New Theorems on the Equations of Nonlinear DC Transistor Networks

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It has long been recognized that equations describing dc transistor networks do not necessarily have unique solutions. The Eccles-Jordan (flip-flop) circuit is an excellent example of one for which the dc equations may have more than one solution.

Only recently, however, has a comprehensive theory concerning matters such as the existence and uniqueness of solutions of the dc equations of general transistor networks begun to take shape. This paper represents another contribution to the evolution of that theory.

A key concept in the development of the recent theory is the concept of a "P_0 matrix." We give a generalization of that concept, showing that one can specify properties possessed by certain pairs of square matrices, analogous to the properties possessed by a single P_0 matrix. Pairs of matrices possessing these properties are called $\mathcal{W}_0$ pairs. Use is made of this $\mathcal{W}_0$ pair concept to prove results which are more general than some of the existing ones. We provide an extension of much of the existing theory in such a manner that a broader class of dc transistor networks may be considered. In particular, the new results provide one with the ability to answer certain questions concerning the existence, uniqueness, boundedness, and so on, of solutions of the equations for any *network which is comprised of transistors, diodes, resistors, and independent sources.*

I. INTRODUCTION

Suppose a network is constructed by connecting in an arbitrary manner any number of transistors, diodes, resistors, and independent voltage and current sources. Without loss of generality, we may consider the network to have the canonical form shown in Fig. 1; that is, we may consider the network to be a multiport containing resistors and inde-

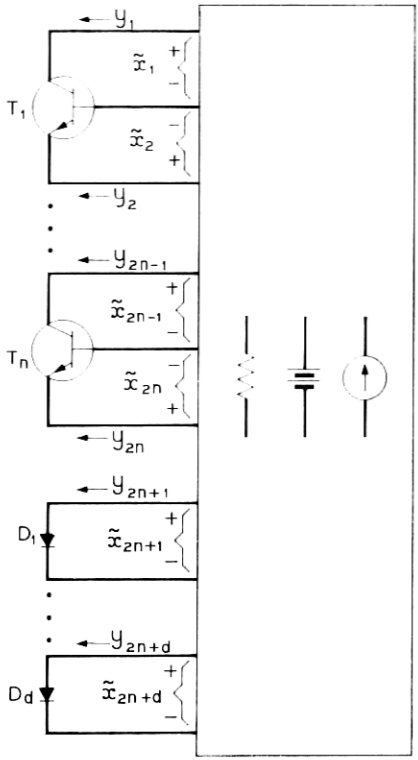

Fig. 1—Canonical form of a transistor network.

pendent sources, with transistors and diodes connected to the ports.*

There are some fundamental questions that one should then, hopefully, be able to answer. For example: Do the equations that describe this dc network have a unique solution? With the exception of certain uniqueness results for a special (but none the less important) class of transistor networks, all of the previous explicit results in Refs. 1, 2, and 3, which have shown methods for obtaining answers to such questions, have been concerned only with the class of transistor networks for which, after setting the value of each independent source to zero, there exists a short-circuit admittance matrix (a G matrix) to characterize the linear

* It will become apparent that the theory can also accommodate many other structures which are of the Fig. 1 type except that the multiport contains additional linear elements (such as controlled sources). We do not stress this point though, since in the present context such elements seem somewhat unnatural.

multiport of Fig. 1. It is the primary purpose of this paper to show how that restriction can be removed. We shall show in fact that almost all of the previous results are but special cases of results that follow from a more general theory in which the assumption of the existence of a G matrix for the linear multiport is unnecessary.*

Section II concerns methods for characterizing a general multiport containing resistors and independent sources. In Section III, we consider the model for a transistor. An equation for dc transistor networks is then developed in Section IV and, after explaining some notation in Section V, we develop the $\mathcal{W}_0$ pair concept in Section VI. Sections VII, VIII and IX show how the $\mathcal{W}_0$ pair concept provides a generalization of the existing results concerning dc transistor networks. Finally, we consider an example network in Section X.

II. LINEAR MULTIPORT CHARACTERIZATION

A multiport having n ports (an n-port) is *characterized* by determining every combination of the $2n$ port voltages and currents that the network admits (see Ref. 4). We discuss here two methods of characterizing multiports that contain resistors and independent sources. The first method makes use of the familiar concept of a hybrid matrix. The second method uses a pair of matrices in a manner that was apparently first suggested—for multiports containing no independent sources—by V. Belevitch.[5]

2.1 *The Hybrid Formalism*

When the value of each independent source is set to zero, for a multiport containing only resistors and independent sources, the multiport becomes, of course, a *resistive* multiport. H. C. So has proved (as a special case of a theorem in Ref. 6) that *any resistive multiport has a hybrid matrix description.* That is, for any resistive n-port, it is always possible to label the port voltage and current variables in such a way that there

* Pragmatists might argue that in any "physical" network, there will always be enough "stray" resistance present which, if taken into account, will guarantee the existence of, say, a G matrix. It seems to this writer, however, that by taking such a point of view, one does not obtain an entirely satisfactory understanding of matters (even *practical* matters). To know that fundamental results do not *depend* (if, in fact, they don't) upon such fortunate occurrences as these (and for many transistor networks this is the case) seems to be the more satisfactory situation. Furthermore, it should be noted that in the analysis of a physical network, to obtain a tractable problem, it often behooves one to neglect the presence of unimportant elements. Thus, it is not necessarily true that such stray resistors will always be present in the model of the network which the analyst desires to consider.

exists an integer m, $0 \leqq m \leqq n$, a pair of n-vectors*

$$x = (i_1, \cdots, i_m, v_{m+1}, \cdots, v_n)^T,$$

$$y = (v_1, \cdots, v_m, i_{m+1}, \cdots, i_n)^T,$$

and a real $n \times n$ matrix H, the hybrid matrix, such that the network admits the port variables v_k, i_k as the voltage and current, respectively, at the kth port, for $k = 1, \cdots, n$, if and only if the vectors x and y satisfy

$$y = Hx. \tag{1}$$

Thus, a resistive multiport may always be characterized by a hybrid matrix.

When independent sources whose values are nonzero are present in an otherwise resistive multiport, a hybrid matrix will not generally suffice to characterize the multiport. Clearly the vectors $x = y = (0, 0, \cdots, 0)^T$ which satisfy equation (1) for any matrix H do not always specify an admissible combination of port variables when independent sources are present. One might hope, however, that a characterization of the type

$$y = Hx + c, \tag{2}$$

where c is some constant vector (whose elements are real numbers), might always be possible. Indeed, we are about to show that this is the case. There is one problem, however, that was not present in the consideration of resistive n-ports that must first be dealt with: there are ways to interconnect independent sources and resistors such that the resulting structure doesn't make sense. That is, the independent sources might impose self-contradictory constraints on the network. We rule out such possibilities by agreeing that, when we refer to "a multiport containing resistors and independent sources," we always assume that the multiport possesses the following property:

Assumption: The linear graph that is formed by associating an edge with each resistor, each independent source, and each port, has no cut-sets containing only current source edges for which the values of the current sources cause a violation of Kirchhoff's current law. Similarly, no circuits of voltage source edges for which the values of the voltage sources cause a violation of Kirchhoff's voltage law are present.

This assumption in no way restricts the generality of our work. We

* We use the superscript T to denote the transpose of a vector or a matrix. Thus, the vectors x and y above are both column vectors.

are simply ruling out multiports, like the 2-port of Fig. 2, for which the set of admissible port voltage and current combinations is empty.

We have worded the Assumption so that the presence of, say, a series connection of two 1-ampere current sources in an otherwise resistive multiport does not cause the multiport to be inadmissible. We have done this because no violation of Kirchhoff's laws results from such interconnections of resistors and sources; the network is perfectly legitimate. One should be aware, however, that if "superfluous" sources are present in a network, it will follow that one cannot uniquely determine the value of each branch voltage and current in the network. That is, even though one might be able to uniquely determine the value of the voltage across the *pair* of 1-ampere sources, there is no way to determine the value of the voltage across each individual source. Aside from such ambiguities, it follows (see below and the proof of Theorem 1 in Ref. 6) that the value of all branch voltages and currents can be uniquely determined for a multiport satisfying the Assumption, whenever the values of the "independent" port variables are known.

Theorem: *Any multiport containing resistors and independent sources can be characterized by equation* (2), *where H is a hybrid matrix characterization of the corresponding resistive multiport that is obtained by setting all independent source values to zero, and c is a vector of real numbers.*

A proof of this theorem can be constructed by incorporating a few simple observations and minor modifications into the arguments used by So in Ref. 6. We therefore simply sketch the main ideas: First, if the linear graph mentioned in the Assumption contains any current source cut-sets, then it must be the case (because of that Assumption) that these sources have values such that Kirchhoff's current law is satisfied. That being the case, the port behavior of the multiport will clearly be unaltered if a sufficient number of current sources are removed (by coalescing appropriate nodes) to eliminate such cut-sets. A similar observation applies to voltage source circuits. Therefore without any loss of generality, we may consider the linear graph to have no current source cut-sets and no voltage source circuits. Next, by Lemmas 1 and 2 of Ref. 6, it then follows that there exists a tree* for the linear graph for which all voltage source edges are branches and all current source edges are links. At each port, one of the two port variables is then designated as "independent," the choice depending upon whether the edge corresponding to that port is a branch or a link. The existence of the

* In case the linear graph is not *connected* each reference to the word *tree* should, of course, be changed to *forest*.

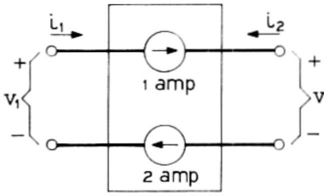

Fig. 2—An inadmissible n-port.

hybrid matrix H and the vector c for the characterization (2) then follows in the same manner as the existence of a hybrid matrix for a resistive multiport follows from So's arguments.

2.2 *Belevitch's Formalism*

For some multiports, it might be that (after setting all independent source values to zero) a hybrid matrix exists such that the vectors x and y in equation (1) satisfy $x = v \equiv (v_1, \cdots, v_n)^T$ and $y = i \equiv (i_1, \cdots, i_n)^T$. In this case the hybrid matrix is given the special name, *admittance matrix*. Similarly, if it happens that H exists such that $x = i$ and $y = v$, then H is called the *impedance matrix*. For many resistive multiports, neither an impedance matrix nor an admittance matrix exists. It is still possible, however, to characterize any n-port for which a hybrid matrix exists in terms of the vectors v and i. Obviously, x and y satisfy equation (1) if and only if v and i satisfy

$$[I_l \,\vdots\, -H_r]v = [H_l \,\vdots\, -I_r]i, \tag{3}$$

where the $n \times m$ matrix H_l and the $n \times (n - m)$ matrix H_r are defined by $H = [H_l \,\vdots\, H_r]$, and similarly $[I_l \,\vdots\, I_r]$ is the $n \times n$ identity matrix.

The characterization (3), being equivalent to equation (1), is perfectly adequate for any resistive n-port. It is, however, but a special case of a more general characterization due to Belevitch, namely:

$$Pv = Qi, \tag{4}$$

where P and Q are $n \times n$ real matrices. Belevitch's characterization can be used for quite a broad class of networks, including some rather pathological ones which require dependent sources, or gyrators and negative resistors to realize, and for which no hybrid characterization exists. For example, the one-port called a *norator*, for which the set of admissible port voltage and current combinations is the set of all pairs of real numbers, may be characterized by $[0]v = [0]i$. We should note, however, that if one allows the aforementioned elements to be present in an n-port, then even equation (4) cannot always provide a characterization. The *nullator*, for example, a one-port whose only admissible combination of port voltage and current variables is the pair (0, 0), is such an n-port.

When an n-port contains independent sources it can often be characterized by the equation

$$Pv = Qi + c, \tag{5}$$

where P and Q are real $n \times n$ matrices, and c is a constant vector. Clearly, any n-port containing only resistors and independent sources has such a characterization. It is this class of n-ports which is our primary concern in the study of transistor networks. We note, however, that equation (5) is adequate for characterizing a much broader class of n-ports.

III. NONLINEAR TRANSISTOR CHARACTERIZATION

In Fig. 3, a commonly used large signal dc transistor model is displayed. It is easily verified that the voltage and current variables defined in that figure obey the following relationships:

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_r \\ -\alpha_f & 1 \end{bmatrix} \begin{bmatrix} f_1(v_1) \\ f_2(v_2) \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} - \begin{bmatrix} r_e + r_b & r_b \\ r_b & r_c + r_b \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}. \tag{7}$$

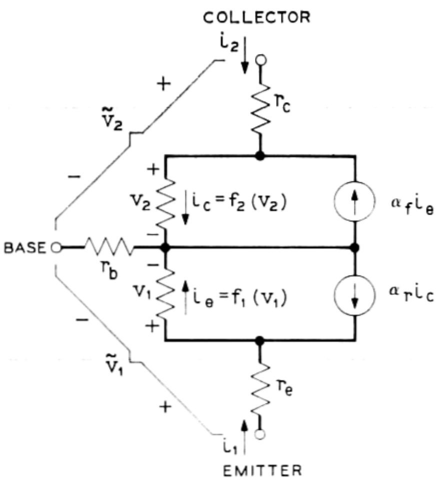

Fig. 3—Large signal dc transistor model.

Each of the parameters α_f and α_r may assume any value in the open interval (0, 1). The parameters r_b , r_c , and r_e , which account for lead resistances, are sometimes omitted by device modelers (their presence is sometimes accounted for by including appropriate additional resistors in the network to which the transistor model is connected). To accommodate these various points of view we specify only, therefore, that the values of the parameters r_b , r_c , and r_e be nonnegative. Thus any or all of them may be zero.

Depending upon whether the transistor being modeled is a pnp or an npn, the graph of each of the functions f_1 and f_2 has one of the general shapes shown in Fig. 4 (at least for values of $|v|$ that are "not too large"). Often these functions are described by an equation of the form

$$f_k(v) = m_k[\exp(n_k v) - 1], \qquad (k = 1, 2), \tag{8}$$

where m_k and n_k are appropriately chosen constants, both being positive for a pnp transistor, and both negative for an npn. On the other hand, for example, a piecewise-linear representation is sometimes specified for f_1 and f_2 .

The nature of the functions f_1 and f_2 for large values of $|v|$ depends upon which assumptions the modeler is willing to make, and which effects he is interested in considering. For large negative (in the pnp case) values of v, for example, the graph of f_k approaches—according to equation (8)—the horizontal asymptote $i = -m_k$. Thus, if the modeler chooses to use equation (8) to describe f_k for all values of v, the range of f_k will not be the entire real line. If, on the other hand, the effect of ohmic surface leakage across the p-n junction is included in the model, the graph of the function f_k will approach asymptotically a straight line having a small, but positive, slope. The range of such a function is, obviously, the whole real line. One might also wish to include the effect of avalanche breakdown in the reverse-biased region. If this is done, the graph of f_k will have a shape reminiscent of that of a Zener diode in the $v < 0$ part of its domain.

In the forward-biased region there are also effects, particularly apparent for large values of v, which the modeler may or may not wish to recognize. For example, there is the so-called high-level injection phenomenon which tends to decrease the value of the forward current and which, using equation (8), is usually accounted for by a decrease in the magnitude of n_k for large values of v. In addition, there is the effect of the ohmic resistance of the crystal which tends to reduce the value of forward current for large values of v.

From the point of view of the device modeler, the question of whether or not to include some of the effects mentioned above is often a minor issue. For many networks the behavior will be essentially the same whether or not, say, surface leakage is accounted for in the transistor model. From the point of view of the network analyst, however, the situation is somewhat different. For example, the matter of whether or not the functions f_k map the real line *onto* the real line can, in some cases, make the difference between whether or not there exists a solution of the network's equations. Similarly, other results that have been obtained recently (presented later, beginning in Section VII) also seem to depend upon the graphs of the functions f_k having certain special properties.

It seems safe to say that no matter which "special effects" are included (or omitted) in the description of the transistor, the functions f_k may at least be considered to be strictly monotone increasing mappings of the real line into itself. For the purpose of formulating the equations for transistor networks, this is the only hypothesis that we shall make. When additional hypotheses regarding the nature of these functions are needed (to obtain certain results concerning properties of these equations) those hypotheses will be mentioned explicitly. In each case it will be clear that the additional hypotheses are, in some appropriate sense, rather weak.

Similar remarks can be made for the diodes that are shown in Fig. 1, which might also be present in transistor networks. Thus, we assume that each diode is described by an equation of the type $i = f(v)$ where, at this point, we only assume that the function f is a strictly monotone increasing mapping of the real line into itself.

IV. EQUATIONS FOR TRANSISTOR NETWORKS

Suppose we are given a dc network consisting of transistors, diodes, resistors, and independent voltage and current sources, connected together in an arbitrary manner. Let there be n transistors and d diodes. Clearly, there is no loss of generality if we consider the network to be of the type shown in Fig. 1. Using the results of Section III, we may describe the nonlinear devices in the network by the equations

$$y = TF(x), \qquad x = \tilde{x} - Ry, \tag{9}$$

where $T = \text{diag}[T_1, T_2]$, with T_1 a block diagonal matrix with n 2×2 diagonal blocks of the form

$$\begin{bmatrix} 1 & -\alpha_r^{(k)} \\ -\alpha_f^{(k)} & 1 \end{bmatrix}, \quad \text{for} \quad k = 1, \cdots, n, \tag{10}$$

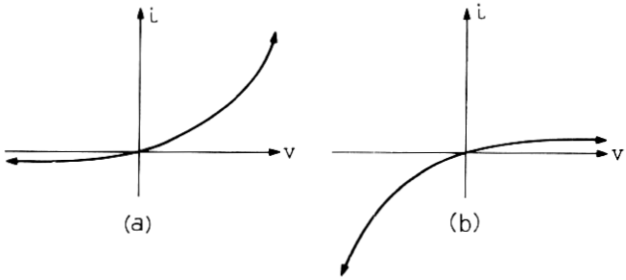

Fig. 4—General shape of the functions f_1 and f_2; (a) pnp transistor, (b) npn transistor.

and T_2 the $d \times d$ identity matrix. Also, $R = \text{diag}\,[R_1\,,\, R_2]$, with R_1 a block diagonal matrix with n 2×2 diagonal blocks of the form

$$\begin{bmatrix} r_e^{(k)} + r_b^{(k)} & r_b^{(k)} \\ r_b^{(k)} & r_c^{(k)} + r_b^{(k)} \end{bmatrix}, \qquad k = 1, \cdots, n, \tag{11}$$

and R_2 the $d \times d$ matrix whose elements are all zeros. The function F has the form $F(x) \equiv [f_1(x_1), \cdots, f_{2n+d}(x_{2n+d})]^T$, where each of the f_k is a strictly monotone increasing mapping of the real line into itself.

Using the results of Section II, the effect of the linear multiport in Fig. 1 is to constrain the vectors of port variables, $\tilde{x}$ and y, to obey the relationship

$$P\tilde{x} = -Qy + c, \tag{12}$$

where P and Q are $(2n + d) \times (2n + d)$ real matrices and c is a real $(2n + d)$-vector. The minus sign appears in equation (12) as a consequence of having chosen the reference direction for the port currents (the elements of the vector y) to be opposite to that which is usually assumed.

By using equations (9), we may easily eliminate the variables $\tilde{x}$ and y from equation (12), resulting in the equation

$$(PR + Q)TF(x) + Px = c. \tag{13}$$

The central problem in determining the values of all branch voltages and currents in a dc transistor network is the determination of a solution of equation (13). The rest is relatively straightforward, for if x is a (unique) solution of equation (13), then the (unique) vectors $\tilde{x}$ and y, such that equations (9) and (12) are satisfied, may immediately be computed from equations (9).

Since the matrix T is nonsingular, it follows that whenever either $(PR + Q)$ or P is nonsingular, equation (13) can be transformed into, respectively, one of the equations

$$F(x) + Ax = b, \tag{14}$$

$$AF(x) + x = b. \tag{15}$$

The first of these equations has been studied rather extensively (see Refs. 1–3 and 7) and for most of the results obtained there, it can be shown that parallel results are possible for equation (15). Both of these equations, however, are but special cases of the equation

$$AF(x) + Bx = c, \tag{16}$$

which accommodates equation (13) directly. It is, therefore, this equation to which we shall now direct our attention. It will be shown that most of the results which have been obtained to date for equation (14) have rather natural (though not obvious) extensions to equation (16). It is important that such extensions be possible because one is often forced to deal with equations like (16) in the analysis of transistor networks. Clearly, this is the case whenever both of the matrices $(PR + Q)$ and P of equation (13) are singular—and this can easily happen (for example, if the matrix R contains all zeros, then it will happen whenever there exists no admittance matrix nor impedance matrix for the linear multiport of Fig. 1).

V. NOTATION

The following notation shall be used throughout the remainder of the paper: For each positive integer n we denote by E^n the n-dimensional Euclidean space, the elements of which are ordered n-tuples of real numbers, which we consider to be column vectors. The origin in E^n is denoted by θ. If $x = (x_1, \cdots, x_n)^T$ and $y = (y_1, \cdots, y_n)^T$ are elements of E^n we denote their inner product by $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$. The norm of each $x \in E^n$ is denoted by $||x|| = \langle x, x \rangle^{\frac{1}{2}}$.

If A is an $n \times n$ matrix, then for $k = 1, \cdots, n$, A_k denotes the kth column of A. A principal submatrix of a square matrix A is any square submatrix of A whose main diagonal is contained in the main diagonal of A. A principal minor of A is the determinant of any principal submatrix of A. If D is a diagonal matrix, then $D > 0$ means that each element of the main diagonal is a positive number; similarly, $D \geqq 0$ denotes that each element of the main diagonal is nonnegative. We denote the $n \times n$ identity matrix by either I_n or, when the dimension is unimportant or is clear from the context, simply by I. The direct sum of two matrices A, B is denoted by $A \oplus B$. A square matrix of real numbers A is said to be strongly row-sum dominant if its elements a_{ij} satisfy $a_{ii} > \sum_{j \neq i} |a_{ij}|$ for $i = 1, \cdots, n$.

If f is a real valued function defined on E^1 then f is said to be monotone increasing if for all $x < y$ it follows that $f(x) \leqq f(y)$. We say that f is strictly monotone increasing if $f(x) < f(y)$ for all $x < y$. For each positive integer n, we denote by $\mathfrak{F}^n$ that collection of mappings of E^n onto itself defined by: $F \in \mathfrak{F}^n$ if and only if there exist, for $i = 1, \cdots, n$, strictly monotone increasing functions f_i mapping E^1 onto E^1 such that for each $x = (x_1, \cdots, x_n)^T \in E^n$, $F(x) = [f_1(x_1), \cdots, f_n(x_n)]^T$.

VI. PAIRS OF MATRICES OF TYPE $\mathcal{W}_0$

Many of the recent results referred to above, concerning equation (14), have relied heavily upon certain properties that a matrix is known to possess whenever it is a member of a class of matrices that has been given the name P_0. In a similar way the results that follow rely upon useful properties that are possessed by certain *pairs of matrices*. We shall define a class, the elements of which are these pairs of matrices, and give it the name $\mathcal{W}_0$.

The class of matrices called P_0 was defined by M. Fiedler and V. Pták.[8] They proved that the following properties of a square matrix of real numbers, A, are equivalent:

(*i*) All principal minors of A are nonnegative.

(*ii*) For each vector $x \neq \theta$ there exists an index k such that $x_k \neq 0$ and $x_k(Ax)_k \geqq 0$.

(*iii*) For each vector $x \neq \theta$ there exists a diagonal matrix $D_x \geqq 0$ such that $\langle x, D_x x\rangle > 0$ and $\langle Ax, D_x x\rangle \geqq 0$.

(*iv*) Every real eigenvalue of A, as well as of each principal submatrix of A, is nonnegative.

Sandberg and Willson proved that another property can be added to this list of equivalent properties,[2,3] namely:

(*v*) $\det (D + A) \neq 0$ for every diagonal matrix $D > 0$.

The class of all matrices possessing one (and hence all) of the above properties is called P_0.

We shall now state a theorem which provides a useful generalization of the concept of the class of P_0 matrices.

Definition: For each pair of $n \times n$ matrices (A, B) we shall denote by $\mathcal{C}(A, B)$ the collection of all the $n \times n$ matrices that can be constructed by juxtaposing columns taken from either A or B while maintaining the original relative ordering of the columns. Thus, $M \in \mathcal{C}(A, B)$ if and only if for each $k = 1, \cdots, n$, either $M_k = A_k$ or $M_k = B_k$.

Obviously $\mathcal{C}(A, B)$ contains 2^n matrices (for certain pairs (A, B)—namely for those having $A_k = B_k$ for one or more values of k—it can happen that two or more matrices in $\mathcal{C}(A, B)$ are identical).

Definition: The pair of $n \times n$ matrices (M, N) is said to be a complementary pair taken from $\mathcal{C}(A, B)$ if and only if both M and N are members of $\mathcal{C}(A, B)$ and for each $k = 1, \cdots, n$, either $M_k = A_k$ and $N_k = B_k$, or else $M_k = B_k$ and $N_k = A_k$.

It is obvious that (A, B) is a complementary pair taken from $\mathcal{C}(A, B)$. It is also clear that $\mathcal{C}(A, B) = \mathcal{C}(B, A)$ and, moreover, that if (M, N) is any complementary pair taken from $\mathcal{C}(A, B)$, then $\mathcal{C}(M, N) = \mathcal{C}(A, B)$. Furthermore, for each $M \in \mathcal{C}(A, B)$ there exists $N \in \mathcal{C}(A, B)$ such that (M, N) is a complementary pair.

Theorem 1: *The following properties of a pair of $n \times n$ matrices of real numbers (A, B) are equivalent*:

(*i*) *det* $(AD + B) \neq 0$ *for every diagonal matrix* $D > 0$.

(*ii*) *There exists a matrix* $M \in \mathcal{C}(A, B)$ *such that det* $M \neq 0$ *and such that det* $M \cdot$ *det* $N \geqq 0$ *for all* $N \in \mathcal{C}(A, B)$.

(*iii*) *For each vector* $x \neq \theta$ *there exists an index k such that either* $(A^Tx)_k \neq 0$ *or* $(B^Tx)_k \neq 0$, *and such that* $(A^Tx)_k(B^Tx)_k \geqq 0$.

(*iv*) *For each vector* $x \neq \theta$ *there exists a diagonal matrix* $D_x \geqq 0$ *such that either* $\langle A^Tx, D_xA^Tx\rangle > 0$ *or* $\langle B^Tx, D_xB^Tx\rangle > 0$ (*that is, such that* $\langle A^Tx, D_xA^Tx\rangle + \langle B^Tx, D_xB^Tx\rangle > 0$), *and such that* $\langle A^Tx, D_xB^Tx\rangle \geqq 0$.

(*v*) *For each complementary pair of matrices* (M, N) *taken from* $\mathcal{C}(A, B)$, *each real value of* λ *that satisfies det*$(M - \lambda N) = 0$ *is nonnegative*.

(*vi*) *There exists a complementary pair of matrices* (M, N) *taken from* $\mathcal{C}(A, B)$ *such that* $M^{-1}N \in P_0$.

(*vii*) *There exists a matrix* $M \in \mathcal{C}(A, B)$ *such that det* $M \neq 0$; *and, for any complementary pair of matrices* (M, N) *taken from* $\mathcal{C}(A, B)$ *with det* $M \neq 0$, $M^{-1}N \in P_0$.

In this paper, we do not make use of properties (*iii*), (*iv*), or (*v*) of Theorem 1. The proof that the remaining four properties are equivalent is given in the Appendix. A complete proof of Theorem 1 is given elsewhere.[9]

Definition: The class of all pairs of matrices which possess one (and hence all) of the properties listed in Theorem 1 is called $\mathcal{W}_0$.

To see that properties (*i*) and (*ii*) of Theorem 1 are in fact generalizations of the previously mentioned properties (*v*) and (*i*), respectively, that define P_0 is a simple matter. It happens that for any $n \times n$ matrix B the pair $(I_n, B) \in \mathcal{W}_0$ if and only if $B \in P_0$. (This follows from property (*vii*) of Theorem 1.) With our attention restricted to pairs of matrices of the type (I_n, B), it is clear that property (*i*) of Theorem 1 is equivalent to property (*v*) which determines those matrices B that that are in P_0. Concerning property (*ii*) of Theorem 1, an arbitrary matrix $N \in \mathcal{C}(I_n, B)$ is either the matrix I_n or else, a matrix formed from B by replacing some of the columns of B by the corresponding columns of I_n. Consequently, $\det N = \det B_N$ where B_N is the principal submatrix of B that is formed by removing from B the columns that are not present in N and then removing the corresponding rows. Hence, since $\det I_n \neq 0$, we may take I_n to be the matrix M in property (*ii*) of Theorem 1, and observe that this property then becomes: $\det B_N \geqq 0$ for all $N \in \mathcal{C}(I_n, B)$. It is now clear that this property is equivalent to the property (*i*) that defines the class of P_0 matrices. (Note that there are exactly $2^n - 1$ principal minors for each $n \times n$ matrix, and that the set $\mathcal{C}(I_n, B) \setminus \{I_n\}$ contains exactly $2^n - 1$ members.)

VII. THEOREMS ON EXISTENCE AND UNIQUENESS

7.1 *First Existence and Uniqueness Theorem*

The following theorem, which is proved in Ref. 2, provides a necessary and sufficient condition for the existence of a unique solution of equation (14) for all F that are strictly monotone increasing "diagonal" mappings of E^n *onto* E^n and for all $b \in E^n$.

Theorem 2: *If A is an $n \times n$ matrix of real numbers, then there exists a unique solution of equation* (14) *for each $F \in \mathfrak{F}^n$ and for each $b \in E^n$ if and only if $A \in P_0$.*

Using this theorem along with the results of Section VI we can prove the following (more general) theorem.

Theorem 3: *If A and B are $n \times n$ matrices of real numbers, then there exists a unique solution of equation* (16) *for each $F \in \mathfrak{F}^n$ and each $c \in E^n$ if and only if $(A, B) \in \mathcal{W}_0$.*

Proof: (*if*) Let $(A, B) \in \mathcal{W}_0$. Then, by Theorem 1, there exists a complementary pair (M, N) taken from $\mathcal{C}(A, B)$ such that $M^{-1}N \in P_0$. For each $F \equiv [f_1(\cdot), \cdots, f_n(\cdot)]^T \in \mathfrak{F}^n$ let $G \equiv [g_1(\cdot), \cdots, g_n(\cdot)]^T$ denote the mapping (also in $\mathfrak{F}^n$) defined by

$$g_k(\cdot) = \begin{cases} f_k(\cdot) & \text{if} \quad M_k = A_k , \\ f_k^{-1}(\cdot) & \text{if} \quad M_k \neq A_k , \end{cases} \quad \text{for} \quad k = 1, \cdots, n.$$

Clearly, the vectors x and y satisfy

$$AF(x) + Bx = MG(y) + Ny$$

if they satisfy the relation

$$y_k = \begin{cases} x_k & \text{if} \quad M_k = A_k , \\ f_k(x_k) & \text{if} \quad M_k \neq A_k , \end{cases} \quad \text{for} \quad k = 1, \cdots, n, \tag{17}$$

and since this relation defines a homeomorphism of E^n onto itself, it follows that there exists a unique solution of equation (16) for each $c \in E^n$ if there exists a unique solution of the equation

$$MG(y) + Ny = c \tag{18}$$

for each $c \in E^n$. But, that this is so follows immediately from Theorem 2 and from the fact that $M^{-1}N \in P_0$.

(*only if*) Suppose $(A, B) \notin \mathcal{W}_0$. Then, by Theorem 1, there exists a diagonal matrix $D > 0$ such that $\det(AD + B) = 0$. Choosing $F(x) \equiv Dx$, we have $F \in \mathfrak{F}^n$, while equation (16) does not have, with this choice of F, a unique solution for all $c \in E^n$. □

There are corollaries to Theorem 2, given in Ref. 2, that also may be generalized in a similar manner. For example, the following result is a generalization of an important special case of Corollary 1 of Ref. 2; it shows that the condition $(A, B) \in \mathcal{W}_0$ is still sufficient to insure the uniqueness of a solution of equation (16) (if a solution exists) even when the mapping F is not *onto*.

Theorem 4: If $F(x) \equiv [f_1(x_1), \cdots, f_n(x_n)]^T$, *where each* f_k *is a strictly monotone increasing mapping of* E^1 *into* E^1, *and if* $(A, B) \in \mathcal{W}_0$, *then there exists at most one solution of equation* (16) *for each* $c \in E^n$.

Proof: Suppose that, for some $c \in E^n$, x^1 and x^2 are solutions of equation (16) with $x^1 - x^2 \neq \theta$. Then, $A[F(x^1) - F(x^2)] + B(x^1 - x^2) = \theta$. But then, since F is a strictly monotone increasing "diagonal" mapping, there exists a diagonal matrix $D > 0$ such that $F(x^1) - F(x^2) = D(x^1 - x^2)$, and hence $(AD + B)(x^1 - x^2) = \theta$. Since $x^1 - x^2 \neq \theta$ it follows that $\det(AD + B) = 0$, which implies that $(A, B) \notin \mathcal{W}_0$. □

7.2 *A Nonuniqueness Theorem*

From the proof of the "only if" part of Theorem 2 (given in Ref. 2) it follows that whenever $A \notin P_0$, there exists a mapping $F \in \mathfrak{F}^n$ and a vector $b \in E^n$ such that equation (14) has more than one solution. On the other hand, even if $A \notin P_0$, if the mapping $F \in \mathfrak{F}^n$ is "fixed," then it is easy to see that the nonuniqueness of solutions of equation (14) need not necessarily follow for any $b \in E^n$[take $F(x) \equiv x$ and $Ax \equiv -2x$, for example]. I. W. Sandberg has shown,[10] however, that if one assumes that the "fixed" mapping F has another special property, rather than assuming that $F \in \mathfrak{F}^n$, then the nonuniqueness of solutions of equation (14) follows, for some $b \in E^n$, whenever $A \notin P_0$. Moreover, he has shown that under these hypotheses and for any $\delta > 0$, there exists some $b \in E^n$ such that equation (14) has two solutions, x and y, which satisfy $||x - y|| = \delta$. The special property that F is assumed to have is given in the following definition (in words, the property is: that it be possible to draw a straight line having any given positive slope, and any given length, between some pair of points on the graph of each of the functions f_k).

Definition: For each positive integer n we denote by $\mathcal{E}^n$ that collection of mappings of E^n into itself defined by: $F \in \mathcal{E}^n$ if and only if there exist, for $k = 1, \cdots, n$, continuous functions f_k mapping E^1 into E^1 such that for each $x \in E^n$, $F(x) = [f_1(x_1), \cdots, f_n(x_n)]^T$, with each of the f_k satisfying, for all $\beta > 0$,

$$\inf \{f_k(\alpha + \beta) - f_k(\alpha): -\infty < \alpha < \infty\} = 0,$$

$$\sup \{f_k(\alpha + \beta) - f_k(\alpha): -\infty < \alpha < \infty\} = \infty.$$

By using Theorem 1 it is possible to prove the following generalization of Sandberg's result:

Theorem 5: Let $F \in \mathcal{E}^n$, let $(A, B) \notin \mathcal{W}_0$ be a pair of real $n \times n$ matrices, and let δ be a positive constant. Then, for some $c \in E^n$ there exist solutions of equation (16), *x and y, satisfying $||x - y|| = \delta$.*

Proof: Since $(A, B) \notin \mathcal{W}_0$ there exists a diagonal matrix $D = \text{diag}(d_1, \cdots, d_n) > 0$, such that $\det(AD + B) = 0$. Therefore, there exists $x^* \in E^n$, with $||x^*|| = \delta$, such that $(AD + B)x^* = \theta$. Since $F \in \mathcal{E}^n$ there exists $x \in E^n$ such that

$$f_k(x_k) - f_k(x_k - x_k^*) = x_k^* d_k, \qquad \text{for} \qquad k = 1, \cdots, n.$$

Let $c = AF(x) + Bx$, and let $y = x - x^*$. Then

$$A[F(x) - F(y)] + B(x - y) = A[F(x) - F(x - x^*)] + Bx^*$$

$$= (AD + B)x^* = \theta. \quad \square$$

For a mapping F to be a member of $\mathcal{E}^n$, it is not necessary that $F \in \mathfrak{F}^n$. It follows from the above definition of $\mathcal{E}^n$ that $F \in \mathcal{E}^n$ implies that each of the functions f_k is a monotone increasing function from E^1 onto some interval in E^1 whose length is infinite; the f_k need not, however, be *strictly* monotone increasing, nor *onto* E^1. For those $F \in \mathcal{E}^n$ for which each of the functions f_k is strictly monotone increasing, we have the following corollary to the two preceding theorems.

Corollary: *Let* $F(x) \equiv [f_1(x_1), \cdots, f_n(x_n)]^T \in \mathcal{E}^n$ *and let each of the functions* f_k *be strictly monotone increasing. Then there exists at most one solution of equation* (16) *for each* $c \in E^n$ *if and only if* $(A, B) \in \mathcal{W}_0$.

VIII. RESULTS ON CONTINUITY AND BOUNDEDNESS

For many systems whose behavior is described by an equation having the form (16), the vector c may be regarded as the system's input and the vector x may be regarded as the system's response or output. Those properties that one might expect well-behaved systems to possess are likely to include continuity and boundedness. Thus, one might expect (*i*) "small" changes to result in the value of the system's output when "small" changes are made in the value of the system's input, and (*ii*) a bounded sequence of input vectors to yield a bounded sequence of outputs. We now show that such properties are indeed possessed by the type of system that is the main concern of this paper.

8.1 *Continuity*

When the $n \times n$ matrix A is a member of the class P_0 and the mapping $F \in \mathfrak{F}^n$, it follows that the solution x of equation (14) is a continuous function of the (input) vector b.[2] Using this fact, it is easy to prove the following theorem.

Theorem 6: *For each* $F \in \mathfrak{F}^n$ *and each pair of* $n \times n$ *matrices* $(A, B) \in \mathcal{W}_0$ *the solution* x *of equation* (16) *is a continuous function of the vector* c.

Proof: Proceeding as in the "if" part of the proof of Theorem 3, we see that the theorem follows immediately from the facts that equation (17) is a homeomorphism and that the aforementioned result guarantees that y, the solution of equation (18), is a continuous function of c. □

8.2 *Boundedness*

In Ref. 2 a theorem (Theorem 5) is proved which shows that, when $F \in \mathfrak{F}^n$ and $A \in P_0$, bounds can be obtained for the solution of equation (14) whenever bounds for $b \in E^n$ are given. The proof of a more general theorem concerning equation (16) can be constructed quite easily by using that theorem, and by using the same technique that was used in the proof of the preceding theorem, along with the trivial observations:

(*i*) For any nonsingular $n \times n$ matrix of real numbers, M, and any real numbers $\alpha_i \leqq \beta_i$, $i = 1, \cdots, n$, there exist real numbers, $\alpha'_i \leqq \beta'_i$, $i = 1, \cdots, n$, such that when each of the components c_i of the vector c satisfies $\alpha_i \leqq c_i \leqq \beta_i$, it follows that $\alpha'_i \leqq (M^{-1}c)_i \leqq \beta'_i$, for $i = 1, \cdots, n$.

(*ii*) For any given real numbers $\gamma_i \leqq \delta_i$, $i = 1, \cdots, n$, there exist for the homeomorphism (17), real numbers $\gamma'_i \leqq \delta'_i$, $i = 1, \cdots, n$, such that whenever x, y satisfy equation (17) with $\gamma_i \leqq y_i \leqq \delta_i$, for $i = 1, \cdots, n$, it follows that $\gamma'_i \leqq x_i \leqq \delta'_i$, for $i = 1, \cdots, n$.

The more general theorem, whose quite obvious proof is omitted, is the following:

Theorem 7: *Let* $F \in \mathfrak{F}^n$, *let* $(A, B) \in \mathfrak{W}_0$ *be a pair of* $n \times n$ *matrices, and, for* $i = 1, \cdots, n$, *let* $\alpha_i \leqq \beta_i$ *be given. There exist, for* $i = 1, \cdots, n$, *real numbers* $\gamma_i \leqq \delta_i$ *such that for any* $c = (c_1, \cdots, c_n)^T \in E^n$ *with* $\alpha_i \leqq c_i \leqq \beta_i$ *for* $i = 1, \cdots, n$, *if* x *satisfies equation* (16), *then* $\gamma_i \leqq x_i \leqq \delta_i$ *for* $i = 1, \cdots, n$.

According to Theorem 7, $(A, B) \in \mathfrak{W}_0$ is a sufficient condition for a bounded sequence of vectors c to yield a bounded sequence of solution vectors of equation (16), for all $F \in \mathfrak{F}^n$. The following theorem shows that $(A, B) \in \mathfrak{W}_0$ is also a necessary condition.

Theorem 8: *If* (A, B) *is a pair of real* $n \times n$ *matrices, then* $(A, B) \in \mathfrak{W}_0$ *if and only if for each* $F \in \mathfrak{F}^n$ *and each unbounded sequence of points* $x^1, x^2, x^3, \cdots$ *in* E^n, *the corresponding sequence* $c^1, c^2, c^3, \cdots$ $[c^k = AF(x^k) + Bx^k$, $k = 1, 2, 3, \cdots]$ *is unbounded.*

This theorem, which is a generalization of Theorem 4 of Ref. 2, can be proved in a manner which is a quite obvious generalization of the proof, given there, of that theorem. Thus, an appeal to Theorem 7 proves the "only if" part, and the "if" part is proved by assuming that $(A, B) \notin \mathfrak{W}_0$ and then choosing the same kind of mapping $F \in \mathfrak{F}^n$ as was chosen in Ref. 2, for which an unbounded sequence of vectors x^k yields a bounded sequence of vectors c^k.

IX. COMPUTATION OF THE SOLUTION

A. Gersho[7] has shown that whenever $F \in \mathfrak{F}^n \cap C^1$ (that is, whenever each of the functions f_k is a continuously differentiable strictly monotone increasing mapping of the real line onto itself), it is possible to compute the solution of equation (14), for any $A \in P_0$ and any $b \in E^n$, by making use of a gradient descent algorithm due to A. A. Goldstein.[11] The following theorem extends this result to the class of equations of the type (16).

Theorem 9: Let M be an arbitrary positive definite symmetric matrix, and let $Q : E^n \to E^1$ be defined by

$$Q(x) = [AF(x) + Bx - c]^T M[AF(x) + Bx - c],$$

where $F \in \mathfrak{F}^n \cap C^1$, $(A, B) \in \mathfrak{W}_0$, and $c \in E^n$. For each $x \in E^n$ and each $\gamma \geqq 0$ let

$$g(x, \gamma) = \begin{cases} \dfrac{Q(x) - Q[x - \gamma \nabla Q(x)]}{\gamma \, \| \nabla Q(x) \|^2}, & \gamma > 0; \\ 1, & \gamma = 0; \end{cases}$$

where $\nabla Q(x)$ denotes the gradient of Q at the point x. Then, if δ is any real number satisfying $0 < \delta \leqq \frac{1}{2}$, and if x^0 is an arbitrary point in E^n, the sequence $\{x^k : k = 0, 1, 2, \cdots\}$ converges to the solution of equation (16), where (for $k = 0, 1, 2, \cdots$) the x^k satisfy

$$x^{k+1} = x^k - \gamma^k \nabla Q(x^k),$$

each γ^k being any real number that satisfies $\delta \leqq g(x^k, \gamma^k) \leqq 1 - \delta$ if $g(x^k, 1) < \delta$, or $\gamma^k = 1$ if $g(x^k, 1) \geqq \delta$.

Proof: This proof uses generalizations of some of the ideas in Ref. 7 and relies ultimately upon the Goldstein algorithm.[11]

We first remark that the sequence $\{x^k\}$ is well-defined: It is easy to show (see the first part of the proof of Theorem 1, p. 31, Ref. 11) that for each $x \in E^n$, $g(x, \cdot)$ is a continuous function on $[0, \infty)$. This being the case, it is clear that if $g(x^k, 1) < \delta$, then for each ξ in the interval $[\delta, 1)$—and, in particular, for each ξ in the interval $[\delta, 1 - \delta]$—there is some γ^k in the interval $(0, 1)$ such that $g(x^k, \gamma^k) = \xi$.

Let $S = \{x \in E^n : Q(x) \leqq Q(x^0)\}$. Using the fact that M is a positive definite symmetric matrix, and using the fact that $F \in \mathfrak{F}^n$, $(A, B) \in \mathfrak{W}_0$ implies that $\| AF(x) + Bx \| \to \infty$ if and only if $\| x \| \to \infty$ (Theorem 8) we have that the set $S \subset E^n$ is bounded. By continuity of Q, S is closed. Thus, S is compact and, therefore, the gradient ∇Q (which is continuous on E^n, since $F \in C^1$) is uniformly continuous on S, and ∇Q is bounded on S. Also, Q is bounded below on S. [Indeed, we have $Q \geqq 0$ on E^n and by the existence and uniqueness theorem, Theorem 3, there exists exactly one point x^* ($x^* \in S$) at which $Q(x^*) = 0$.]

It is easily verified that, for each $x \in E^n$,

$$\nabla Q(x) = 2(AD_x + B)^T M[AF(x) + Bx - c],$$

where, for $k = 1, \cdots, n$, the kth diagonal element of the diagonal matrix $D_x > 0$ has the value of the derivative of the function f_k, evaluated at the point x_k. Since $(A, B) \in \mathfrak{W}_0$ implies that det $(AD_x + B) \neq 0$, and since det $M \neq 0$, it follows that $\nabla Q(x) = \theta$ if and only if x is the solution of equation (16).

In view of the above, it follows directly from Goldstein's theorem that the sequence $\{x^k\}$ converges to the solution of equation (16). □

Other methods of computing the solution of equation (16), in certain cases, also exist. If one performs a transformation of the type (17) on the independent variable x (in theory this can always be done) then the solution of equation (16) can easily be computed by first computing the solution of an equation of the type $G(y) + M^{-1}Ny = M^{-1}c$, where $G \in \mathfrak{F}^n$ and $M^{-1}N \in P_0$. Methods of computing the solution of certain equations of this type may be found in Refs. 1–3.

X. EXAMPLE

With the aid of the modern computing facilities that are commonly available today, it is clearly a rather routine matter to obtain an equation of the type (16) for any given transistor network. Moreover, it is not unfeasible, even for networks of moderately large size (say, up to 4 or 5 transistors), to consider the straightforward evaluation of the 2^n determinants specified in property (2) of Theorem 1, and thereby resolve the issue of whether or not the matrices involved in the equation are a $\mathfrak{W}_0$ pair. Due regard would of course have to be paid to the matter of performing sufficiently accurate computations.

On the other hand, even without the aid of a computer, it should often be possible to use a little ingenuity and a few devices* to reduce the computations involved in the application of the above theory to many specific problems to a point where they will just about fit onto the back of an envelope. Consider, for example, the following analysis of a three-transistor network:

For the network of Fig. 5, the voltage and current variables defined there must satisfy the following equations:

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} T^{(1)} & & \\ & T^{(2)} & \\ & & T^{(3)} \end{bmatrix} \begin{bmatrix} f_1(v_1) \\ f_2(v_2) \\ f_3(v_3) \\ f_4(v_4) \\ f_5(v_5) \\ f_6(v_6) \end{bmatrix}, \tag{19}$$

* According to R. Bellman: "a device is a trick that works at least twice." [12]

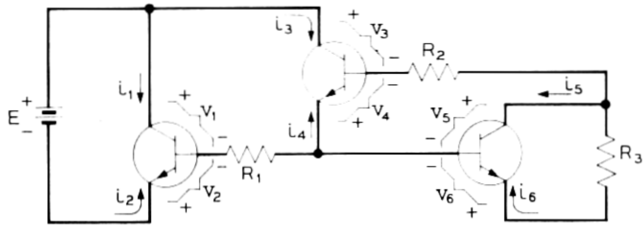

Fig. 5—Example of a three transistor network.

$$\begin{pmatrix} v_1 \\ v_3 \\ v_6 \\ -i_2 \\ -i_4 \\ -i_5 \end{pmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & R_1 & 0 & 1 & 1 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 1 \\ \hline -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & G_2 & G_2 \\ 0 & 0 & -1 & 0 & G_2 & G_2 \end{array}\right] \begin{pmatrix} -i_1 \\ -i_3 \\ -i_6 \\ v_2 \\ v_4 \\ v_5 \end{pmatrix} + \begin{pmatrix} E \\ E \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{20}$$

where (we are using the transistor model of Fig. 3, with $r_b = r_c = r_e = 0$) each of the 2×2 matrices $T^{(k)}$, $k = 1, 2, 3$, is of the form (10). A hybrid characterization has been used for the linear part of the network. As indicated in equation (3), this hybrid characterization can easily be converted into a characterization of the Belevitch type. Thus, denoting the 3×3 blocks of the hybrid matrix in equation (20) by H_{11}, H_{12}, H_{21}, H_{22}, in the usual manner, one obtains

$$\begin{bmatrix} I & -H_{12} \\ 0 & -H_{22} \end{bmatrix} v = -\begin{bmatrix} H_{11} & 0 \\ H_{21} & -I \end{bmatrix} i + c, \tag{21}$$

where $v = (v_1, v_3, v_6, v_2, v_4, v_5)^T$ and i is similarly defined. We could now simply reorder the columns of each matrix in equation (21) in such a way that the resulting equation would have the same form, except that the subscripts on the components of the vectors v and i would occur in the natural order (1, 2, 3, 4, 5, 6) and then use that equation, along with equation (19), to produce an equation of the type (16) for our network. In this example, though, it's probably easier to reorder the rows and columns of the matrix T (recall, $T = T^{(1)} \oplus T^{(2)} \oplus T^{(3)}$) to obtain from equation (19) an equation that is compatible with equation (21). Thus,

$$i = \begin{bmatrix} I & -P \\ -Q & I \end{bmatrix} F(v), \tag{22}$$

where

$$F(v) \equiv [f_1(v_1), f_3(v_3), f_6(v_6), f_2(v_2), f_4(v_4), f_5(v_5)]^T,$$

and

$$P = \text{diag}\,[\alpha_r^{(1)}, \alpha_r^{(2)}, \alpha_f^{(3)}], \qquad Q = \text{diag}\,[\alpha_f^{(1)}, \alpha_f^{(2)}, \alpha_r^{(3)}].$$

Eliminating i from equations (21) and (22), we obtain

$$\begin{bmatrix} H_{11} & 0 \\ H_{21} & -I \end{bmatrix} \begin{bmatrix} I & -P \\ -Q & I \end{bmatrix} F(v) + \begin{bmatrix} I & -H_{12} \\ 0 & -H_{22} \end{bmatrix} v = c. \tag{23}$$

Note that since $\det H_{11} = \det H_{22} = 0$, it is impossible to put this equation into either of the forms (14) or (15). Clearly this would be the same situation no matter which ordering of subscripts was chosen for the components of v. The cause of the difficulty is simply the fact that neither an impedance matrix nor an admittance matrix exists for the linear part of our network.

Let us determine whether or not the pair of matrices

$$\left(\begin{bmatrix} H_{11} & 0 \\ H_{21} & -I \end{bmatrix} \begin{bmatrix} I & -P \\ -Q & I \end{bmatrix}, \begin{bmatrix} I & -H_{12} \\ 0 & -H_{22} \end{bmatrix} \right)$$

is a $\mathcal{W}_0$ pair. We shall try to verify property (1) of Theorem 1. Let $\delta_1, \cdots, \delta_6$ denote arbitrary positive real numbers, and let $\Delta_I = \text{diag}\,(\delta_1, \delta_2, \delta_3)$, $\Delta_{II} = \text{diag}\,(\delta_4, \delta_5, \delta_6)$. We wish to show that

$$\det \left\{ \begin{bmatrix} H_{11} & 0 \\ H_{21} & -I \end{bmatrix} \begin{bmatrix} I & -P \\ -Q & I \end{bmatrix} \begin{bmatrix} \Delta_I^{-1} & 0 \\ 0 & \Delta_{II} \end{bmatrix} + \begin{bmatrix} I & -H_{12} \\ 0 & -H_{22} \end{bmatrix} \right\} \neq 0.$$

By multiplying the above matrix on the left by the (nonsingular) matrix $\text{diag}\,(I_3, -I_3)$ and then multiplying on the right by $\text{diag}\,(\Delta_I, I_3)$, we obtain the equivalent statement:

$$\det \left[\begin{array}{c|c} H_{11} + \Delta_I & -H_{12} - H_{11} P \, \Delta_{II} \\ \hline -H_{21} - Q & H_{22} + (I + H_{21} P) \, \Delta_{II} \end{array} \right] \neq 0.$$

The 3×3 submatrix in the upper left corner is nonsingular and diagonal. The 3×3 submatrix in the lower left corner can be diagonalized by performing a single elementary row operation on the matrix; namely, by subtracting $1/(\delta_2 + R_1)$ times the second row from the fourth row. Having done this, our problem reduces to one of showing that

$$\det \left[\begin{array}{ccc|ccc} \delta_1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \delta_2 + R_1 & 0 & -1 & -(1 + \alpha_r^{(2)} R_1 \, \delta_5) & 0 \\ 0 & 0 & \delta_3 + R_3 & 0 & 0 & -(1 + \alpha_f^{(3)} R_3 \, \delta_6) \\ \hline 1 - \alpha_f^{(1)} & 0 & 0 & (1 - \alpha_r^{(1)}) \, \delta_4 + \dfrac{1}{\delta_2 + R_1} & \dfrac{1 - \alpha_r^{(2)} \, \delta_2 \, \delta_5}{\delta_2 + R_1} & 0 \\ 0 & 1 - \alpha_f^{(2)} & 0 & 0 & G_2 + (1 - \alpha_r^{(2)}) \, \delta_5 & G_2 \\ 0 & 0 & 1 - \alpha_r^{(3)} & 0 & G_2 & G_2 + (1 - \alpha_f^{(3)}) \, \delta_6 \end{array}\right] \neq 0.$$

It is easy to verify that whenever $\det A_{11} \neq 0$, then

$$\det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \neq 0$$

if and only if $\det (A_{22} - A_{21} A_{11}^{-1} A_{12}) \neq 0$. In our case both A_{11} and A_{21} are diagonal and hence we can immediately reduce our problem to:

$$\det \begin{bmatrix} (1-\alpha_r^{(1)})\delta_4 + \dfrac{1}{\delta_2+R_1} + \dfrac{1-\alpha_f^{(1)}}{\delta_1} & \dfrac{1-\alpha_r^{(2)}\delta_2\delta_5}{\delta_2+R_1} & 0 \\ \dfrac{1-\alpha_f^{(2)}}{\delta_2+R_1} & G_2+(1-\alpha_r^{(2)})\delta_5 + \dfrac{1-\alpha_f^{(2)}}{\delta_2+R_1}(1+\alpha_r^{(2)}R_1\delta_5) & G_2 \\ 0 & G_2 & G_2+(1-\alpha_f^{(3)})\delta_6 + \dfrac{1-\alpha_r^{(3)}}{\delta_3+R_3}(1+\alpha_f^{(3)}R_3\delta_6) \end{bmatrix} \neq 0.$$

It is obvious that the above determinant is always positive. First, note that every term in the matrix is nonnegative except, possibly, the (1, 2) term, which may be either positive or negative (or zero). In the event that the (1, 2) term is positive (or zero), we have $1/(\delta_2 + R_1) \geqq (1 - \alpha_r^{(2)} \delta_2 \delta_5)/(\delta_2 + R_1)$, and hence we observe that the matrix is strongly row-sum dominant. This implies that its determinant is positive.

In the event that the (1, 2) term is negative, we do not necessarily have dominance; however, considering an expansion of the determinant along its first row we see that, because of the assumption that the (1, 2) term is negative, the value of the determinant is computed as the sum of two *positive* terms.

We have thus shown that, no matter which (positive) values are assigned to R_1, R_2, R_3, or which values the transistor's current gains assume $[0 < \alpha_f^{(k)} < 1, 0 < \alpha_r^{(k)} < 1]$, the pair of 6×6 matrices that appear in equation (23) is a $\mathcal{W}_0$ pair. Thus, all of the results concerning a solution's existence, uniqueness, continuity, boundedness, and so on, hold for this equation.

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APPENDIX

Proof of Part of Theorem 1

In this appendix we prove the equivalence of properties (*i*), (*ii*), (*vi*), and (*vii*) of Theorem 1, which define the class of pairs of matrices $\mathcal{W}_0$. We omit the proof of the equivalence of the three remaining properties, since those properties are not referred to in this paper. A complete proof of Theorem 1 is given elsewhere.[9] We begin by proving a useful lemma:

Lemma 1: *For each positive integer n the polynomial*

$$(c_0)\, d_1\, d_2 \cdots d_n + (c_1)\, d_1\, d_2 \cdots d_{n-1} + \cdots + (c_n)\, d_2 \cdots d_n$$
$$+ (c_{n+1})\, d_1\, d_2 \cdots d_{n-2} + \cdots + (c_{n(n+1)/2})\, d_3 \cdots d_n + \cdots + (c_{2^n-1})$$

in the n variables $d_1, d_2, \cdots, d_n$ is nonzero for all positive values of the variables if and only if at least one of the coefficients $c_0, \cdots, c_{2^n-1}$ is nonzero, and all nonzero coefficients have the same sign.

Proof: (By induction) For $n = 1$ the statement is obviously true. Let N be a positive integer. Then any polynomial of the above type in $N + 1$ variables, $(c_0)d_1 \cdots d_{N+1} + \cdots + (c_{2^{N+1}-1})$, can be written as $P(d_1, \cdots, d_N) \cdot d_{N+1} + Q(d_1, \cdots, d_N)$ where P and Q are both polynomials of the above type in N variables. Then, assuming that the statement is true for $n = N$, $P + Q \neq 0$ and $P \cdot Q \geqq 0$ for all positive values of the variables $d_1, \cdots, d_N$ if and only if at least one of the coefficients $c_0, \cdots, c_{2^{N+1}-1}$ is nonzero and all nonzero coefficients have the same sign. But, we know that $P \cdot d_{N+1} + Q \neq 0$ for all $d_{N+1} > 0$ if and only if $P + Q \neq 0$ and $P \cdot Q \geqq 0$. □

*A.*1 *Property (i) is Equivalent to (ii)*

Let $D = \text{diag}\,(d_1, \cdots, d_n)$. By expanding $\det\,(AD + B)$ along the first column we have

$$\det\,(AD + B) = d_1 \cdot \det P + \det Q,$$

where the first columns of P and Q satisfy $P_1 = A_1$, $Q_1 = B_1$, and for $k = 2, \cdots, n$, $P_k = Q_k = (AD + B)_k$. Both P and Q are independent of d_1. We now expand $\det P$ and $\det Q$ along their second columns, resulting in

$$\det P = d_2 \cdot \det R + \det S,$$

$$\det Q = d_2 \cdot \det U + \det V,$$

and hence,

$$\det (AD + B) = d_1 d_2 \cdot \det R + d_1 \cdot \det S + d_2 \cdot \det U + \det V,$$

where

$$\begin{aligned} R_1 &= A_1 , & R_2 &= A_2 , \\ S_1 &= A_1 , & S_2 &= B_2 , \\ U_1 &= B_1 , & U_2 &= A_2 , \\ V_1 &= B_1 , & V_2 &= B_2 , \end{aligned}$$

and for $k = 3, \cdots, n$,

$$R_k = S_k = U_k = V_k = (AD + B)_k .$$

Proceeding in this manner until all columns of $(AD + B)$ have been encountered, we obtain an expansion of $\det (AD + B)$ as a polynomial in the variables $\{d_1, d_2, \cdots, d_n\}$ whose coefficients are the determinants of the matrices in $\mathcal{C}(A, B)$. By using Lemma 1 it thus follows that (*i*) and (*ii*) are equivalent.

A.2 *Property (vi) Follows from (i) and (ii)*

According to (*ii*) there exists a complementary pair of matrices (M, N) taken from $\mathcal{C}(A, B)$ such that $\det M \neq 0$. Let $D = \operatorname{diag} (d_1, \cdots, d_n) > 0$, then $\det (M^{-1}N + D) \neq 0$ if and only if $\det (MD + N) \neq 0$. But, using property (*i*), $\det (MD + N) = \det (A\hat{D} + B) \cdot \det \tilde{D} \neq 0$, where the matrices $\hat{D} = \operatorname{diag} (\hat{d}_1, \cdots, \hat{d}_n) > 0$ and $\tilde{D} = \operatorname{diag} (\tilde{d}_1, \cdots, \tilde{d}_n) > 0$ are defined by $\hat{d}_k = d_k$ and $\tilde{d}_k = 1$ if $M_k = A_k$, and $\hat{d}_k = 1/d_k$, $\tilde{d}_k = d_k$ otherwise (for $k = 1, \cdots, n$). Thus, $M^{-1}N \in P_0$.

A.3 *Property (i) Follows from (vi)*

Using the notation above, it is clear that for each diagonal matrix $D > 0$, $\det (AD + B) = \det (M\hat{D} + N) \cdot \det \tilde{D}$. Thus, if $M^{-1}N \in P_0$ it follows that $\det (AD + B) \neq 0$.

A.4 *Property (vii) is Equivalent to (vi)*

Clearly property (*vi*) follows from property (*vii*). Thus, we need only prove that (*vi*) implies (*vii*). Let (M, N) and (P, Q) both be complementary pairs taken from $\mathcal{C}(A, B)$ with $M^{-1}N \in P_0$ and $\det P \neq 0$. For any $D = \operatorname{diag} (d_1, \cdots, d_n) > 0$, $\det (P^{-1}Q + D) \neq 0$ if and only if $\det (PD + Q) \neq 0$. But $\det (PD + Q) = \det (M\hat{D} + N) \cdot \det \tilde{D} \neq 0$, where the matrices $\hat{D} = \operatorname{diag} (\hat{d}_1, \cdots, \hat{d}_n) > 0$ and $\tilde{D} = \operatorname{diag} (\tilde{d}_1, \cdots, \tilde{d}_n) > 0$ are defined by $\hat{d}_k = d_k$ and $\tilde{d}_k = 1$ if $P_k = M_k$, and $\hat{d}_k = 1/d_k$, $\tilde{d}_k = d_k$ otherwise (for $k = 1, \cdots, n$). Thus, $P^{-1}Q \in P_0$. □

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