$ we have

$$I(m; Y_T) \leqq \frac{1}{2} E \int_0^T E^2[s + z \mid m, Y_t] \, dt - \frac{1}{2} E \int_0^T \widehat{\widehat{(s + z)}}^2 \, dt$$

but for a gaussian signal this inequality is an equality. In addition the signal power is unchanged and the feedback processor need only be linear. Therefore one need consider only gaussian input and linear processing in calculating capacity.

## II. GENERALITY OF THE MODEL

The restrictions on $N(\omega)$ are in fact only needed for $N^0(\omega)$. If a noise spectrum is such that the logarithmic integral of $N^0(\omega)$ is minus infinity then the capacity of the channel is infinite without feedback. Therefore the bound applies to any channel which has a finite capacity without feedback.

The bounds are all valid for noisy feedback as well, however it is not clear that gaussian signals are optimum in that case.

## III. ACKNOWLEDGMENT

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## REFERENCES

1. Shannon, C. E., "The Zero Error Capacity of a Noisy Channel," IRE Trans. Inform. Theory, *IT-2*, No. 3 (September 1956), pp. 8–19.
2. Kadota, T. T., Zakai, M., and Ziv, J., "On the Capacity of Continuous Memoryless Channels with Feedback," to be published in the IEEE Trans. Inform. Theory.
3. Mazo, J. E. and Salz, J., "Spectral Properties of Single-Sideband Angle Modulation," IEEE Trans. on Comm. Tech., *Com-16*, No. 1 (February 1968), pp. 52–62.
4. Shannon, C. E., "A Mathematical Theory of Communication," B.S.T.J., *27*, No. 4 (October 1948), pp. 623–656.