Information Theory and Approximation of Bandlimited Functions

By DAVID JAGERMAN

(Manuscript received April 15, 1970)

For bandlimited functions, simultaneous approximation of a function and several of its derivatives is considered. Concomitant entropy estimates are obtained. A feasible algorithm for the transmission of information is discussed. This algorithm has been applied to the design of a class of PCM systems.[1]

I. INTRODUCTION

It is the purpose of this paper to discuss both the best approximation of sets of bandlimited functions under Sobolev norms and the concomitant information-theoretic estimates. The Sobolev norms are useful when it is desired to approximate simultaneously the function and some of its derivatives. This requires an amount of information beyond that for approximating only the function. Section II gives the necessary background definitions of width, entropy, and capacity; theorems providing representations of bandlimited functions, as well as a form of Mitjagin's inequality relating approximability to entropy, are proved. The distinction between capacity and entropy is comparable to that between communication and storage, since capacity refers to the number of distinguishable functions transmitted from a signal source while entropy measures a bit requirement for the reproduction of a function to within a specified accuracy. A constructive approach to communication requirements implies an explicit means of representing any function of the signal source by numbers with a uniformly bounded number of digits. The procedure or algorithm used is usually obtained from an infinite series representation with subsequent truncation and quantization. Pulse code modulation systems provide examples of this procedure. Section II gives a precise definition, while Section III presents an explicit construction of a feasible algorithm. This algorithm has been applied to the design of a class of PCM systems.[1]

Sections III and IV contain the theorems and proofs which provide upper bounds on widths and entropies. Section III discusses signal sources with finite instantaneous power. Section IV considers signal sources in which the total energy is finite.

II. PRELIMINARIES

Let A be a subset of a Banach space X; it is desired to approximate A, that is, uniformly all elements of A by means of n-dimensional subspaces X_n of X. The deviation $E_{X_n}(A)$ of A from X_n is defined by

$$E_{X_n}(A) = \sup_{f \varepsilon A} \inf_{g \varepsilon X_n} \| f - g \|. \tag{1}$$

The deviation provides information on how well A may be uniformly approximated by elements of the given space X_n ; however, another choice of X_n might provide a smaller deviation. Accordingly the nth width, $d_n^X(A)$, of A relative to the space X is defined by[2]

$$d_n^X(A) = \inf_{X_n \subset X} E_{X_n}(A). \tag{2}$$

If the infimum is attained, then a corresponding X_n is called an extremal subspace. The following properties are immediate.

$$0 \leqq d_{n+1}^X(A) \leqq d_n^X(A), \qquad n \geqq 0, \tag{3}$$

$$d_0^X(A) = \sup_{X \varepsilon A} \| x \|, \tag{4}$$

$$B \subset A \Rightarrow d_n^X(B) \leqq d_n^X(A). \tag{5}$$

If X has finite dimension m, then $d_n^X(A) = 0$ for $n \geqq m$.

A set of sets whose diameters do not exceed $2\epsilon(\epsilon > 0)$ and whose union contains A is called an ϵ-covering of A. A finite set $S \subset X$ such that for $f \varepsilon A$ there is a $g \varepsilon S$ with $\| f - g \| \leqq \epsilon$ is called an ϵ-net of A. Clearly $d_n^X(A) \leqq \epsilon$ for a set A possessing an ϵ-net of n elements. If A is totally bounded then $\lim_{n \to \infty} d_n^X(A) = 0$. To see this, choose a covering of A consisting of n ϵ-balls, then their centers constitute an ϵ-net of A.

Let $N_\epsilon(A)$ (presumed finite) be the number of sets in a minimal ϵ-covering of A; then the *absolute* ϵ*-entropy*, $H_\epsilon(A)$, of A is defined by

$$H_\epsilon(A) = \log N_\epsilon(A) \tag{6}$$

in which the logarithm is taken to base two.[2-4]

Let $N_\epsilon^X(A)$ be the number of elements in a minimal ϵ-net $S \subset X$ of A; then the relative ϵ-entropy, $H_\epsilon^X(A)$, is defined by

$$H_\epsilon^X(A) = \log N_\epsilon^X(A) \tag{7}$$

in which the logarithm is taken to base two.[2-4] For A totally bounded, let $x_1, \cdots, x_n$ be the elements of an ϵ-net, and let $B_j (1 \leqq j \leqq n)$ be a ball of radius ϵ about x_j; then the sets $U_j = B_j \cap A$ constitute an ϵ-covering of A; hence

$$H_\epsilon(A) \leqq H_\epsilon^X(A). \tag{8}$$

The minimum number of binary digits, d, of an integer expressed in radix two needed to identify uniquely every element in a minimal ϵ-covering of A satisfies

$$[H_\epsilon(A)] \leqq d \leqq [H_\epsilon(A)] + 1 \tag{9}$$

in which $[x]$ designates the *integral part* of x, that is, the unique integer satisfying $x - 1 < [x] \leqq x$. Thus $H_\epsilon(A)$ may serve as an absolute measure of efficiency for processes designed for the storage and transmission of information.

Let a set ω of n real numbers be chosen, and also a mapping from A onto $\Omega_p = \omega \times \cdots \times \omega$ (p times); that is,

$$x \in A \to \alpha = (\alpha_1, \cdots, \alpha_p) \in \Omega_p, \alpha_1, \cdots, \alpha_p \in \omega.$$

Let the algorithm Γ define a one-to-one and onto mapping of Ω_p to an ϵ-net S of A in which $\Gamma(\alpha) \in S$ approximates $x \in A$ to within ϵ; then the volume $V(\Gamma)$ is defined by

$$V(\Gamma) = p \log n. \tag{10}$$

In view of expression (8), one has

$$V(\Gamma) \geqq H_\epsilon^X(A) \geqq H_\epsilon(A). \tag{11}$$

Thus the greater $V(\Gamma)$ is, the less efficient is the algorithm Γ compared to the absolute standard $H_\epsilon(A)$.

If $D \subset A$ has the property that

$$f \neq g, \qquad f, g \in D \Rightarrow \| f - g \| > \epsilon, \tag{12}$$

then D is called ϵ-distinguishable. Let $M_\epsilon(A)$ be the number of elements (presumed finite) in a maximal ϵ-distinguishable subset of A, then the *ϵ-capacity*, $C_\epsilon(A)$ is defined by

$$C_\epsilon(A) = \log M_\epsilon(A), \tag{13}$$

the logarithm being again taken to base two.[3] For a transmission system, $C_\epsilon(A)$ measures the number of distinguishable signals of the source or of the processed signal at the output of the receiver depending on the identification of A. The following inequalities hold between

ϵ-capacity and ϵ-entropy:

$$C_{2\epsilon}(A) \leqq H_\epsilon(A) \leqq C_\epsilon(A). \tag{14}$$

To show this, consider the inequality on the right. Let D be a maximal ϵ-distinguishable subset of A; then ϵ-balls about each element of D constitute an ϵ-covering of A for, otherwise, there would be an $x \varepsilon A$ not covered and hence more than ϵ away from every element of D. This would contradict the maximality of D. For the inequality on the left, let D be a 2ϵ-distinguishable subset of A, then the number of elements of D cannot exceed the number of covering sets of diameter 2ϵ or less in an ϵ-covering of A for, otherwise, there would be at least two elements of D in the same covering set. This would contradict the 2ϵ-distinguishability of D.

It is possible to bound $H_\epsilon^X(A)$ above and below in terms of $d_n^X(A)$ (refer to Ref. 2 where Mitjagin's inequalities are given). An improved form of Mitjagin's upper bound is proved below.

Theorem 1: Let A be a totally bounded subset of a real, normed, vector space X. Let the n*th widths relative to X be* $d_n^X(A)$, *and let*

$$N = \max_n \, [n : d_{n-1}^X(A) \geqq (1 - \alpha)\epsilon]$$

with α an arbitrary number satisfying $0 < \alpha < 1$; then

$$H_\epsilon^X(A) \leqq N \log \left(\frac{2d_0^X}{\alpha\epsilon} + \frac{2 - \alpha}{\alpha} \right).$$

Proof: Let E_N be an N-dimensional subspace of X for which $E_{E_N}(A) < (1 - \alpha)\epsilon$, then $\forall\, x \,\varepsilon\, A\ \exists\, y \,\varepsilon\, E_N \ni \| x - y \| < (1 - \alpha)\epsilon$. Let A_N be the set of all such y for every $x \varepsilon A$. An $\alpha\epsilon$ net of A_N is an ϵ-net of A; hence $H_\epsilon^X(A) \leqq H_{\alpha\epsilon}^X(A_N) \leqq C_{\alpha\epsilon}(A_N)$. Let $y_1, \cdots, y_M$ be an $\alpha\epsilon$-distinguishable subset of A_N, and let $B_k \subset E_N$ be balls with centers y_k and radius $\frac{1}{2}\alpha\epsilon$, then they are disjoint and are all contained in the ball B with center the origin and radius $d_0^X + (1 - \frac{1}{2}\alpha)\epsilon$. Let λ_N be the volume element in E_N; then $\lambda_N M(\frac{1}{2}\alpha\epsilon)^N \leqq \lambda_N[d_0^X + (1 - \frac{1}{2}\alpha)\epsilon]^N$. The inequality of the theorem follows on taking logorithms.

The class of functions to be studied consists of the space B_σ defined by the conditions that $f(t) \varepsilon B_\sigma$ be analytically continuable into the complex plane as an entire function of exponential order one and type σ, and that it be bounded on the whole real axis $-\infty < t < \infty$. The following inequality is valid for B_σ:[5]

$$\sup_{-\infty<t<\infty} | \dot{f}(t) | \leqq \sigma \sup_{-\infty<t<\infty} | f(t) |. \tag{15}$$

Important subspaces of the space B_σ are the space C_σ defined by

$$f \varepsilon C_\sigma \Rightarrow A(\eta) = o(e^{\sigma|\eta|}) \tag{16}$$

in which

$$A(\eta) = \sup_{-\infty<\xi<\infty} | f(\xi + i\eta) |, \qquad \xi, \eta \quad \text{real}, \tag{17}$$

and the space W_σ defined by

$$f \varepsilon W_\sigma \Rightarrow f \varepsilon L^2(-\infty, \infty). \tag{18}$$

Several representations for B_σ exist;[6] however, the following representations are needed for the present investigation. Let

$$\phi(t, \sigma) = \frac{\sin \sigma t}{\sigma t}, \tag{19}$$

$$\phi_j(t, \sigma) = \phi(t - jh, \sigma), \qquad h = \pi/\sigma, \tag{20}$$

then one has the following:
Theorem 2:

$$f(t) \varepsilon C_\sigma \Leftrightarrow f(t) = \sum_{j=-\infty}^{\infty} f(jh)\phi_j(t, \sigma)$$

for all complex t. The series converges uniformly in every closed, bounded region.
Proof: Consider the integral

$$I_N = \frac{1}{2\pi i} \int_{C_N} \frac{f(\zeta)}{(\zeta - t) \sin \sigma\zeta} d\zeta, \qquad \zeta = \xi + i\eta, \tag{21}$$

taken over a square C_N with corners at $(N + \frac{1}{2})(\pm 1 \pm i)h$, and N so large that t is in the interior of the region bounded by C_N. The theorem is clearly true when $t = kh$ (k integral); it will hence be assumed $t \neq kh$ for any integral k. The index $N \geqq 0$ is an integer. Evaluation of I_N by use of residues yields

$$f(t) = \sum_{j=-N}^{N} f(jh)\phi_j(t, \sigma) + I_N \sin \sigma t; \tag{22}$$

thus, to prove the implication to the right, it is sufficient to show $I_N \to 0$, $N \to \infty$. Let $I_N^{(1)}$ be the integral (21) extended over that part of C_N given by $\xi = (N + \frac{1}{2})h$; then

$$| I_N^{(1)} | \leqq \frac{1}{2\pi}$$

$$\int_{-(N+\frac{1}{2})h}^{(N+\frac{1}{2})h} \frac{A(\eta)}{| (N + \frac{1}{2})h + i\eta - t | \, | \sin (\pi(N + \frac{1}{2}) + i\sigma\eta) |} d\eta. \tag{23}$$

Since

$$| (N + \tfrac{1}{2})h + i\eta - t | \geqq (N + \tfrac{1}{2})h - | t | ,$$

$$| \sin (\pi(N + \tfrac{1}{2}) + i\sigma\eta) | = \cosh \sigma\eta \geqq \tfrac{1}{2}e^{\sigma|\eta|},$$

one has

$$| I_N^{(1)} | \leqq \frac{1}{\pi} \frac{1}{(N + \frac{1}{2})h - | t |} \int_{-(N+\frac{1}{2})h}^{(N+\frac{1}{2})h} e^{-\sigma|\eta|} A(\eta) \, d\eta. \tag{25}$$

Writing equation (25) in the form

$$| I_N^{(1)} | \leqq \frac{2}{\pi} \frac{(2N + 1)h}{(2N + 1)h - 2 | t |} \frac{1}{(2N + 1)h} \int_{-(N+\frac{1}{2})h}^{(N+\frac{1}{2})h} e^{-\sigma|\eta|} A(\eta) \, d\eta, \tag{26}$$

using equation (16) and the following lemma[7]

$$f(\eta) \rightarrow 0, \qquad | \eta | \rightarrow \infty \Rightarrow \lim_{T\rightarrow\infty} \frac{1}{2T} \int_{-T}^{T} f(\eta) \, d\eta = 0, \tag{27}$$

shows that $I_N^{(1)} \rightarrow 0$ uniformly in t. The same conclusion applies to the integral extended over $\xi = -(N + \frac{1}{2})h$.

Let $I_N^{(2)}$ be the integral (21) extended over $\eta = (N + \frac{1}{2})h$; then

$$| I_N^{(2)} | \leqq \frac{1}{2\pi} \cdot \int_{-(N+\frac{1}{2})h}^{(N+\frac{1}{2})h} \frac{f(\xi + i(N + \frac{1}{2})h)}{| \xi + i(N + \frac{1}{2})h - t | \, | \sin (\sigma\xi + i\pi(N + \frac{1}{2})) |} \, d\xi. \tag{28}$$

Since

$$| \xi + i(N + \tfrac{1}{2})h - t | \geqq (N + \tfrac{1}{2})h - | t |, \tag{29}$$

$$| \sin (\sigma\xi + i\pi(N + \tfrac{1}{2})) | \geqq \frac{1 - e^{-\pi}}{2} e^{\pi(N+\frac{1}{2})},$$

one has

$$| I_N^{(2)} | \leqq \frac{2}{\pi(1 - e^{-\pi})} \frac{(2N + 1)h}{(2N + 1)h - 2 | t |} e^{-\pi(N+\frac{1}{2})} A((N + \tfrac{1}{2})h). \tag{30}$$

In view of equation (16), $I_N^{(2)} \rightarrow 0$ uniformly in t. The same conclusion applies to the integral extended over $\eta = -(N + \frac{1}{2})h$. For the implication to the left, one may observe that $\phi_i(t, \sigma) \in C_\sigma$, and that the series converges uniformly.

The series of Theorem 2, which is clearly interpolatory, is called the *cardinal series.*[8]

For $f(t) \in L^2(-\infty, \infty)$, the Fourier transform relations are given by

$$F(u) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-iut} f(t)\, dt, \tag{31}$$

$$f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{iut} F(u)\, du. \tag{32}$$

The Fourier transform of $\phi_j(t, \sigma)$ is

$$\begin{aligned} \Phi_j(u, \sigma) &= \frac{1}{\sigma}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-iujh}, \qquad | u | < \sigma; \\ &= 0, \qquad | u | > \sigma. \end{aligned} \tag{33}$$

The Parseval relation now shows that the sequence $\phi_j(t, \sigma)$, $-\infty < j < \infty$ is orthogonal over $(-\infty, \infty)$ with respect to unit weight; thus,

$$\begin{aligned} \int_{-\infty}^{\infty} \phi_j(t, \sigma)\phi_k(t, \sigma)\, dt &= 0, \qquad j \neq k; \\ &= h, \qquad j = k. \end{aligned} \tag{34}$$

The following theorem may now be stated for $f \in W_\sigma$.

Theorem 3: $f \in W_\sigma$

$$\Rightarrow \int_{-\infty}^{\infty} | f(t) |^2\, dt = h \sum_{j=-\infty}^{\infty} | f(jh) |^2,$$

$$f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\sigma}^{\sigma} e^{iut} F(u)\, du,$$

$$F(u) = \frac{h}{(2\pi)^{\frac{1}{2}}} \sum_{j=-\infty}^{\infty} f(jh) e^{-iujh}, \qquad | u | < \sigma.$$

Proof: The Paley–Wiener theorem[9] shows that $f \in W_\sigma$ has the representation given in Theorem 3; hence, by the Cauchy–Schwartz inequality

$$| f(\xi + i\eta) | \leqq \left\{\frac{\sinh \sigma\eta}{2\pi\eta} \int_{-\sigma}^{\sigma} | F(u) |^2\, du\right\}^{\frac{1}{2}} = o(e^{\sigma|\eta|}). \tag{35}$$

Equation (35) shows that $W_\sigma \subset C_\sigma$; thus, by Theorem 2, f is in the closure of the system $\phi_j(t, \sigma)$, $-\infty < j < \infty$. The Parseval relation now follows from equation (34). To establish the formula for $F(u)$, it is only necessary to show

$$\int_{-\sigma}^{\sigma} e^{iut} \sum_{j=M}^{N} f(jh) e^{-iujh}\, du \to 0, \qquad M, N \to \infty, \quad M, N \to -\infty, \tag{36}$$

because each term is the Fourier transform of the corresponding ϕ_j term of the cardinal series. One has

$$\left| \int_{-\sigma}^{\sigma} e^{iut} \sum_{j=M}^{N} f(jh)e^{-iujh}\, du \right|^2 \leqq 2\sigma \int_{-\sigma}^{\sigma} \left| \sum_{j=M}^{N} f(jh)e^{-iujh} \right|^2 du, \tag{37}$$

$$\left| \int_{-\sigma}^{\sigma} e^{iut} \sum_{j=M}^{N} f(jh)e^{-iujh}\, du \right|^2 \leqq 4\sigma^2 \sum_{j=M}^{N} | f(jh) |^2 \rightarrow 0. \tag{38}$$

The limit zero is obtained as a consequence of the Parseval relation of Theorem 3.

To obtain a representation for the class B_σ ,[10] let

$$\theta(t) = \phi\left(t, \frac{\delta\sigma}{(1-\delta)m}\right)^m \phi\left(t, \frac{\sigma}{1-\delta}\right) \tag{39}$$

$$m > 0 \text{ integral}, \qquad 0 < \delta < 1,$$

$$\theta_j(t) = \theta(t - jh), \qquad h = \frac{\pi}{\sigma}(1 - \delta); \tag{40}$$

then one has

Theorem 4: $f \varepsilon B_\sigma$

$$\Rightarrow f(t) = \sum_{j=-\infty}^{\infty} f(jh)\theta_j(t).$$

The series converges absolutely and uniformly in every closed, bounded region.

Proof: The function

$$f(t)\left\{\frac{\sin \frac{\delta\sigma}{(1-\delta)m}(s-t)}{\frac{\delta\sigma}{(1-\delta)m}(s-t)}\right\}^m \tag{41}$$

belongs to $W_{\sigma/(1-\delta)}$ for each positive integer m and arbitrary s, hence the cardinal series applied to this function yields the expansion

$$f(t)\left\{\frac{\sin \frac{\delta\sigma}{(1-\delta)m}(s-t)}{\frac{\delta\sigma}{(1-\delta)m}(s-t)}\right\}^m$$

$$= \sum_{j=-\infty}^{\infty} f(jh)\left\{\frac{\sin \frac{\delta\sigma}{(1-\delta)m}(s-jh)}{\frac{\delta\sigma}{(1-\delta)m}(s-jh)}\right\}^m \phi_j\left(t, \frac{\sigma}{1-\delta}\right). \tag{42}$$

Let $s = t$; then the required representation is obtained. The absolute convergence follows from the boundedness of $| f(jh) |$ and

$$| \theta_j(t) | = 0(|j|^{-m-1}). \tag{43}$$

Approximation will be studied in the uniform norm and the following Sobolev norm

$$|| f ||_s = \left\{ \int_{-T/2}^{T/2} (| f(t) |^2 + \mu_1 | \dot{f}(t) |^2 + \cdots + \mu_s | f^{(s)}(t) |^2) \, dt \right\}^{\frac{1}{2}} \tag{44}$$

in which $\mu_1, \cdots, \mu_s$ are positive numbers. For the space B_σ, the symbol $B^T_{\sigma,s}$ will be used for the corresponding normed space. The symbol $B^T_{\sigma,s}(M)$ will be used for the subset defined by $| f(t) | \leqq M, -\infty < t < \infty$. For the space W_σ, the corresponding normed space will be denoted by $W^T_{\sigma,s}$, and $W^T_{\sigma,s}(B)$ for the subset in which

$$\left\{ \int_{-\infty}^{\infty} | f(t) |^2 \, dt \right\}^{\frac{1}{2}} \leqq B. \tag{45}$$

III. THEORETICAL INVESTIGATION OF B_σ

Let B^T_σ designate the vector space B_σ normed by

$$|| f ||_u = \max_{-T/2 \leqq t \leqq T/2} | f(t) |, \tag{46}$$

and let $B^T_\sigma(M)$ be the subset of B^T_σ satisfying

$$| f(t) | \leqq M, \qquad -\infty < t < \infty. \tag{47}$$

The completion of B^T_σ is the space C^T of functions continuous over $[-T/2, T/2]$ and normed by equation (46).

Let

$$c = \frac{\sigma T}{2}, \qquad \delta_n = 1 - \left(\frac{2c}{\pi n}\right)^{\frac{1}{2}}, \qquad n > \frac{2c}{\pi}, \tag{48}$$

$$m = \left[\frac{\pi \delta_{n-1}^2}{2e} (n - 1) \right], \qquad m \geqq 1;$$

then the following theorem provides a bound on the nth width, $d_n^{C^T}(B^T_\sigma(M))$, of $B^T_\sigma(M)$ relative to the space C^T.

Theorem 5: $d_n^{C^T}(B^T_\sigma(M) \leqq (2M/\pi m)e^{-m}$.

Proof: The series representation of Theorem 4 will be used. The function

$$g(t) = \sum_{|j| \leqq N} f(jh) \theta_j(t) \tag{49}$$

establishes an approximation to $f(t)$ whose error is given by

$$f(t) - g(t) = \sum_{|j|>N} f(jh)\theta_j(t). \tag{50}$$

From equations (39) and (40), one has

$$|\theta_j(t)| \leqq \frac{1}{\pi}\left(\frac{m}{\pi\delta}\right)^m \frac{1}{\left(|j| - \frac{T}{2h}\right)^{m+1}}, \qquad |j| > \frac{T}{2h}, \quad |t| \leqq \frac{T}{2}. \tag{51}$$

Define the function $\rho(x)$ by

$$\begin{aligned} \rho(x) &= \tfrac{1}{2} - x \qquad 0 \leqq x < 1, \\ &= \rho(x+1) \quad \text{for all} \quad x, \end{aligned} \tag{52}$$

then the Sonin (Euler–Maclaurin) summation formula[11] is

$$\sum_{a<j\leqq b} W(j) = \int_a^b W(x)\,dx + \rho(x)W(x)\Big]_a^b - \int_a^b \rho(x)W'(x)\,dx \tag{53}$$

in which $a < b$ are arbitrary numbers. Use of equation (53) with

$$W(x) = \frac{1}{\pi}\left(\frac{m}{\pi\delta}\right)^m \frac{1}{\left(x - \frac{T}{2h}\right)^{m+1}}, \qquad x > T/h, \tag{54}$$

$$a = N + \tfrac{1}{2}, \qquad b = \infty$$

yields

$$\sum_{|j|>N} |\theta_j(t)| \leqq \frac{2}{\pi m}\left[\frac{m}{\pi\delta\left(N + \frac{1}{2} - \frac{T}{2h}\right)}\right]^m. \tag{55}$$

Let

$$m = \left[\frac{\pi\delta}{e}\left(N + \frac{1}{2} - \frac{T}{2h}\right)\right] \geqq 1; \tag{56}$$

then

$$\sum_{|j|>N} |\theta_j(t)| \leqq \frac{2}{\pi m} e^{-m}. \tag{57}$$

Thus, from equation (50), one obtains

$$\| f - g \|_u \leqq \frac{2M}{\pi m} e^{-m}, \tag{58}$$

and hence

$$d_{2N+1}^{C\,T}\left(B_\sigma^T(M)\right) \leqq \frac{2M}{\pi m} e^{-m}. \tag{59}$$

For n odd, one has

$$d_n^{C\,T}\left(B_\sigma^T(M)\right) \leqq \frac{2M}{\pi} \frac{\exp\left\{-\left[\frac{\pi\delta}{2e}\left(n - \frac{T}{h}\right)\right]\right\}}{\left[\frac{\pi\delta}{2e}\left(n - \frac{T}{h}\right)\right]}; \tag{60}$$

while if n is even, one has $d_n^{C\,T} \leqq d_{n-1}^{C\,T}$; hence, in all cases

$$d_n^{C\,T} \leqq \frac{2M}{\pi} \frac{\exp\left\{-\left[\frac{\pi\delta}{2e}\left(n - 1 - \frac{T}{h}\right)\right]\right\}}{\left[\frac{\pi\delta}{2e}\left(n - 1 - \frac{T}{h}\right)\right]}. \tag{61}$$

The fractional guardband δ is now chosen as in equation (48) from which the inequality of the theorem follows.

When n is large, a more accurate estimate of $d_n^{C\,T}$ may be obtained by using a polynomial approximation to B_σ^T. Let

$$f(t) = f\left(\frac{T}{2} x\right) = g(x), \tag{62}$$

and let $L(x)$ be the Lagrange interpolation polynomial established for $g(x)$ on the zeros of $T_n(x)$, the nth Tchebysheff polynomial of first kind, over $[-1, 1]$; then the standard error formula for Lagrange interpolation[11] yields

$$\max_{-1 \leqq x \leqq 1} |\, g(x) - L(x) \,| \leqq \frac{1}{n!\, 2^{n-1}} \max_{-1 \leqq x \leqq 1} |\, g^{(n)}(x) \,|. \tag{63}$$

Bernstein's inequality (15) and equation (62) now yield

$$\left\| f(t) - L\left(\frac{2}{T} t\right) \right\|_u \leqq \frac{2M}{n!} \left(\frac{c}{2}\right)^n; \tag{64}$$

hence one has

Theorem 6:

$$d_n^{C\,T}\left(B_\sigma^T(M)\right) \leqq \frac{2M}{n!} \left(\frac{c}{2}\right)^n, \qquad n \geqq 0.$$

Let H_s^T be the space of functions $f(t)$ possessing derivatives up to order s satisfying $f, \dot{f}, \cdots, f^{(s)} \in L^2(-T/2, T/2)$ and normed by equation (44); then Theorem 7 provides an estimate of the nth width of $B^T_{\sigma,s}(M)$ relative to H^T_s.

Theorem 7: Let

$$\Gamma = \left\{ \sum_{r=0}^{s-1} \frac{\mu_r}{T^{2r}(s-r-1)!^2\,(2s-2r-1)(s-r)} + \frac{2}{T^{2s}}\,\mu_s \right\}^{\frac{1}{2}},$$

in which $\mu_0 = 1$ *and the sum is considered zero when* $s = 0$*, then*

$$d_{n+s}^{H_s^T}(B^T_{\sigma,s}(M)) \leqq \frac{M\Gamma\, T^{\frac{1}{2}}(2c)^{n+s}}{n!\binom{2n}{n}(2n+1)^{\frac{1}{2}}}.$$

Proof: For the function $g(x)$ of equation (62), the identity

$$g(x) = P(x) + \int_{-1}^{x} \frac{(x-u)^{s-1}}{(s-1)!}\, g^{(s)}(u)\, du, \tag{65}$$

(in which $P(x)$ is a polynomial of degree not exceeding $s - 1$), will be used to obtain a polynomial approximation to $g(x)$ in the Sobolev norm (44). Let $L(x)$ be the Lagrange interpolation polynomial for $g^{(s)}(x)$ formed with n nodal points on $[-1, 1]$ and $\omega(x)$ the corresponding fundamental poynomial; then one has

$$g^{(s)}(x) = L(x) + \frac{1}{n!}\, g^{(n+s)}(\xi)\omega(x), \qquad \xi \in [-1, 1]. \tag{66}$$

The polynomial $I(x)$ defined by

$$I(x) = P(x) + \int_{-1}^{x} \frac{(x-u)^{s-1}}{(s-1)!}\, L(u)\, du \tag{67}$$

will be used to approximate $g(x)$ in the Sobolev norm; its degree does not exceed $n + s - 1$. Let

$$| g^{(j)}(x) | \leqq M_j, \qquad | x | \leqq 1; \tag{68}$$

then, from equation (66), one has

$$| g^{(r)}(x) - I^{(r)}(x) | \leqq \frac{M_{n+s}}{n!} \int_{-1}^{x} \frac{(x-u)^{s-r-1}}{(s-r-1)!}\, | \omega(u) |\, du, \qquad 0 \leqq r < s, \tag{69}$$

$$| g^{(s)}(x) - I^{(s)}(x) | \leqq \frac{M_{n+s}}{n!}\, | \omega(x) |. \tag{70}$$

The norm (44) for the interval $[-1, 1]$ may be written

$$\| g - I \|_s^2 = \int_{-1}^{1} \sum_{r=0}^{s} \nu_r \mid g^{(r)}(x) - I^{(r)}(x) \mid^2 dx, \tag{71}$$

in which $\nu_0, \cdots, \nu_s \geqq 0$; the ν_r and μ_r are related through the change of variable $t = (T/2)x$. Using equations (69) and (70), one has

$$\| g - I \|_s^2 \leqq \frac{M_{n+s}^2}{n!^2} \cdot \int_{-1}^{1} \left\{ \sum_{r=0}^{s-1} \nu_r \left(\int_{-1}^{x} \frac{(x-u)^{s-r-1}}{(s-r-1)!} \mid \omega(u) \mid du \right)^2 + \nu_r \omega(x)^2 \right\} dx. \tag{72}$$

Define the function $k(u, v)$ by

$$k(u, v) = \int_{\max(u,v)}^{1} (x-u)^{s-r-1}(x-v)^{s-r-1} \, dx; \tag{73}$$

then equation (72) may be written:

$$\| g - I \|_s^2 \leqq \frac{M_{n+s}^2}{n!^2} \left\{ \sum_{r=0}^{s-1} \frac{\nu_r}{(s-r-1)!^2} \cdot \int_{-1}^{1} \int_{-1}^{1} k(u, v) \mid \omega(u)\omega(v) \mid du \, dv + \nu_s \int_{-1}^{1} \omega(x)^2 \, dx \right\}. \tag{74}$$

The Cauchy–Schwartz inequality shows that

$$k(u, v) \leqq \frac{(1-u)^{s-r-\frac{1}{2}}(1-v)^{s-r-\frac{1}{2}}}{2s - 2r - 1}; \tag{75}$$

hence,

$$\| g - I \|_s^2 \leqq \frac{M_{n+s}^2}{n!^2} \left\{ \sum_{r=0}^{s-1} \frac{\nu_r}{(s-r-1)!^2 (2s-2r-1)} \cdot \left(\int_{-1}^{1} (1-u)^{s-r-\frac{1}{2}} \mid \omega(u) \mid du \right)^2 + \nu_s \int_{-1}^{1} \omega(x)^2 \, dx \right\}. \tag{76}$$

Further application of the Cauchy–Schwartz inequality yields

$$\| g - I \|_s^2 \leqq \frac{M_{n+s}^2}{n!^2} \cdot \left\{ \sum_{r=0}^{s-1} \frac{\nu_r 2^{2s-2r}}{2(s-r-1)!^2 (2s-2r-1)(s-r)} + \nu_s \right\} \int_{-1}^{1} \omega(x)^2 \, dx. \tag{77}$$

A good choice for $\omega(x)$ is

$$\omega(x) = \frac{2^n}{\binom{2n}{n}} P_n(x), \tag{78}$$

where $P_n(x)$ is the nth Legendre polynomial. The coefficient of $P_n(x)$ in equation (78) makes $\omega(x)$ monic. Since

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1}, \tag{79}$$

one obtains

$$\| g - I \|_s \leqq \frac{2^n M_{n+s}}{n! \binom{2n}{n}} \left(\frac{2}{2n+1} \right)^{\frac{1}{2}} \cdot \left\{ \sum_{r=0}^{s-1} \frac{\nu_r 2^{2s-2r}}{2(s-r-1)!^2 (2s-2r-1)(s-r)} + \nu_s \right\}^{\frac{1}{2}}. \tag{80}$$

The Bernstein inequality (15) shows that

$$M_{n+s} \leqq M c^{n+s}; \tag{81}$$

hence

$$\| g - I \|_s \leqq \frac{M 2^n c^{n+s}}{n! \binom{2n}{n}} \left(\frac{2}{2n+1} \right)^{\frac{1}{2}} \cdot \left\{ \sum_{r=0}^{s-1} \frac{\nu_r 2^{2s-2r}}{2(s-r-1)!^2 (2s-2r-1)(s-r)} + \nu_s \right\}^{\frac{1}{2}}. \tag{82}$$

Finally the change of variable $t = (T/2)x$ and the replacement of ν_r by the original μ_r yield the result of the theorem.

The results of Theorems 5, 6, and 7, may be translated into estimates of entropy by use of the Mitjagin inequality of Theorem 1. The estimates so obtained will apply only to the subset of $B_\sigma(M)$ for which $f(t)$ is real. Doubling the bounds will provide estimates for complex valued $f(t)$.

Theorem 8: Let $0 < \alpha < 1$, $(1 - \alpha)\epsilon < (2M/\pi e)$, $f(t)$ *real,*

$$m = \left[\ln \frac{2M}{\pi(1-\alpha)\epsilon} \cdot \frac{1 + \ln \dfrac{2M}{\pi(1-\alpha)\epsilon}}{1 + \ln \dfrac{2M}{\pi(1-\alpha)\epsilon} + \ln \ln \dfrac{2M}{\pi(1-\alpha)\epsilon}} \right];$$

then

$$H_\epsilon(B_\sigma^T(M)) \leqq \left\{2 + \left[\frac{2c}{\pi}\left(1 + \left(\frac{e}{c}(m+1)\right)^{\frac{1}{2}}\right)^2\right]\right\} \log\left(\frac{2M}{\alpha\epsilon} + \frac{2-\alpha}{\alpha}\right).$$

Proof: According to Theorems 1 and 5, one must solve the inequality

$$\frac{2M}{\pi m} e^{-m} \geqq (1-\alpha)\epsilon \tag{83}$$

for the largest integer m; thus

$$me^m \leqq \frac{2M}{\pi(1-\alpha)\epsilon}. \tag{84}$$

Consider the function

$$F(x) = \delta - x - \ln x, \qquad \delta > 1. \tag{85}$$

One has

$$F'(x) = -1 - \frac{1}{x}; \tag{86}$$

hence, by the mean value theorem,

$$F(\delta - h) = -\ln \delta + h\left(1 + \frac{1}{\xi}\right), \qquad \delta - h < \xi < \delta. \tag{87}$$

Let

$$F(\delta - h) = 0; \tag{88}$$

then, since h is positive,

$$0 < -\ln \delta + h\left(1 + \frac{1}{\delta - h}\right), \tag{89}$$

$$0 < -\delta \ln \delta + h(1 + \delta + \ln \delta) - h^2; \tag{90}$$

thus

$$h > \frac{\delta \ln \delta}{1 + \delta + \ln \delta}, \tag{91}$$

and

$$\delta - h < \delta \frac{1 + \delta}{1 + \delta + \ln \delta}. \tag{92}$$

The inequality

$$x + \ln x \leqq \delta \tag{93}$$

is thus satisfied by

$$x < \delta \frac{1 + \delta}{1 + \delta + \ln \delta}; \tag{94}$$

hence, setting

$$\delta = \ln \frac{2M}{\pi(1 - \alpha)\epsilon} \tag{95}$$

and taking cognizance of the integral character of m, one has

$$m \leqq \left[\ln \frac{2M}{\pi(1 - \alpha)\epsilon} \cdot \frac{1 + \ln \frac{2M}{\pi(1 - \alpha)\epsilon}}{1 + \ln \frac{2M}{\pi(1 - \alpha)\epsilon} + \ln \ln \frac{2M}{\pi(1 - \alpha)\epsilon}} \right] \tag{96}$$

provided

$$(1 - \alpha)\epsilon < \frac{2M}{\pi e}. \tag{97}$$

For the computation of d_{n-1}, one has from equation (48)

$$m = \left[\frac{\pi}{2e} \delta_{n-2}^2 (n - 2) \right]. \tag{98}$$

Hence

$$\frac{\pi}{2e} \delta_{n-2}^2 (n - 2) < m + 1, \tag{99}$$

$$\left\{ 1 - \left(\frac{2c}{\pi(n - 2)} \right)^{\frac{1}{2}} \right\}^2 (n - 2) < \frac{2e}{\pi} (m + 1). \tag{100}$$

Let

$$n = 2 + \frac{2c}{\pi} \nu; \tag{101}$$

then

$$\nu \left(1 - \frac{1}{\nu^{\frac{1}{2}}} \right)^2 < \frac{e}{c} (m + 1); \tag{102}$$

accordingly

$$\nu < \left\{ 1 + \left(\frac{e}{c} (m + 1) \right)^{\frac{1}{2}} \right\}^2. \tag{103}$$

Thus n satisfies

$$n \leqq 2 + \left[\frac{2c}{\pi}\left\{1 + \left(\frac{e}{c}(m+1)\right)^{\frac{1}{2}}\right\}^2\right]; \tag{104}$$

the theorem now follows from equations (96), (97), (104), and Theorem 1.

When ϵ is small, a more accurate estimate of entropy may be obtained by use of Theorem 6 in place of Theorem 5. Accordingly one has

Theorem 9: *Let* $0 < \alpha < 1$, $\eta = (2M/(1-\alpha)\epsilon(\pi ec)^{\frac{1}{2}})$,

$$\eta \geqq \max\left(\frac{1}{ec}, e^{c/2}\right), \qquad f(t) \quad \textit{real};$$

then

$$H_\epsilon(B_\sigma^T(M))$$

$$\leqq \left\{1 + \left[\frac{2 \ln \eta + \frac{1}{2} - \frac{1}{2}\ln\left(\frac{2}{ec}\ln \eta\right)}{\ln\left(\frac{2}{ec}\ln \eta\right) + 1 + \frac{1}{2 \ln \eta}}\right]\right\} \log\left(\frac{2M}{\alpha\epsilon} + \frac{2-\alpha}{\alpha}\right).$$

Proof: According to Theorem 6, one may consider

$$\frac{2M}{n!}\left(\frac{c}{2}\right)^n \geqq (1-\alpha)\epsilon. \tag{105}$$

Stirling's formula provides the inequality

$$n! > n^n e^{-n}(2\pi n)^{\frac{1}{2}}, \tag{106}$$

and hence one may consider

$$\frac{2M}{(2\pi n)^{\frac{1}{2}}}\left(\frac{ec}{2n}\right)^n \geqq (1-\alpha)\epsilon. \tag{107}$$

Let

$$n = \frac{ec}{2}x, \qquad \eta = \frac{2M}{(1-\alpha)\epsilon(\pi ec)^{\frac{1}{2}}}; \tag{108}$$

then equation (107) becomes

$$x^{x+1/ec} \leqq \eta^{2/ec}. \tag{109}$$

Consider the function

$$F(x) = \delta - (x+a)\ln x; \tag{110}$$

then, by the mean value theorem

$$F(\delta - h) = \delta - (\delta + a)\ln\delta + h\left(\ln\delta + 1 + \frac{a}{\delta}\right) - \frac{h^2}{2}\left(\frac{1}{\xi} - \frac{a}{\xi^2}\right),$$
$$\delta - h < \xi < \delta. \qquad (111)$$

Let

$$x = \delta - h \geqq a, \qquad F(\delta - h) = 0; \qquad (112)$$

then

$$0 \leqq \delta - (\delta + a)\ln\delta + h\ln\left(\delta + 1 + \frac{a}{\delta}\right). \qquad (113)$$

Let

$$\delta \geqq 1/e; \qquad (114)$$

then

$$h \geqq \frac{(\delta + a)\ln\delta - \delta}{\ln\delta + 1 + \frac{a}{\delta}}, \qquad (115)$$

and hence

$$x \leqq \frac{2\delta + a - a\ln\delta}{\ln\delta + 1 + \frac{a}{\delta}}. \qquad (116)$$

Thus, in terms of n and η, one has

$$n \leqq \left[\frac{2\ln\eta + \frac{1}{2} - \frac{1}{2}\ln\left(\frac{2}{ec}\ln\eta\right)}{\ln\left(\frac{2}{ec}\ln\eta\right) + 1 + \frac{1}{2\ln\eta}}\right]. \qquad (117)$$

The lower bound on η in the theorem assures the satisfaction of the conditions on x and δ in equations (112) and (114). Use of Theorem 1 now provides the inequality of the theorem.

Theorem 10 provides an entropy estimate deduced from the width result of Theorem 7.

Theorem 10: $0 < \alpha < 1$, $\eta = (M\Gamma(2c)^s/(1 - \alpha)\epsilon(e\sigma)^{\frac{1}{2}})$

$$\eta \geqq \max\left(\frac{1}{ec}, e^{c/2}\right), \qquad \gamma = (1 + \mu_1\sigma^2 + \cdots + \mu_s\sigma^{2s})^{\frac{1}{2}}, \quad f(t) \ \ \textit{real};$$

then

$$H_\epsilon(B^T_{\sigma,s}(M)) \leqq \left\{s + 1 + \left[\frac{2 \ln \eta + \frac{1}{2} - \frac{1}{2} \ln\left(\frac{2}{ec} \ln \eta\right)}{\ln\left(\frac{2}{ec} \ln \eta\right) + 1 + \frac{1}{2 \ln \eta}}\right]\right\} \log\left(\frac{2M\gamma T^{\frac{1}{2}}}{\alpha\epsilon} + \frac{2-\alpha}{\alpha}\right).$$

Proof: The investigation parallels that of Theorem 9. A difference occurs in the estimation of d_0. The Bernstein inequality of (15) shows that

$$d_0 \leqq M\gamma T^{\frac{1}{2}}; \tag{118}$$

hence the estimate of the theorem.

The case $m = 1$ of the representation given in Theorem 4 may be used to obtain an explicit ϵ-net for $B_\sigma(M)$, and hence to provide a constructive algorithm for the transmission of information from such a source. The representation for $f(t) \varepsilon B_\sigma(M)$ takes the form

$$f(t) = \sum_{j=-\infty}^{\infty} f(jh) \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)},$$

$$\sigma h = \pi(1 - \delta). \tag{119}$$

In order to proceed, it is necessary to estimate the quantity $A(\delta)$ given by

$$A(\delta) = \sup_{-\infty < t < \infty} \sum_{j=-\infty}^{\infty} \left| \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)} \right|. \tag{120}$$

Theorem 11: $A(\delta) \leqq 1/\delta^{\frac{1}{2}}$ for $0 < \delta < 1$.

Proof: The Cauchy–Schwartz inequality yields

$$A(\delta)^2 \leqq \sup_{-\infty < t < \infty} \sum_{j=-\infty}^{\infty} \left[\frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)}\right]^2 \cdot \sup_{-\infty < t < \infty} \sum_{j=-\infty}^{\infty} \left[\frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)}\right]^2. \tag{121}$$

From the Parseval relation of Theorem 3, one has

$$\sum_{j=-\infty}^{\infty} \left[\frac{\sin \frac{\sigma}{1-\delta}(t-jh)}{\frac{\sigma}{1-\delta}(t-jh)}\right]^2 = \frac{1}{h}\int_{-\infty}^{\infty}\left[\frac{\sin \frac{\sigma}{1-\delta}(t-s)}{\frac{\sigma}{1-\delta}(t-s)}\right]^2 ds = 1, \quad (122)$$

$$\sum_{j=-\infty}^{\infty} \left[\frac{\sin \frac{\delta\sigma}{1-\delta}(t-jh)}{\frac{\delta\sigma}{1-\delta}(t-jh)}\right]^2 = \frac{1}{h}\int_{-\infty}^{\infty}\left[\frac{\sin \frac{\delta\sigma}{1-\delta}(t-s)}{\frac{\delta\sigma}{1-\delta}(t-s)}\right]^2 ds = \frac{1}{\delta}. \quad (123)$$

The theorem is established.

Let

$$S = \sup_{-\infty<j<\infty} |(fjh)|; \quad (124)$$

then a corollary to Theorem 11 is

Corollary:

$$\sup_{-\infty<t<\infty} |f(t)| \leqq S/\delta^{\frac{1}{2}}.$$

Proof: From equations (119) and (120), one has

$$\sup_{-\infty<t<\infty} |f(t)| \leqq SA(\delta). \quad (125)$$

The result follows from Theorem 11.

The function

$$g(t) = \sum_{|j|\leqq N} f(jh) \frac{\sin \frac{\delta\sigma}{1-\delta}(t-jh)}{\frac{\delta\sigma}{1-\delta}(t-jh)} \frac{\sin \frac{\sigma}{1-\delta}(t-jh)}{\frac{\sigma}{1-\delta}(t-jh)} \quad (126)$$

constitutes an approximation to $f(t)$. The error may be assessed by application of equation (51) for $m = 1$, and Sonin's formula (53); thus

$$\| f - g \|_u \leqq \frac{M}{\pi^2\delta}\left(\left(N + \frac{1}{2} - \frac{T}{2h}\right)^{-1} + \left(N + \frac{1}{2} + \frac{T}{2h}\right)^{-1}\right). \quad (127)$$

For $0 < \alpha < 1$, let

$$N = \left[\frac{M}{(1-\alpha)\epsilon\pi^2\,\delta}\left(1 + \left\{1 + \left(\frac{c(1-\alpha)\epsilon\,\delta\pi}{(1-\delta)M}\right)^2\right\}^{\frac{1}{2}}\right) - \frac{1}{2}\right] + 1; \quad (128)$$

then direct verification establishes

$$\| f - g \|_u < (1 - \alpha)\epsilon. \quad (129)$$

It may be observed that for large c, one has

$$N \cong \frac{c}{\pi(1 - \delta)} = \frac{T}{2h}; \tag{130}$$

that is, N is approximately the number of nodal points jh in $(-T/2, T/2)$.

Let

$$\beta_j(f) = \left[\frac{A(\delta)f(jh)}{2\alpha\epsilon}\right] \tag{131}$$

and

$$\beta(f) = (\beta_{-N}(f), \cdots, \beta_N(f)); \tag{132}$$

then the set U_β is defined to consist of all $f(t)$ generating the same vector $\beta = \beta(f)$. It will now be shown that the diameter of U_β does not exceed 2ϵ. Let $f_1(t), f_2(t) \in U_\beta$; then

$$| f_1(jh) - f_2(jh) | \leqq \frac{2\alpha\epsilon}{A(\delta)}. \tag{133}$$

One has

$$f_1(t) - f_2(t) = \sum_{j=-\infty}^{\infty} (f_1(jh) - f_2(jh)) \cdot \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)}; \tag{134}$$

and hence, by equation (129),

$$| f_1(t) - f_2(t) | \leqq \sum_{|j| \leqq N} | f_1(jh) - f_2(jh) | \cdot \left| \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)} \right| + 2(1 - \alpha)\epsilon \tag{135}$$

in which N is chosen as in equation (128). From equation (133), one has

$$| f_1(t) - f_2(t) | \leqq \frac{2\alpha\epsilon}{A(\delta)} \cdot \sum_{|j| \leqq N} \left| \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)} \right| + 2(1 - \alpha)\epsilon. \tag{136}$$

Use of equation (120) shows that

$$\| f_1 - f_2 \|_u \leqq 2\epsilon. \tag{137}$$

The sets U_β are centerable with respect to themselves; that is, there exists an element $g(t) \in U_\beta$ whose distance from any other element of U_β does not exceed ϵ. Consider the function $g(t)$ defined by

$$g(t) = \frac{2\alpha\epsilon}{A(\delta)} \sum_{|j| \leqq N} (\beta_j(f) + \tfrac{1}{2}) \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)}. \tag{138}$$

One has

$$| f(jh) - g(jh) | \leqq \frac{\alpha\epsilon}{A(\delta)}, \qquad | j | \leqq N, \tag{139}$$

and

$$\begin{aligned} f(t) - g(t) &= \sum_{|j| \leqq N} (f(jh) - g(jh)) \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)} \\ &\quad + \sum_{|j| > N} f(jh) \frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)}; \end{aligned} \tag{140}$$

hence, by equations (120), (129), and (139)

$$\| f - g \|_u \leqq \epsilon. \tag{141}$$

The required constructive algorithm, Γ, is thus given by the mapping $f \to g$ in equations (131) and (138).

Theorem 12: $V(\Gamma) = (2N + 1) \log \{[A(\delta)M/2\alpha\epsilon] - [-A(\delta)M/2\alpha\epsilon] + 1\}$, *in which N is given in equation* (128).

Proof: It is necessary to enumerate the number of distinct $g(t)$ which are generated by $\Gamma(B_\sigma(M))$. Since

$$\frac{A(\delta) \mid f(jh) \mid}{2\alpha\epsilon} \leqq \frac{A(\delta)M}{2\alpha\epsilon}, \tag{142}$$

the number of distinct values of $\beta_i(f)$ is

$$\left[\frac{A(\delta)M}{2\alpha\epsilon}\right] - \left[-\frac{A(\delta)M}{2\alpha\epsilon}\right] + 1, \tag{143}$$

and hence the number of distinct vectors $\beta(f)$ is

$$\left\{\left[\frac{A(\delta)M}{2\alpha\epsilon}\right] - \left[-\frac{A(\delta)M}{2\alpha\epsilon}\right] + 1\right\}^{2N+1}. \tag{144}$$

The theorem follows from equation (144).

Corollary 1: $V(\Gamma) \leqq (2N + 1) \log ([M/2\alpha\epsilon\delta^{\frac{1}{2}}] - [-M/2\alpha\epsilon\delta^{\frac{1}{2}}] + 1)$.
Proof: Theorem 11.

Corollary 2. $V(\Gamma) \leqq (2N + 1) \log (M/\alpha\epsilon(\delta)^{\frac{1}{2}} + 2)$.
Proof: Corollary 1 and the inequalities

$$\left[\frac{M}{2\alpha\epsilon(\delta)^{\frac{1}{2}}}\right] \leqq \frac{M}{2\alpha\epsilon(\delta)^{\frac{1}{2}}}, \tag{145}$$

$$-\left[-\frac{M}{2\alpha\epsilon(\delta)^{\frac{1}{2}}}\right] < \frac{M}{2\alpha\epsilon(\delta)^{\frac{1}{2}}} + 1.$$

IV. THEORETICAL INVESTIGATION OF W_σ

Using Theorem 3 for $f, g \in W_\sigma$, the Sobolev inner product

$$(f, g)_s = \int_{-T/2}^{T/2} (f\bar{g} + \mu_1 \dot{f}\dot{\bar{g}} + \cdots + \mu_s f^{(s)}\overline{g^{(s)}})\, dt \tag{146}$$

takes the form

$$(f, g)_s = \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \frac{\sin \frac{T}{2}(u - v)}{\pi(u - v)} \cdot (1 + \mu_1 uv + \cdots + \mu_s u^s v^s) F(u)\bar{G}(v)\, du\, dv, \tag{147}$$

in which $F(u)$, $G(u)$ are the Fourier transforms of f, g respectively. The corresponding positive definite quadratic form Q is

$$Q = \| f \|_s^2 = \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \frac{\sin \frac{T}{2}(u - v)}{\pi(u - v)} \cdot (1 + \mu_1 uv + \cdots + \mu_s u^s v^s) F(u)\bar{F}(v)\, du\, dv, \tag{148}$$

and an operator K generating Q is given by

$$KF = \int_{-\sigma}^{\sigma} \frac{\sin \frac{T}{2}(u - v)}{\pi(u - v)} \cdot (1 + \mu_1 uv + \cdots + \mu_s u^s v^s) F(v) \, dv, \qquad |u| \leqq \sigma; \tag{149}$$

thus

$$Q = \int_{-\sigma}^{\sigma} \bar{F} K F \, du. \tag{150}$$

The equation defining the eigenvalues and eigenfunctions of K is

$$K\Phi_k = \lambda_k \Phi_k , \qquad k \geqq 0, \tag{151}$$

in which the ordering $\lambda_0 \geqq \lambda_1 \geqq \lambda_2 \geqq \cdots$ is used. It follows from the Hilbert–Schmidt theory[12] that the eigenvalues are denumerable and of finite multiplicity and the eigenfunctions form an orthonormal set which, from the positive definite character of K, is complete in $L^2(-\sigma, \sigma)$.

Let

$$\varphi_k(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\sigma}^{\sigma} e^{iut} \Phi_k(u) \, du; \tag{152}$$

then the Parseval relation for Fourier transforms shows that the sequence $\phi_0(t), \phi_1(t), \phi_2(t), \cdots$ is orthonormal over $(-\infty, \infty)$; further, from equations (147), (150), and (151), one has

$$(\phi_j , \phi_k)_s = \int_{-\sigma}^{\sigma} \bar{\Phi}_k K \Phi_j \, du = \lambda_j \int_{-\sigma}^{\sigma} \Phi_j \bar{\Phi}_k \, du = 0 \qquad j \neq k$$

$$= \lambda_j \qquad j = k. \tag{153}$$

Thus the sequence $\{\phi_k(t)\}_0^\infty$ forms an orthogonal system with respect to the Sobolev inner product (146). The system $\{\phi_k(t)\}_0^\infty$ is also complete in $W_{\sigma,s}^T$ as a consequence of the completeness of the system $\{\Phi_k(u)\}_0^\infty$ in $L^2(-\sigma, \sigma)$.

Define the n-dimensional subspace $X_n \subset W_\sigma$ by

$$X_n = X_n(\phi_0 , \cdots , \phi_{n-1}) \tag{154}$$

then Theorem 13 provides the nth width of $W_{\sigma,s}^T(B)$, relative to H_s^T, in terms of the eigenvalues of K.

Theorem 13: $d_n^{H_s T}(W_{\sigma,s}^T(B)) = B\lambda_n^{\frac{1}{2}}$.

Proof: Let $f(t) \in W_{\sigma,s}(B)$; then

$$f(t) = \sum_{k=0}^{\infty} a_k \phi_k(t). \tag{155}$$

Let

$$g(t) = \sum_{k=0}^{n-1} a_k \phi_k(t) \in X_n ; \tag{156}$$

then the orthogonality of the $\phi_k(t)$, (153), yields

$$\| f - g \|_s^2 = \sum_{k=n}^{\infty} | a_k |^2 \lambda_k . \tag{157}$$

Thus

$$\inf_{g \in X_n} \| f - g \|_s^2 \leqq \sum_{k=n}^{\infty} | a_k |^2 \lambda_k . \tag{158}$$

From the monotonicity of the λ_k, one has

$$\inf_{g \in X_n} \| f - g \|_s^2 \leqq \lambda_n \sum_{k=n}^{\infty} | a_k |^2; \tag{159}$$

however, the orthonormality of the $\phi_k(t)$ over $(-\infty, \infty)$ shows that

$$\int_{-\infty}^{\infty} | f(t) |^2 \, dt = \sum_{k=0}^{\infty} | a_k |^2 \leqq B^2, \tag{160}$$

and hence from equation (159)

$$E_{X_n}(W_{\sigma,s}^T(B)) = \sup_{f \in W_{\sigma,s}(B)} \inf_{g \in X_n} \| f - g \|_s \leqq B\lambda_n^{\frac{1}{2}}. \tag{161}$$

Thus

$$d_n(W_{\sigma,s}^T(B)) \leqq B\lambda_n^{\frac{1}{2}}. \tag{162}$$

Consider the ball U_{n+1} defined by

$$g(t) = \sum_{k=0}^{n} a_k \phi_k(t), \qquad \| g \|_s \leqq B\lambda_n^{\frac{1}{2}}; \tag{163}$$

then, by a theorem on balls in a finite dimensional subspace of a Banach space,[2]

$$d_n(U_{n+1}) = B\lambda_n^{\frac{1}{2}}. \tag{164}$$

Thus the theorem will be established if it is shown that the ball U_{n+1} defined in equation (163) is contained in $W_{\sigma,s}(B)$. It is only necessary to verify that

$$\int_{-\infty}^{\infty} | g(t) |^2 \, dt = \sum_{k=0}^{n} | a_k |^2 \leqq B^2. \tag{165}$$

One has from

$$\| g \|_s^2 = \sum_{k=0}^{n} | a_k |^2 \lambda_k \leqq B^2 \lambda_n \tag{166}$$

that

$$\sum_{k=0}^{n} | a_k |^2 \leqq \sum_{k=0}^{n} | a_k |^2 \frac{\lambda_k}{\lambda_n} \leqq B^2, \tag{167}$$

and hence the theorem is proved.

Use of the series representation of Theorem 4 permits one to estimate $d_n^{H_0 T}(W_{\sigma,0}^T(B))$. The quantities m and δ_n are as in equation (48); additionally, the corresponding interval h_n is defined by $h_n = \pi(1 - \delta_n)/\sigma$.

Theorem 14:

$$d_n^{H_0 T}(W_{\sigma,0}^T(B)) \leqq \frac{2}{\pi} \frac{B}{h_{n-1}^{\frac{1}{2}}} \frac{e^{-m}}{\left((2m+1)\left(n - 1 - \frac{T}{h_{n-1}}\right)\right)^{\frac{1}{2}}}.$$

Proof: Let

$$f(t) = \sum_{j=-\infty}^{\infty} f(jh)\theta_j(t), \tag{168}$$

and

$$g(t) = \sum_{|j| \leqq N} f(jh)\theta_j(t); \tag{169}$$

then

$$| f(t) - g(t) | \leqq \sum_{|j|>N} | f(jh) | \, | \theta_j(t) |. \tag{170}$$

Since, by Parseval's relation of Theorem 3

$$\int_{-\infty}^{\infty} | f(t) |^2 \, dt = h \sum_{j=-\infty}^{\infty} | f(jh) |^2 \leqq B^2, \tag{171}$$

Schwartz's inequality applied to equation (170) yields

$$| f(t) - g(t) |^2 \leqq \frac{B^2}{h} \sum_{|j|>N} \theta_j(t)^2. \tag{172}$$

One has, from equation (51)

$$\| f - g \|_U^2 \leqq \frac{2B^2}{\pi^2 h} \left(\frac{m}{\pi \delta}\right)^m \sum_{j>N} \frac{1}{\left(j - \frac{T}{2h}\right)^{2m+2}}, \qquad N > \frac{T}{2h}. \tag{173}$$

One may use Sonin's formula, equation (53), to effect the summation in equation (173); thus

$$\| f - g \|_U \leqq \frac{B}{\pi} \frac{\sqrt{2}}{\left(h(2m+1)\left(N + \frac{1}{2} - \frac{T}{2h}\right)\right)^{\frac{1}{2}}} \left[\frac{m}{\pi\delta\left(N + \frac{1}{2} - \frac{T}{2h}\right)}\right]^m . \tag{174}$$

The choice

$$m = \left[\frac{\pi\delta}{e}\left(N + \frac{1}{2} - \frac{T}{2h}\right)\right] \tag{175}$$

leads to

$$\| f - g \|_U \leqq \frac{B}{\pi} \frac{\sqrt{2}\, e^{-m}}{\left(h(2m+1)\left(N + \frac{1}{2} - \frac{T}{2h}\right)\right)^{\frac{1}{2}}} . \tag{176}$$

Thus equation (176) shows that

$$d_{2N+1}^{H_0 T}(W_{\sigma,0}^T(B)) \leqq \frac{B}{\pi}\left(\frac{2}{h}\right)^{\frac{1}{2}} \frac{e^{-m}}{\left((2m+1)\left(N + \frac{1}{2} - \frac{T}{2h}\right)\right)^{\frac{1}{2}}} ; \tag{177}$$

and, hence, for n odd

$$d_n^{H_0 T}(W_{\sigma,0}^T(B)) \leqq \frac{2}{\pi}\frac{B}{h^{\frac{1}{2}}} \frac{e^{-m}}{\left((2m+1)\left(n - \frac{T}{h}\right)\right)^{\frac{1}{2}}} . \tag{178}$$

For n even, one has

$$d_n^{H_0 T} W_{\sigma,0}^T(B) \leqq \frac{2}{\pi}\frac{B}{h^{\frac{1}{2}}} \frac{e^{-m}}{\left((2m+1)\left(n - 1 - \frac{T}{h}\right)\right)^{\frac{1}{2}}} ; \tag{179}$$

thus equation (179) applies in all cases. The fractional guardband is now chosen as in equation (48), and the inequality of the theorem follows.

Theorem 13 permits an immediate corollary to be obtained from Theorem 14.

Corollary: For $s = 0$, one has

$$\lambda_n \leqq \frac{4}{\pi^2 h_{n-1}} \frac{e^{-2m}}{(2m+1)\left(n - 1 - \frac{T}{h_{n-1}}\right)} .$$

As was done in Theorem 7, polynomial approximation will be used to estimate $d_n^{H^T s}(W_{\sigma,s}^T(B))$. The estimate is given in Theorem 15.

Theorem 15:

$$d_{n+s}^{H^T s}(W_{\sigma,s}^T(B)) \leqq B\Gamma\left(\frac{2c}{\pi}\right)^{\frac{1}{2}} \frac{(2c)^{n+s}}{n!\binom{2n}{n}((2n+1)(2n+2s+1))^{\frac{1}{2}}}.$$

Proof: The estimate will be obtained from equation (80). In order to estimate M_{n+s}, consider

$$f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\sigma}^{\sigma} e^{iut} F(u)\, du, \tag{180}$$

from which one has

$$g(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\sigma}^{\sigma} e^{iu(T/2)x} F(u)\, du. \tag{181}$$

Accordingly

$$g^{(r)}(x) = \left(\frac{T}{2}\right)^r \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\sigma}^{\sigma} e^{iu(T/2)x} (iu)^r F(u)\, du. \tag{182}$$

By use of the Schwartz inequality, one obtains

$$|\ g^{(r)}(x)\ |^2 \leqq \left(\frac{T}{2}\right)^{2r} \frac{1}{2\pi} \int_{-\sigma}^{\sigma} u^{2r}\, du \int_{-\sigma}^{\sigma} |\ F(u)\ |^2\, du. \tag{183}$$

The Parseval relation for Fourier transforms

$$\int_{-\infty}^{\infty} |\ f(t)\ |^2\, dt = \int_{-\sigma}^{\sigma} |\ F(u)\ |^2\, du \leqq B^2 \tag{184}$$

and equation (184) now yields

$$|\ g^{(r)}(x)\ |^2 \leqq B^2 \frac{\sigma}{\pi} \frac{c^{2r}}{2r+1}. \tag{185}$$

Thus

$$M_{n+s} \leqq B\left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \frac{c^{n+s}}{(2n+2s+1)^{\frac{1}{2}}}. \tag{186}$$

The remainder of the analysis is the same as in Theorem 7.

Theorem 13 again permits an immediate corollary to be obtained from Theorem 15.

Corollary:

$$\lambda_{n+s} \leqq \frac{2c}{\pi} \Gamma^2 \frac{(2c)^{2n+2s}}{n!^2 \binom{2n}{n}^2 (2n+1)(2n+2s+1)}.$$

Theorems 14 and 15 lead to corresponding estimates of entropy through use of Theorem 1.

Theorem 16: Let

$$n \geqq 2 + \left(1 + \left(\frac{2c}{\pi}\right)^{\frac{1}{2}}\right)^2, \qquad 0 < \alpha < 1, \quad f(t) \quad \textit{real},$$

and

$$m = \left[\ln\left(\frac{2B}{\pi\sqrt{3}\,(1-\alpha)\epsilon}\left(\frac{2c}{\pi}\right)^{\frac{1}{4}}\right)\right];$$

then

$$H_\epsilon(W_{\sigma,0}^T(B)) \leqq \left\{2 + \left[\frac{2c}{\pi}\left(1 + \left(\frac{e}{c}(m+1)\right)^{\frac{1}{2}}\right)^2\right]\right\} \log\left(\frac{2B\lambda_0^{\frac{1}{2}}}{\alpha\epsilon} + \frac{2-\alpha}{\alpha}\right).$$

Proof: From Theorem 14, one has

$$d_{n-1}(W_{\sigma,0}^T(B)) \leqq \frac{2}{\pi} B\left(\frac{1}{h_{n-2}}\right)^{\frac{1}{2}} \frac{e^{-m}}{\left((2m+1)\left(n-2-\frac{T}{h_{n-2}}\right)\right)^{\frac{1}{2}}}. \tag{187}$$

From equation (48), one has

$$2m + 1 \geqq 3, \qquad \frac{T}{h_{n-2}} = \left(\frac{2c}{\pi}(n-2)\right)^{\frac{1}{2}}; \tag{188}$$

hence

$$d_{n-1}(W_{\sigma,0}^T(B)) \leqq \frac{2}{\pi\sqrt{3}} B\left(\frac{2c}{\pi}\right)^{\frac{1}{4}} e^{-m} \frac{1}{\left((n-2)^{\frac{1}{2}} - \left(\frac{2c}{\pi}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}}. \tag{189}$$

Since

$$\frac{1}{\left((n-2)^{\frac{1}{2}} - \left(\frac{2c}{\pi}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}} \leqq 1 \quad \text{for} \quad n \geqq 2 + \left(1 + \left(\frac{2c}{\pi}\right)^{\frac{1}{2}}\right)^2, \tag{190}$$

$d_{n-1}(W^T_{\sigma,0}(B))$ obeys the inequality

$$d_{n-1}(W^T_{\sigma,0}(B)) \leqq \frac{2}{\pi\sqrt{3}} B\left(\frac{2c}{\pi}\right)^{\frac{1}{4}} e^{-m}. \tag{191}$$

According to Theorem 1, one may consider

$$\frac{2}{\pi\sqrt{3}} B\left(\frac{2c}{\pi}\right)^{\frac{1}{4}} e^{-m} \geqq (1-\alpha)\epsilon; \tag{192}$$

and hence

$$m = \left[\ln\left(\frac{2B}{\pi\sqrt{3}\,(1-\alpha)\epsilon}\left(\frac{2c}{\pi}\right)^{\frac{1}{4}}\right)\right]. \tag{193}$$

The remaining analysis is the same as that of Theorem 8. The inequality of the theorem now follows.

Theorem 17: *Let* $0 < \alpha < 1$, $\eta = \Gamma B(2c)^s/(1-\alpha)\epsilon e(\pi c)^{\frac{1}{2}}$,

$$\eta \geqq \max\left(\frac{4}{e^2c^2}, e^{c/2}\right), \qquad f(t) \quad \textit{real},$$

then

$$H_\epsilon(W^T_{\sigma,s}(B)) \leqq \left\{s + 1 + \left[\frac{2\ln\eta + 1 - \ln\left(\frac{2}{ec}\ln\eta\right)}{\ln\left(\frac{2}{ec}\ln\eta\right) + 1 + \frac{1}{\ln\eta}}\right]\right\} \log\left(\frac{2B\lambda_0^{\frac{1}{2}}}{\alpha\epsilon} + \frac{2-\alpha}{\alpha}\right).$$

Proof: The proof parallels that of Theorem 9.

It may be useful to observe

$$\lambda_0 \leqq \frac{2c}{\pi}\gamma^2. \tag{194}$$

IV. ACKNOWLEDGMENTS

I should like to express my appreciation for the careful review of the manuscript given by the referee in which he brought to my attention an important error. I should also like to thank Dr. H. Heffes for his thorough reading of the paper and for his suggestions.

REFERENCES

1. Heffes, H., Horing, S., and Jagerman, D., "On the Design and Analysis of a Class of PCM Systems," to be published in the March 1971 B.S.T.J.
2. Lorentz, G. G., *Approximation of Functions*, New York: Holt, Rinehart and Winston, 1966.

3. Kolmogorov, A. N., and Tikhomirov, W. M., *Arbeiten Zur Informationstheorie III*, Mathematische Forschungsberichte, X, Berlin: VEB Deutscher Verlag der Wissenschaften, 1960.
4. Vitushkin, A. G., *Theory of the Transmission and Processing of Information*, New York: Pergaman Press, 1961.
5. Achieser, N. I., *Theory of Approximation*, New York: Frederick Ungar 1956.
6. Timan, A. F., *Theory of Approximation of Functions of a Real Variable*, New York: Macmillan, 1963, Chapter IV.
7. Hardy, G. H., *Divergent Series*, London: Clarendon Press, 1949, p. 50.
8. Whittaker, J. M., *Interpolatory Function Theory*, London: Cambridge University Press, 1935.
9. Paley, R. E. A. C., and Wiener, N., "Fourier Transforms in the Complex Domain," American Mathematical Society Colloquium Publications, Vol. XIX, 1934.
10. Helms, H. D., and Thomas, J. B., "Truncation Error of Sampling-Theorem Expansions," Proc. of the I.R.E., *50*, No. 1 (February 1962), pp. 179–182.
11. Fort, Tomlinson, *Finite Differences and Difference Equations in the Real Domain*, England: Oxford University Press, 1948.
12. Tricomi, F. G., *Integral Equations*, New York: Interscience, 1965.