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A Method for Measurement of the Duration of Picosecond Pulses by Beat Frequency Detection of Laser Output

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This paper describes a new method for measuring the duration of mode-locked picosecond laser pulses. It is similar to the Two Photon Fluorescence and Second Harmonic Generation methods in that it measures not the pulse duration directly but rather an autocorrelation function of the signal. It has the advantage in that it can be used for very low-power repetitive light signals.

I. INTRODUCTION

Two methods for determining the duration of picosecond pulses have been reported. The earlier method, which utilizes second harmonic generation (SHG) in nonlinear crystals, was reported independently and almost simultaneously by M. Maier, W. Kaiser and J. A. Giordmaine;[1] J. A. Armstrong;[2] and H. P. Weber.[3] The second method, which utilizes the two-photon fluorescence (TPF) in certain dyes, was first reported by J. A. Giordmaine[4] and his co-workers. Measurements of pulse durations based on these methods have been reported by numerous authors. Extensive references to these papers are included in a review article by A. J. DeMaria, W. H. Glenn, Jr., M. J. Brienza and M. E. Mack.[5]

Neither of these methods measures pulse duration directly. Both, in fact, measure the autocorrelation function $G(\tau)$ of the intensity $I(t)$, namely

$$G(\tau) = \int_{-\infty}^{\infty} I(t)I(t + \tau)\, dt.$$

A careful measurement of the contrast ratio between $G(\tau)$ at the peak of the pulse and $G(\tau)$ between pulses is necessary in order to determine whether or not the measured values represent the duration of a pure AM pulse. (This was first pointed out for TPF by H. P. Weber[6] and independently by J. R. Klauder, M. A. Duguay, J. A. Giordmaine and S. L. Shapiro.)[7] If the contrast ratio is not the correct value for pure AM pulses, little concerning the signal waveform can be reliably inferred from a knowledge of $G(\tau)$.

The beat frequency detection (BFD) method proposed in this paper measures a different quantity, namely

$$H_k(\tau) = \tfrac{1}{4}F_k(\tau) + \tfrac{1}{4}F_k(-\tau)$$

where

$$F_k(\tau) = \left| \frac{1}{T} \int_{-T/2}^{T/2} \mathcal{E}^*(t)\, \mathcal{E}(t + \tau) \exp\left(-j2\pi \frac{t}{T} k\right) dt \right|^2$$

and $\mathcal{E}(t)$ is the electric field of the optical signal, $1/T$ is the pulse repetition rate and k/T is the frequency of the harmonic chosen for analysis. It too suffers from the fact that if the correct "contrast ratio"* is not observed, the result is ambiguous.

If the proper "contrast ratio" is observed, then $\mathcal{E}(t)$ is real (i.e., there is no angle modulation of the signal) and in cases of practical interest the pulse duration is short compared with the period T. Thus, for small k, $kt/T \ll 1$ over the region where $\mathcal{E}(t)$ contributes significantly to the integral and

$$F_k(-\tau) = F_k(\tau) = \left| \frac{1}{T} \int_{-T/2}^{T/2} \mathcal{E}^*(t)\, \mathcal{E}(t + \tau)\, dt \right|^2 .$$

Thus $H_k(\tau)$ is essentially the autocorrelation function of $\mathcal{E}(t)$ when there is no angle modulation. It is therefore as good a measure of pulse duration as is the autocorrelation function of $I(t)$.

* This term will be defined in Section III.

Under certain conditions there is reason to believe that the pulses are "chirped." That is, they have the form

$$\mathcal{E}(t) = \mathcal{E}_0(t) \exp [j(\omega_0 t + \alpha t^2)]$$

where $\mathcal{E}_0(t)$ is a real, slowly varying envelope function. The measured quantity $H_k(\tau)$ has an interesting and distinctive behavior in this case. This is considered in Section III. It should be possible in some cases to determine by the BFD method how much, if any, "chirp" is present. Neither TPF nor SHG is capable of providing this information.

II. DESCRIPTION AND ANALYSIS OF THE BEAT-FREQUENCY DETECTION METHOD OF PULSE-DURATION MEASUREMENT

The output $\mathcal{E}(t)$ of a laser is a periodic function which can be written in the form

$$\mathcal{E}(t) = \sum_{n=-\infty}^{\infty} A_n \cos \{(n\omega + \omega_0)t + \phi_n\} \tag{1}$$

where $2\pi/\omega = T$ is the period, ω_0 is the center frequency of the spectrum and A_n is the amplitude of the nth line in the spectrum. If the phases ϕ_n are related by

$$\phi_n = n\Phi \tag{2}$$

where Φ is any constant, the signal $\mathcal{E}(t)$ is a sequence of pulses with no angle modulation and the pulse envelope has the minimum possible duration for the set $\{A_n\}$. Such a signal is called "mode locked". If equation (2) is not satisfied, angle modulation (and a wider pulse) occurs. In general, if ϕ_n is a set of random numbers a "noiselike" (but periodic) signal is obtained. The pulse duration of the signal $\mathcal{E}(t)$ is therefore dependent on $\{\phi_n\}$.

Consider the circuit shown in Fig. 1. The laser output $\mathcal{E}(t)$ is divided into equal parts by the 3 dB hybrid mirror (beam splitter) and the two parts are recombined in the same mirror after one path has suffered a delay τ relative to the other. Beam transforming lenses may be required in order to assure alignment of the phase fronts of the two interfering beams. Mirror M_2 is mounted on a track so that the delay in one arm of the device can be varied over a range corresponding to one period of the laser signal. Mirror M_1 is mounted on a piezoelectric acoustic modulator so that its position can be varied periodically over a distance of a few microns for reasons which will be explained below. The output of the hybrid is therefore

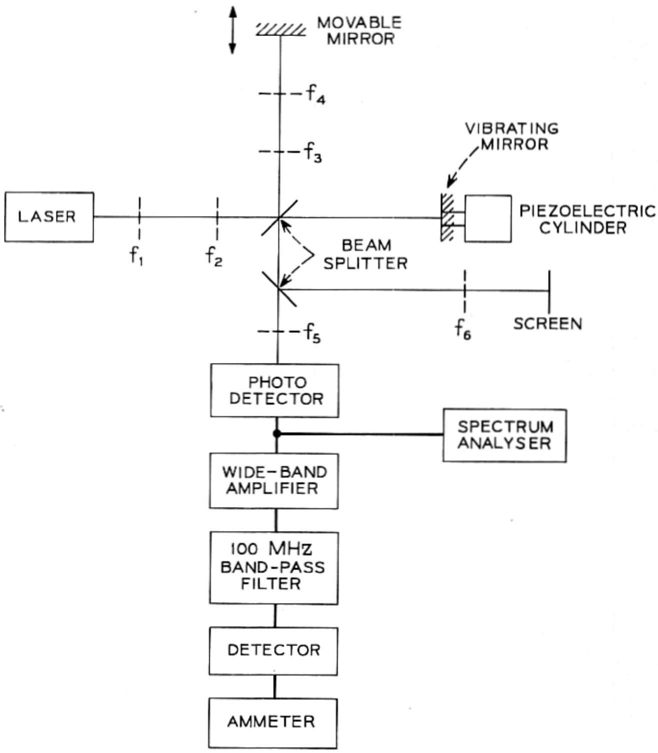

Fig. 1—Diagram of the BFD apparatus.

$$\{\mathcal{E}(t) + \mathcal{E}(t-\tau)\}$$
$$= \sum_{n=-\infty}^{\infty} A_n[\cos\{(n\omega + \omega_0)t + \phi_n\} + \cos\{(n\omega + \omega_0)(t-\tau) + \phi_n\}]. \quad (3)$$

If this signal is now applied to a product demodulator (photodetector), one obtains as output:

$$\begin{aligned} V(t) &= [\mathcal{E}(t) + \mathcal{E}(t-\tau)]^2 \\ &= \sum_{n=-\infty}^{\infty} \sum_{\kappa=-\infty}^{\infty} B_n^{(\kappa)} \cos\{\kappa\omega t + \theta_n^{(\kappa)}\} \\ &\quad \cdot \left[\cos\frac{\kappa\omega\tau}{2} + \cos\left\{\left(n + \frac{\kappa}{2}\right)\omega + \omega_0\right\}\tau\right] \end{aligned} \quad (4)$$

where

$$B_n^{(\kappa)} = A_{n+\kappa}A_n, \qquad \theta_n^{(\kappa)} = \phi_{n+\kappa} - \phi_n$$

and terms in $2\omega_0 t$ have been ignored as has a time shift of amount $\tau/2$. If this signal is now passed through a bandpass filter which is centered at $k\omega$ and is sufficiently narrow that only the $\kappa = k$ term in the sum is passed, one obtains

$$V_k(t) = \sum_{n=-\infty}^{\infty} B_n^{(k)} \cos \{k\omega t + \theta_n^{(k)}\} \cdot \left[\cos \frac{k\omega\tau}{2} + \cos \left\{\left(n + \frac{k}{2}\right)\omega + \omega_0\right\}\tau\right]. \tag{5}$$

It should be noted that since $\theta_n^{(0)} = 0$ for all n, $V_0(t)$ is independent of $\{\phi_n\}$ and independent of t.

$$V_0(t) = \sum_{n=-\infty}^{\infty} B_n^{(0)}[1 + \cos \{(n\omega + \omega_0)\tau\}]. \tag{6}$$

One can readily understand this result by realizing that the $V_k(t)$ term in general results from the beats between all possible pairs of lines separated by an amount $k\omega$. In particular $V_0(t)$ results from "beats" between each spectral line and itself. Since no beats between different lines contribute to $V_0(t)$, no relative phases between lines influence the result.

Returning to the general case, square law detection of $V_k(t)$ gives

$$\begin{aligned} U_k(\tau) &= 2 \mid V_k(\tau) \mid^2 \\ &= \sum_n \sum_m B_n^{(k)} B_m^{(k)} \cos (\theta_n^{(k)} - \theta_m^{(k)}) \\ &\quad \cdot \Bigg[\cos^2 \frac{k\omega\tau}{2} + \tfrac{1}{2} \cos (n - m)\omega\tau + \tfrac{1}{2} \cos \{(n + m + k)\omega + 2\omega_0\}\tau \\ &\quad + \cos \frac{k\omega\tau}{2} \Bigg[\cos \left\{\left(n + \frac{k}{2}\right)\omega + \omega_0\right\}\tau \\ &\quad + \cos \left\{\left(m + \frac{k}{2}\right)\omega + \omega_0\right\}\tau\Bigg]\Bigg]. \end{aligned} \tag{7}$$

We observe that the entire ω_0 dependence is contained in the last three terms. If one of the mirrors in the interferometer is scanned through an optical wavelength, the contribution of these terms is cancelled out (since over an optical wavelength their average value is zero). This is accomplished by mounting one of the mirrors on a piezoelectric transducer and vibrating it. The resultant signal (after passing through a suitable low-pass filter) is given by

$$\bar{U}_k(\tau) = \sum_n \sum_m B_n^{(k)} B_m^{(k)} \cos \{\theta_n^{(k)} - \theta_m^{(k)}\} \cdot \left[\cos^2 \frac{k\omega\tau}{2} + \tfrac{1}{2} \cos (n - m)\omega\tau\right]. \tag{8}$$

We pause to evaluate equation (8) for two extreme cases. Consider the mode-locked case $\phi_n = n\Phi$ which implies $\theta_n^{(k)} - \theta_m^{(k)} = 0$. This gives

$$\bar{U}_k(\tau)_{\substack{\text{mode}\\ \text{locked}}} = \cos^2 \frac{k\omega\tau}{2} \left[\sum_n B_n\right]^2 + \tfrac{1}{2} \left| \sum_n B_n \exp(jn\omega\tau) \right|^2. \tag{9}$$

In this equation and in the remainder of this paper the superscript (k) on B_n and θ_n is to be understood. The mode-locked case represents one extreme in the sense that it provides the shortest possible pulse consistent with a particular set of amplitudes $\{A_n\}$. The "random case", i.e., ϕ_n a random variable uniformly distributed on $[-\pi, \pi]$, represents the opposite extreme—the expectation value of the signal power is constant in time. The expectation value of $\bar{U}_k(\tau)$ for this case can be obtained by observing that the expectation value of $\cos \{\theta_n - \theta_m\}$ when the $\{\phi_n\}$ are uniformly distributed on $[0, 2\pi)$ is given by

$$\begin{aligned} \langle \cos(\theta_n - \theta_m) \rangle &= 1 \quad \text{if} \quad k = 0 \quad (\text{for all } n, m), \\ &= 1 \quad \text{if} \quad n = m \quad (\text{for all } k), \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

This gives:

$$\langle \bar{U}_k(\tau) \rangle = \left[\cos^2 \left(\frac{k\omega\tau}{2} \right) + \frac{1}{2} \right] \sum_n B_n^2 , \qquad k \neq 0. \tag{10}$$

Equations (9) and (10) are illustrated in Fig. 2.

We now return to a consideration of the general case. Equation (8) can be written in the form

$$\begin{aligned} \bar{U}_k(\tau) = {} & \cos^2 \frac{k\omega\tau}{2} \left| \sum_{n=-\infty}^{\infty} \mathcal{B}_n \right|^2 \\ & + \frac{1}{4} \left| \sum_{n=-\infty}^{\infty} \mathcal{B}_n \exp(jn\omega\tau) \right|^2 + \frac{1}{4} \left| \sum_{n=-\infty}^{\infty} \mathcal{B}_n \exp(-jn\omega\tau) \right|^2 \end{aligned} \tag{11}$$

where

$$\mathcal{B}_n = B_n \exp(j\theta_n).$$

Similarly set

$$\mathcal{A}_n = A_n \exp(j\phi_n).$$

Then

$$\mathcal{B}_n = \mathcal{A}_{n+k} \mathcal{A}_n^* .$$

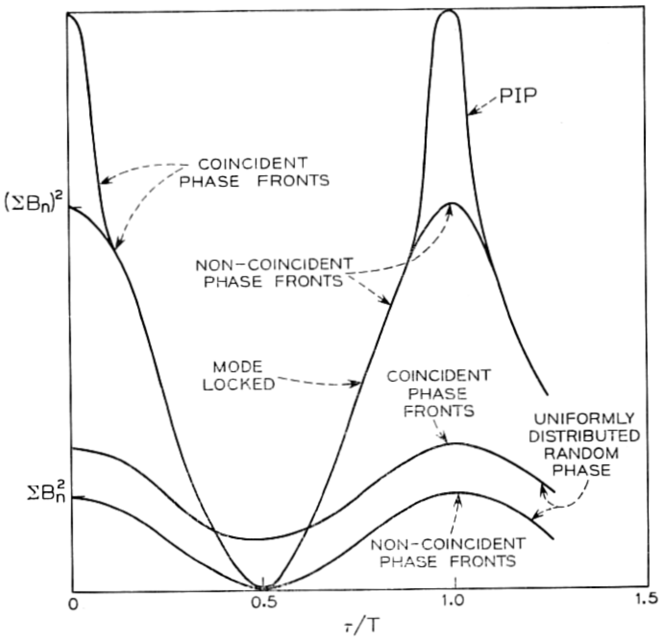

Fig. 2—Expected experimental results for the four cases discussed in the text.

Consider the expression:

$$\begin{aligned}
\left| \sum_{n=-\infty}^{\infty} \mathcal{B}_n \exp\left(j2\pi \frac{x}{T} n\right) \right|^2
&= \left| \sum_{n=-\infty}^{\infty} \mathcal{A}_{n+k}\mathcal{A}_n^* \exp\left(j2 \frac{\pi x n}{T}\right) \right|^2 \\
&= \left| \frac{1}{T} \int_{-T/2}^{T/2} E^*(-t)E(x - t) \exp\left(j2\pi k \frac{t}{T}\right) dt \right|^2 \qquad (12)
\end{aligned}$$

where

$$E(t) = \sum_{n=-\infty}^{\infty} \mathcal{A}_n \exp\left(j \frac{2\pi t}{T} n\right). \qquad (13)$$

Define

$$F_k(x) = \left| \frac{1}{T} \int_{-T/2}^{T/2} E^*(t)E(t + x) \exp\left(-j2\pi k \frac{t}{T}\right) dt \right|^2. \qquad (14)$$

Then

$$\bar{U}_k(\tau) = \cos^2\left(\frac{k\omega\tau}{2}\right)F_k(0) + \tfrac{1}{4}F_k(\tau) + \tfrac{1}{4}F_k(-\tau). \qquad (15)$$

From equation (15) we can draw an interesting comparison between the BFD method and the TPF and SHG methods. It is clear from equation (14) that $F_k(0)$ is the power spectrum of the intensity and therefore the autocorrelation function of the intensity is simply

$$G(\tau) = \sum_{k=-\infty}^{\infty} F_k(0) \exp\left(-j2\pi k \frac{\tau}{T}\right).$$

Therefore the quantities measured by all three methods depend only on the quantities $F_k(\tau)$. The results of the BFD method depend on the values for a single k and all τ; the results for TPF and SHG on the values for $\tau = 0$ and all k. From equations (12) and (14), we have

$$F_k(\tau) = \left| \sum_{n=-\infty}^{\infty} \mathcal{B}_n \exp\left(j2\pi \frac{\tau}{T} n\right) \right|^2 \tag{16}$$

while from equation (1)

$$I(t) = \left| \sum_{n=-\infty}^{\infty} \mathcal{A}_n \exp\left(j2\pi n \frac{t}{T}\right) \right|^2.$$

Thus, $F_k(\tau)$ and $I(t)$ are *formally* equivalent with the transformations

$$B_n \leftrightarrow A_n ,$$

$$\theta_n \leftrightarrow \phi_n ,$$

$$\tau \leftrightarrow t.$$

Comparing equation (13) with (1) shows that

$$\mathcal{E}(t) = \text{Re}\,\{E(t) \exp(j\omega_0 t)\}.$$

That is, $E(t)$ is the (complex) signal from the laser with the carrier frequency omitted.

For the mode-locked case, the pulses are very short compared with the repetition rate $1/T$. Thus for at least the first few $k = 1, 2, \cdots$ we can write $(2\pi kt)/T \ll 1$ for those values of t for which $E(t)$ is significantly different from zero. This gives (since from equation (16) $F(\tau)$ is even when the B_n are all real):

$$F_k(-\tau) = F_k(\tau) \doteq \left| \frac{1}{T} \int_{-T/2}^{T/2} E^*(t)E(t+\tau)\, dt \right|^2.$$

Thus the first term in equation (15) is just a raised cosine with period T/k, but the sum of the second and third terms is the square of the autocorrelation function of $E(t)$. This sum provides a good measure of the pulse duration when the signal is mode locked.

Since this technique, like TPF and SHG, possesses some ambiguity if the signal is not mode locked, it is important to be able to distinguish mode-locked behavior from non-mode-locked behavior. In order to do this, consider the quantity

$$\bar{U}_k(0) = \frac{3}{2}\left|\sum_{n=-\infty}^{\infty} \mathcal{B}_n\right|^2,$$

$$= \frac{3}{2}\left|\sum_{n=-\infty}^{\infty} B_n\right|^2 \quad \text{mode-locked case,}$$

$$= \frac{3}{2}\sum_{n=-\infty}^{\infty} B_n^2 \quad \underset{k\neq 0}{\text{random-phase case.}}$$

For $k = 0$, $\theta_n \equiv 0$ and $\mathcal{B}_n = B_n$ is real for any set $\{\phi_n\}$. Thus

$$\bar{U}_0(0) = \frac{3}{2}\left|\sum_{n=-\infty}^{\infty} B_n\right|^2$$

for all cases. By considering the $k = 0$ (dc) term the quantity

$$\left|\sum_{n=-\infty}^{\infty} B_n\right|^2$$

can be measured experimentally. Then if

$$\bar{U}_k(0) \neq \bar{U}_0(0) \qquad k = 1, 2, 3 \cdots$$

the signal is not mode locked.

III. SPECIAL CASES

It is instructive to consider how much $\bar{U}_k(0)$ differs from $\bar{U}_0(0)$ for various cases. To this end we consider three special situations, namely, a pure FM signal, a model of partial mode locking, and a "chirped" pulse. Consider first the sinusoidally modulated pure FM signal:

$$\mathcal{E}(t) = \cos\{\omega_0 t + \phi \sin \omega t\}$$

$$= \sum_{n=-\infty}^{\infty} J_n(\phi) \cos\{(\omega_0 + n\omega)t\}.$$

This is just a special case of equation (1) with

$$A_n = J_n(\phi), \qquad \phi_n = 0 \quad \text{all } n.$$

The resulting expression for $\bar{U}_k(\tau)$ is therefore

$$\bar{U}_k(\tau) = \cos^2\frac{k\omega\tau}{2}\left[\sum_{n=-\infty}^{\infty} J_n(\phi)J_{n+k}(\phi)\right]^2$$

$$+ \frac{1}{2}\left|\sum_{n=-\infty}^{\infty} J_n(\phi)J_{n+k}(\phi)\exp(jn\omega\tau)\right|^2.$$

This can be reduced, by means of a well-known Bessel function identity to

$$\bar{U}_k(\tau) = \tfrac{1}{2} J_k^2\left(2\phi \sin \frac{\omega\tau}{2}\right).$$

Note that $\bar{U}_0(0) = \frac{1}{2}$ whereas $\bar{U}_k(0) = 0$, $k = 1, 2, 3, \cdots$. Also, the largest value that $\bar{U}_k(\tau)$ $k = 1, 2, \cdots$ can assume is more than 9 dB below $\bar{U}_0(0)$ for the FM signal.

The simplest model for partial mode locking is to let the phase ϕ_n of the nth spectral line of the laser be uniformly distributed on the interval $[-\alpha, \alpha] = [-\epsilon\pi, \epsilon\pi]$. Both a Monte Carlo calculation and an analytical calculation were made. The analytical calculation is described below. The results of both calculations are presented in Fig. 3.

Consider equation (8). In order to determine the expectation value (ensemble average) $\langle \bar{U}_k(\tau) \rangle$ we must compute the values of $\langle \cos(\theta_m - \theta_n) \rangle$. Three cases arise (for $k \neq 0$):

Case I: $n = m$

$$\langle \cos(\theta_n - \theta_m) \rangle = 1.$$

Case II: $n = m \pm k$

$$\langle \cos(\theta_n - \theta_m) \rangle = \langle \cos(\phi_{n+2k} - 2\phi_{n+k} + \phi_n) \rangle \equiv a.$$

Case III: All subscripts distinct

$$\langle \cos(\theta_n - \theta_m) \rangle = \langle \cos(\phi_{n+k} - \phi_n + \phi_{m+k} - \phi_m) \rangle \equiv b.$$

Then

$$\begin{aligned}\bar{U}_k(\tau) = {} & b \sum_{\substack{n \\ n \neq m \\ n \neq m \pm k}} \sum_m B_n B_m \left[\cos^2\left(\frac{k\omega\tau}{2}\right) + \tfrac{1}{2}\cos(n - m)\omega\tau\right] \\ & + \left[\cos^2\left(\frac{k\omega\tau}{2}\right) + \frac{1}{2}\right] \sum_n B_n^2 \\ & + 2a \sum_n B_n B_{n+k} \left[\cos^2\left(\frac{k\omega\tau}{2}\right) + \tfrac{1}{2}\cos(k\omega\tau)\right].\end{aligned}$$

This can be rewritten in the form

$$\begin{aligned}\bar{U}_k(\tau) = {} & G_1 \cos^2\left(\frac{k\omega\tau}{2}\right) \\ & + G_2 + \tfrac{1}{2} b \Big[\Big(\sum_n B_n \cos n\omega\tau\Big)^2 + \Big(\sum_n B_n \sin n\omega\tau\Big)^2\Big]\end{aligned}$$

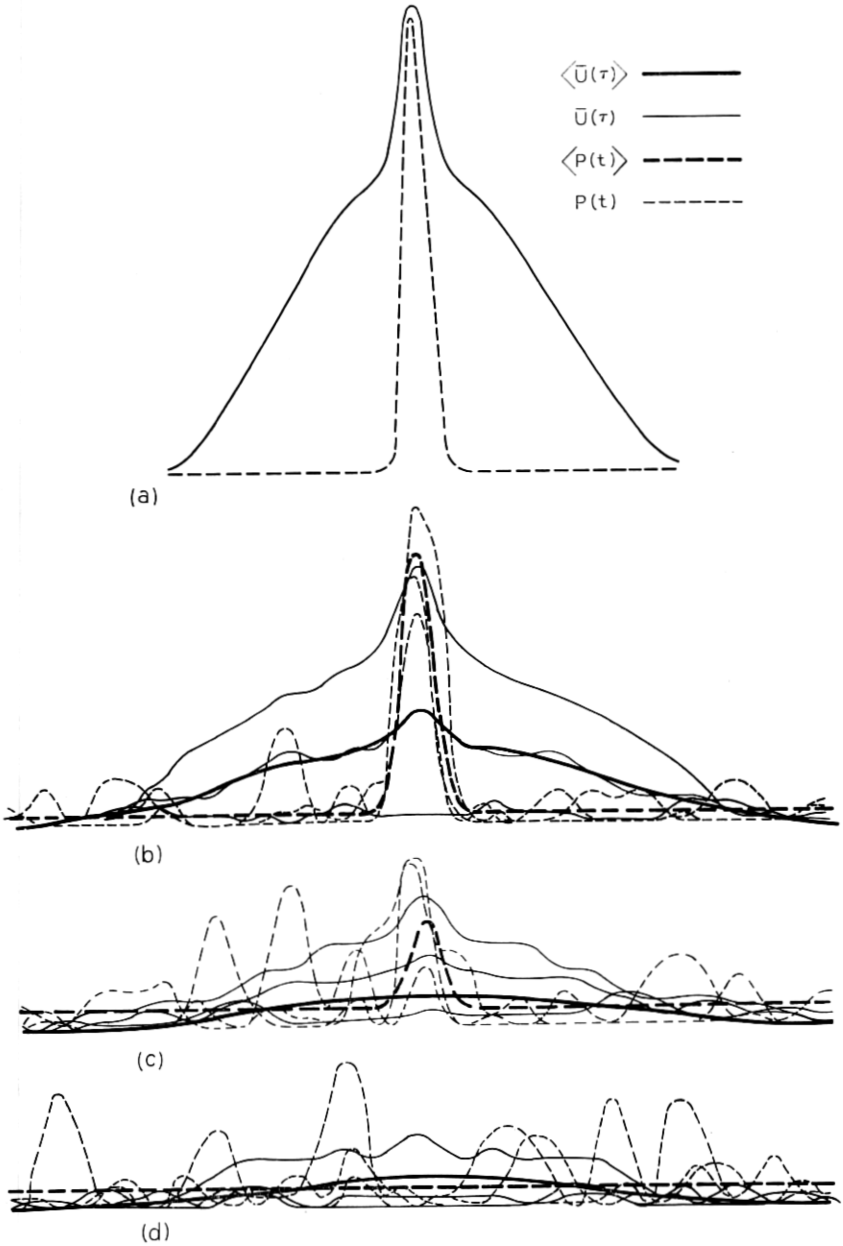

Fig. 3(a)—$\langle \bar{U}(\tau) \rangle$ and $\langle P(t) \rangle$ for ideal mode locking. (b)—$\langle \bar{U}(\tau) \rangle$, $\langle P(t) \rangle$ and three representative curves from the Monte Carlo calculation of $\bar{U}(\tau)$ and $P(t)$ for $\epsilon = 0.5$. (c)—$\langle \bar{U}(\tau) \rangle$, $\langle P(t) \rangle$ and three representative curves from the Monte Carlo calculation of $\bar{U}(\tau)$ and $P(t)$ for $\epsilon = 0.7$. (d)—$\langle \bar{U}(\tau) \rangle$, $\langle P(t) \rangle$ and three representative curves from the Monte Carlo calculation of $\bar{U}(\tau)$ and $P(t)$ for $\epsilon = 1$.

where

$$G_1 = b(\sum_n B_n)^2 + (1 - b) \sum_n B_n^2 + 4(a - b) \sum_n B_n B_{n+k} ,$$

$$G_2 = \tfrac{1}{2}(1 - b) \sum_n B_n^2 - (a - b) \sum_n B_n B_{n+k} .$$

We now turn to the problem of evaluating a and b. It can be shown* that

$$\langle \cos (x) \rangle = \prod_{i=1}^{n} \frac{\sin \epsilon_i}{\epsilon_i}$$

where $x = \sum_{i=1}^{n} t_i$,

t_i is uniformly distributed on $[-\epsilon_i , \epsilon_i]$,

$\{t_i\}$ is a set of statistically independent random variables.

From this, one immediately obtains:

$$a = \left(\frac{\sin \alpha}{\alpha}\right)^2 \left(\frac{\sin 2\alpha}{2\alpha}\right) = \left(\frac{\sin \alpha}{\alpha}\right)^3 \cos (\alpha),$$

$$b = \left(\frac{\sin (\alpha)}{\alpha}\right)^4 .$$

In particular

$$\begin{aligned} \langle \bar{U}_k(0) \rangle &= G_1 + G_2 + \tfrac{1}{2} b \mid \sum_n B_n \mid^2 \\ &= \tfrac{3}{2} b \mid \sum_n B_n \mid^2 + \tfrac{3}{2}(1 - b) \sum_n B_n^2 + 3(a - b) \sum_n B_n B_{n+k} . \end{aligned} \tag{17}$$

Let

$$\begin{aligned} R_k &= \langle \bar{U}_k(0) \rangle / (\tfrac{3}{2} \mid \sum_n B_n \mid^2) = \frac{\langle \bar{U}_k(0) \rangle}{\langle \bar{U}_0(0) \rangle} \\ &= b + \frac{(1 - b) \sum_n B_n^2}{(\sum_n B_n)^2} + 2(a - b) \frac{\sum_n B_n B_{n+k}}{(\sum_n B_n)^2} . \end{aligned} \tag{18}$$

Note that for $\alpha = 0$ (mode-locked case) $R_k = 1$ while for $\alpha = \pi$ (random-phase case)

$$R_k = \frac{\sum_n B_n^2}{(\sum_n B_n)^2} \equiv \rho_k .$$

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* This result, even though it is not new, is derived in the Appendix.

R_k can be thought of as the "contrast ratio" in analogy with the TPF case. Consideration of the above results shows that $R_k \approx 1$ for $\alpha < 0.5$ rad. If we assume $\rho_k = 0.1$ (a reasonable value for about 10 spectral lines contributing to the signal) and neglect the third term in equation (18) [Note that this term is always negative—therefore its omission makes the result somewhat pessimistic] we obtain the following results

α(rad)	R_k
0	1
0.5	0.86
1	0.55
1.5	0.28
2	0.14
2.5	0.10

Thus the "degradation" in "contrast ratio" becomes quite pronounced by the time α exceeds one radian. In this model the expectation value of the actual pulse power is given by

$$P(t) = \frac{1}{2}\left\{\left(1 - \frac{\sin^2 \alpha}{\alpha^2}\right) \sum_{n=-\infty}^{\infty} A_n^2 + \frac{\sin^2 \alpha}{\alpha^2} \left| \sum_{n=-\infty}^{\infty} A_n \exp(jn\omega t) \right|^2 \right\}.$$

Thus as α departs from zero the expectation value of the power separates into two terms, one a constant term proportional to $\sum A_n^2$, the other a replica of the mode locked pulse diminished in amplitude by the factor $\sin^2 \alpha/\alpha^2$. At $\alpha = 1$ radian (where R_k is beginning to fall off as $\sin^4 \alpha/\alpha^4$) the pulse term in $\langle P(t) \rangle$ is down only 1.5 dB and falling half as rapidly.

The expectation values of $U_1(\tau)$ and $P(t)$ are plotted in Fig. 3 for the gaussian case

$$A_n = \exp\left[-\left(\frac{nf}{f_0}\right)^2\right]$$

where f is the pulse repetition frequency and f_0 is related to the bandwidth of the pulse. For this figure a value of 0.1 was used for the quantity f/f_0. The results of a Monte Carlo type calculation are also illustrated in this figure.

Finally we consider the case of "chirped" gaussian pulses, i.e., a repetitive train of pulses of the form

$$E(t) = \exp\left[-\frac{1}{2}\left(\frac{t}{\tau}\right)^2\right] \exp(j\beta t^2).$$

This can be represented by the series

$$E(t) = \sum A_n \exp [j(n\omega t + \gamma n^2)]$$

where $A_n = \exp[-\frac{1}{2}(n\omega/\Omega)^2]$. The following result is derived for arbitrary A_n, however. Here $\phi_n = \gamma n^2$ and $\theta_n = \gamma(n^2 + 2nk + k^2 - n^2) = \gamma k(2n + k)$.

Consider

$$\left| \sum_{n=-\infty}^{\infty} \mathcal{B}_n \exp\left(j\frac{2\pi\zeta n}{T}\right)\right|^2$$

$$= \left| \sum_{n=-\infty}^{\infty} A_n A_{n+k} \exp [j\gamma k(2n + k)] \exp\left(j\frac{2\pi\zeta n}{T}\right)\right|^2$$

$$= \left| \sum_{n=-\infty}^{\infty} A_n A_{n+k} \exp\left(j\frac{2\pi n x}{T}\right)\right|^2$$

where $x = [\zeta + (T/\pi)k\gamma]$.

Comparing this with equation (12), we see that this "chirping" results in a decrease in the amplitude of the raised cosine term and a splitting in the τ term in equation (15). In fact

$$\bar{U}_k(\tau) = \cos^2\left(\frac{k\omega\tau}{2}\right)F_k\left(\frac{Tk\gamma}{\pi}\right) + \tfrac{1}{4}F_k\left(\frac{Tk\gamma}{\pi} + \tau\right) + \tfrac{1}{4}F_k\left(\tau - \frac{Tk\gamma}{\pi}\right)$$

where in this result $F_k(x)$ is evaluated using the unchirped pulse since the phase term was handled explicitly.

This result is readily understood when one recalls that $F(\tau)$ is formally equivalent to $I(t)$ with this substitution

$$A_n \leftrightarrow B_n ,$$

$$\phi_n \leftrightarrow \theta_n .$$

When A_n is gaussian, B_n is also, but when ϕ_n is quadratic in n, θ_n is linear. A phase shift which is linear in frequency corresponds to a shift of the origin in the "time" domain. This result is illustrated in Fig. 4.

The curve for $\gamma = 0.05$ corresponds to the case in which the pulse duration is ten times the reciprocal bandwidth of the signal. This is comparable to the observed values of pulse duration and bandwidth for Nd-Glass lasers. Thus this technique might provide a useful tool for investigating the amount of chirp on pulses from these lasers.

It is of interest to consider the second-order statistics of $\bar{U}(\tau)$ in order to estimate the amount of fluctuation to be expected in the random non-mode-locked case. In order to derive the second-order statistics,

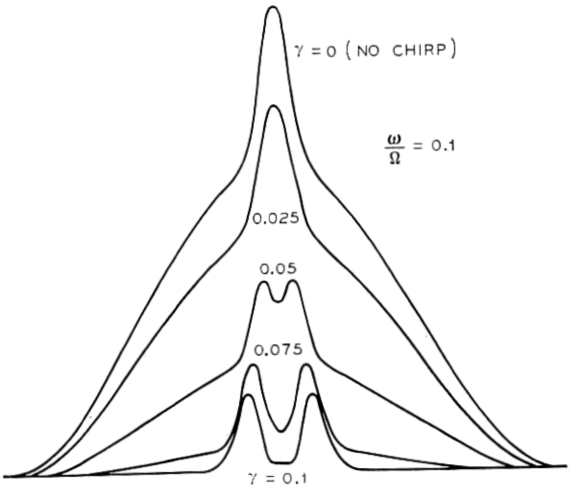

Fig. 4—$U(\tau)$ for a gaussian pulse with various amounts of chirp.

it is necessary to compute $[\bar{U}_k(\tau)]^2$ and to observe that

$$\begin{aligned}\langle\cos\{(\theta_n - \theta_m) + (\theta_{n'} - \theta_{m'})\}\rangle &= 1 \quad n = m, \quad n' = m', \\ &= 1 \quad n = m', \quad m = n', \\ &= 0 \quad \text{otherwise.} \\ \langle\cos\{(\theta_n - \theta_m) - (\theta_{n'} - \theta_{m'})\}\rangle &= 1 \quad n = m, \quad n' = m', \\ &= 1 \quad n = n', \quad m = m', \\ &= 0 \quad \text{otherwise.}\end{aligned}$$

After some tedious manipulation one obtains the variance

$$\sigma^2 = \sum_{n \neq m}\sum \left[\cos^2\left(\frac{k\omega\tau}{2}\right) + \tfrac{1}{2}\cos(n - m)\omega\tau\right] B_n^2 B_m^2 .$$

This result is plotted in Fig. 5 for the signal illustrated in Fig. 3a.

It is also instructive to consider a somewhat different experiment. If we repeat the above experiment for the case where either the phase fronts of the signals do not coincide or the beams do not overlap at the photodetector we obtain, instead of equation (4),

$$\begin{aligned}V(t) &= |\,\mathcal{E}(t)\,|^2 + |\,\mathcal{E}(t - \tau)\,|^2 \\ &= \sum_n \sum_k B_n \cos\{k\omega t + \theta_n\} \cos\frac{k\omega\tau}{2}\end{aligned}$$

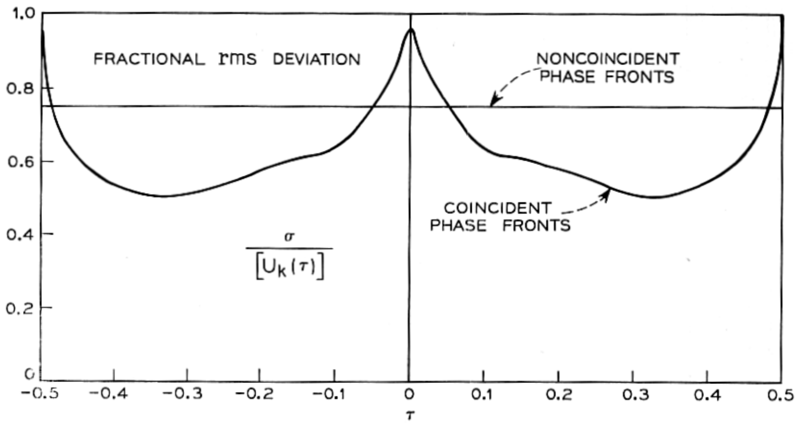

Fig. 5—Fractional rms deviation for the two statistical cases discussed in the text.

and

$$V_k(t) = \sum_n B_n \cos \{k\omega t + \theta_n\} \cos \frac{k\omega\tau}{2}$$

which upon square law detection becomes

$$U_k(\tau) = 2 \mid V_k^{(\tau)} \mid^2 = \cos^2 \frac{k\omega\tau}{2} \sum_n \sum_m B_n B_m \cos (\theta_n - \theta_m).$$

This gives the following result.

Uniformly distributed random phase:

$$\langle U_k(\tau) \rangle = \cos^2 \left(\frac{k\omega\tau}{2}\right) \sum_n B_n^2$$

$$\sigma^2 = \cos^4 \left(\frac{k\omega\tau}{2}\right) [\sum_n{}' \sum_{\substack{n \\ n \neq m}}{}' B_n^2 B_m^2].$$

Mode-locked case $\phi_n = n\Phi$)

$$U_k(\tau) = \cos^2 \left(\frac{k\omega\tau}{2}\right) (\sum_n B_n)^2.$$

This case is identical in form with the Hanbury Brown and Twiss[8] experiment.

The results for the mode-locked case and the random-phase case are summarized in Table I and illustrated in Fig. 2.

TABLE I—SUMMARY OF RESULTS FOR MODE-LOCKED CASE AND UNIFORMLY DISTRIBUTED RANDOM PHASE

	Mode-Locked Case ($\phi_n = k\Phi$)	Uniformly Distributed Random Phase
Light beams add coherently on the detector, i.e., phase fronts coincide	$\bar{U}_k(\tau) = \cos^2\left(\frac{k\omega\tau}{2}\right)\left(\sum_n B_n\right)^2 + \frac{1}{2}\left\lvert \sum_n B_n \exp(jn\omega\tau)\right\rvert^2$	$\langle \bar{U}_n(\tau)\rangle = \left[\cos^2\left(\frac{k\omega\tau}{2}\right) + \frac{1}{2}\right] \sum_n B_n^2$ $\sigma^2 = \sum_n{}' \sum_{\substack{m \\ n\neq m}}{}' \left[\cos^2\left(\frac{k\omega\tau}{2}\right) + \frac{1}{2}\cdot\cos(m-n)\omega\tau\right]^2 B_n^2 B_m^2$
Light beams do not add coherently on the detector, i.e., phase fronts do not coincide	$\bar{U}_k(\tau) = \cos^2\left(\frac{k\omega\tau}{2}\right)\left(\sum_n B_n\right)^2$	$\langle \bar{U}_k(\tau)\rangle = \cos^2\left(\frac{k\omega\tau}{2}\right) \sum_n B_n^2$ $\sigma^2 = \cos^4 \frac{k\omega\tau}{2}\left[\sum_n{}' \sum_{\substack{m \\ n\neq m}}{}' B_n^2 B_m^2\right]$

IV. EXPERIMENTAL VERIFICATION OF THE METHOD

A scaled experiment was performed using a He–Ne laser with a $c/(2\ell)$ frequency of 100 MHz. The pulse duration was measured by the BFD technique and compared with the actual pulse shape as observed by means of a high-speed photodiode and a sampling oscilloscope. Figure 6 shows the results of the experiment and Fig. 7 shows the oscilloscope display of the pulse. The pulse duration of 0.7 ns as determined from the BFD experiment is in excellent agreement with the value measured with the oscilloscope.

V. CONCLUSIONS

The experiment proposed here should be capable of distinguishing between mode-locked and non-mode-locked behavior of the laser. If mode-locked behavior is observed, a measure of the pulse width is available from the shape of the dc voltage $\bar{U}(\tau)$ versus τ curve.

The experiment is relatively simple to instrument and should be readily applicable to any CW laser whose pulse repetition rate is sufficiently low to allow construction of narrowband "IF" circuits at that frequency.

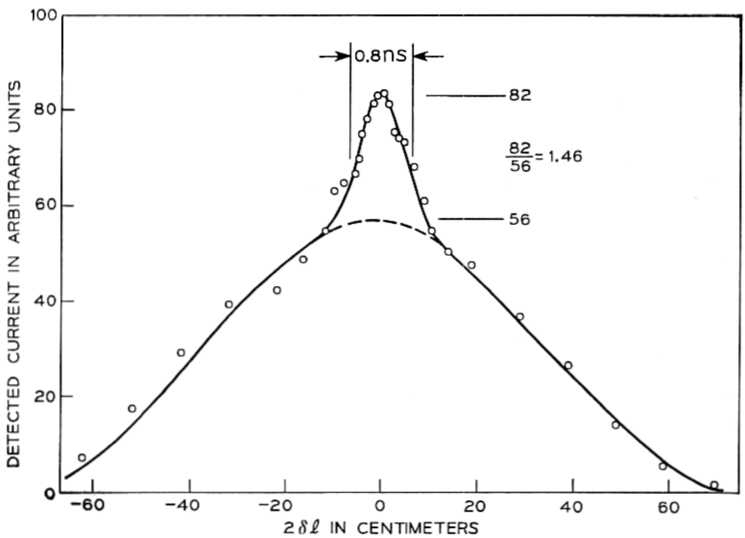

Fig. 6—Experimental results for a mode-locked He-Ne laser.

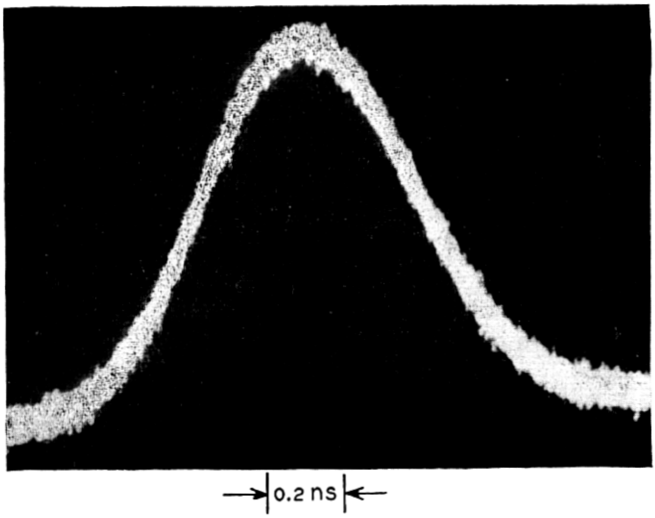

Fig. 7—Mode-locked pulse from He-Ne laser.

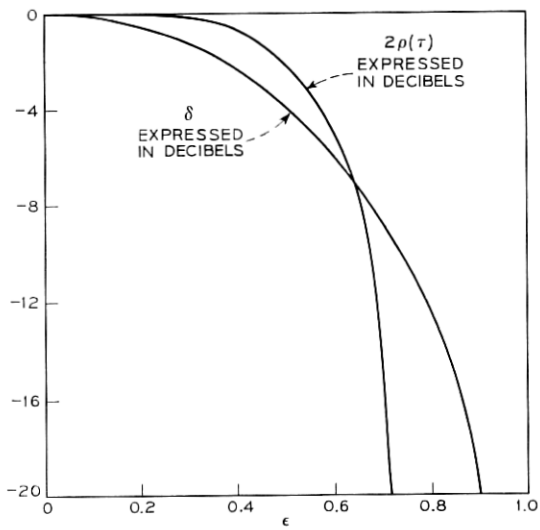

Fig. 8—Contrast ratio and pulse degradation as functions of epsilon.

VI. ACKNOWLEDGMENTS

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APPENDIX

A general form for the expectation values of $\cos(x)$ such as the ones required for the calculation in the text is derived below. In particular, it is shown by induction that

$$\langle \cos (x) \mid n \rangle = \prod_{i=1}^{n} \frac{\sin (\epsilon_i)}{\epsilon_i} \tag{19}$$

where: $x = \sum_{i=1}^{n} t_i$,

each t_i is uniformly distributed on an interval $[-\epsilon_i, \epsilon_i]$, the density function

$$\begin{aligned} f_i(t) &= 1/(2\epsilon_i), \qquad t \in [-\epsilon_i , \epsilon_i] \\ &= 0 \quad \text{otherwise} \end{aligned}$$

all of the t_i's are statistically independent.

$\langle \cos (x) \mid n \rangle$ is the expectation value of $\cos (x)$.

We calculate the $n = 1$ case:

$$\langle \cos (x) \mid 1 \rangle = \frac{1}{2\epsilon_i} \int_{-\epsilon_i}^{\epsilon_i} \cos (x)\, dx = \frac{\sin (\epsilon_1)}{\epsilon_1}. \tag{20}$$

We now assume the form (19) and show that it can be extended to the $n + 1$ case:

$$\langle \cos (x) \mid n + 1 \rangle = \int_{-\infty}^{\infty} \cos (x) F_{n+1}(x)\, dx$$

where $F_{n+1}(x)$ is the distribution function of x for $n + 1$ terms. But

$$\begin{aligned} F_{n+1}(x) &= F_n(x)\ f_{n+1}(x), \\ &= \frac{1}{2\epsilon_{n+1}} \int_{-\epsilon_{n+1}}^{\epsilon_{n+1}} F_n(x - \tau)\, d\tau. \end{aligned}$$

Thus

$$\langle \cos(x) \mid n+1 \rangle$$

$$= \frac{1}{2\epsilon_{n+1}} \int_{-\infty}^{\infty} \cos(x) \int_{-\epsilon_{n+1}}^{\epsilon_{n+1}} F_n(x-\tau)\, d\tau\, dx$$

$$= \frac{1}{2\epsilon_{n+1}} \int_{-\epsilon_{n+1}}^{\epsilon_{n+1}} \int_{-\infty}^{\infty} \{\cos(x)\cos(\tau) + \sin(x)\sin(\tau)\} F_n(x)\, dx\, d\tau$$

but the second term vanishes because $F_n(x)$, being generated by the successive convolution of even functions, is even. The integral reduces to

$$\langle \cos(x) \mid n+1 \rangle = \frac{1}{\epsilon_{n+1}} \int_{-\epsilon_{n+1}}^{\epsilon_{n+1}} \cos(\tau)\, d\tau \int_{-\infty}^{\infty} \cos(x) F_n(x)\, dx$$

$$= \frac{\sin(\epsilon_{n+1})}{\epsilon_{n+1}} \prod_{i=1}^{n} \frac{\sin(\epsilon_i)}{\epsilon_i}$$

which completes the proof.

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