# New Results on Avalanche Multiplication Statistics with Applications to Optical Detection

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*In this paper, we derive statistics of the random gain of two types of avalanche diode optical detectors. A simple optical binary receiver which could employ these devices is analyzed. In particular, we determine the moment generating function of the random gain probability density for a diode with equal hole and electron collision probabilities and for a diode with unilateral gain. For the unilateral gain case, we invert the moment generating function to obtain the probability density which turns out to be a shifted Bose Einstein density. Using the Chernoff bound, we analyze the performance of a simple binary receiver using the above devices. In addition, we exactly analyze a receiver with a deterministic gain device. We upper bound the degradation incurred from the use of a random gain rather than a deterministic gain. For the devices above, the degradation can be as small as a dB or less in certain ranges of parameter values discussed in the text.*

## I. INTRODUCTION

Practical optical direct detection receivers can employ detectors with internal gain (more than one output electron on the average per optically or thermally generated primary electron) to overcome thermal noise in amplifying stages following the detector. Since the gain is a random variable, its statistics affect the system performance. In certain systems (e.g., linear analog intensity modulation with linear processing and a mean square error risk criterion) it is sufficient to know the mean and variance of the random gain to determine performance. Various authors[1,2] have calculated those statistics for a variety of avalanche multiplier models. In digital systems with a probability of error risk, one needs to know the probability density of the gain to evaluate performance.

Up to now, little work has been published on these statistics.* In this paper, we derive the moment generating function (and in one case the density) of the multiplication statistics for two special cases of avalanche diode detectors. We apply results to performance evaluation for a simple binary optical receiver.

## II. A REVIEW OF THE AVALANCHE MULTIPLICATION PROCESS

To understand the results which follow, we briefly review some results which have appeared in the literature.

We can model an avalanche diode as follows.[1] Incident light or thermal agitation causes the generation of a hole-electron pair in a portion of the semiconductor bulk of the device. An electric field across this bulk not necessarily uniform in strength causes the carriers to drift in appropriate directions. In certain regions of the bulk where the field is sufficiently high, carriers of one type or the other pick up enough energy to suffer ionizing collisions which result in the generation of additional hole-electron pairs. Each of these secondaries may of course generate additional secondaries. We wish to find some statistics of the total number of secondaries that result from a given primary (initially generated) pair.

To solve this problem, we must pick the right bookkeeping scheme. That is, after making more detailed assumptions about the above process, we must find a scheme which will lead to an algebraically tractable way of keeping track of all of the secondaries. We shall solve for some important statistics of two particular types of avalanche multipliers in the next section.

First, for both of these types, we shall make the following assumption. We require that the field strength and bulk characteristics be such that with high probability the interval (in time or distance as you wish) between ionizing collisions of a given carrier is sufficiently large that previous collisions do not appreciably affect the statistics of a current collision. That is, we assume that we can model all ionizing collisions of a given carrier on its way through the high field region of the bulk as independent. We shall assume that each carrier type (hole and electron) has an associated probability per unit length of suffering an ionizing collision. These probabilities may be functions of position if the field in the bulk is not uniform. As a result, the

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* To the best of the author's knowledge, the only other results on the complete statistics are due to R. J. McIntyre[1] and were derived independently of these results. At the writing of this manuscript they are not yet published.

number of secondaries directly generated by a given carrier (not including secondaries generated by these secondaries) is Poisson distributed with mean equal to the integral of the generation probability per unit length over the path travelled by that carrier.

We now study two important special cases of the above. By special case it is meant that assumptions on each carrier will be made about the ionization probability per unit length.

Case 1 will correspond to a diode in which holes and electrons have equal collision ionization probabilities per unit length. In such a diode, we have a feedback situation where carriers travelling in one direction create carriers through collision travelling in the opposite direction which in turn create carriers travelling in the original direction, etc. For such a case, we shall see that the mean number of pairs directly resulting from any given pair, which is a function of the applied voltage, must be less than one in order to have a stable (non-infinite) multiplication. Case 2 will correspond to a diode where only one type of carrier causes collision generation of new pairs. Such a diode is said to have unilateral gain.

## III. CASE 1: THE EQUAL IONIZATION PROBABILITY DIODE

Suppose we assume that the ionization probabilities for holes and electrons (at a given velocity) are equal. We also assume that the shape of the high field region of the bulk is such that its length parallel to the field is constant. As a result no matter where in the bulk a hole-electron pair is created the sum of the distances in the high field region travelled by the two carriers is the same (since they travel in opposite directions). More important, the total number of secondaries directly generated by the two carriers is Poisson distributed with mean equal to the integral of the ionization probability per unit length over the total distance in the high field region parallel to the field travelled by both carriers, which is constant (see Fig. 1). The fact that the statistics of the secondaries directly generated by a given hole-electron pair is independent of where that pair itself is generated is the bookkeeping aid we shall next exploit. We discuss a type of random multiplier which is a generalized case of the above multiplication process.

## IV. A RANDOM MULTIPLIER

Consider the following random multiplier. Into the device we send an initial "count" (which for the avalanche diode corresponds to a

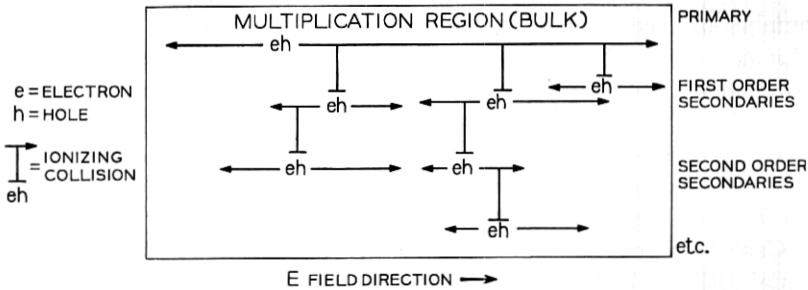


Fig. 1—Model of equal ionization device.

hole-electron pair). This count generates "directly" a random number of first-order secondary counts. What we mean by "directly" shall be clear presently. Each first-order secondary count independently generates a random number of second-order secondary counts. Analogously, each secondary of order $j$ (if any of this order are created) generates a random number of $(j + 1)^{\text{th}}$ order secondaries. For any order $j$, the probability density governing the number $k$ of $(j + 1)$th order secondaries generated by that count is $f(k)$. We emphasize that an important assumption in the ensuing analysis is the fact that a secondary of any order generates higher-order secondaries independently of what order it is and how many other counts of other orders there are.

We are interested in determining the statistics of the total number of counts of all orders (we shall call the input count an order zero secondary for convenience). Our procedure will be as follows. We shall find the statistics of the sum of the counts up to order, say, $n$ in terms of the statistics of the sum of counts up to order $n - 1$. Since we know the statistics of the sum of the counts up to order zero (namely there is one count) and order 1, we use induction to get an expression for the statistics of the total sum of all counts of all orders.

Call the number of counts of order $j$ the random variable $S_j$. Remember that these counts are randomly generated by all counts of order $j - 1$. Now call the sum of the $S_j$ from zero to $n$ inclusive the random variable $G_n$. We seek the statistics of the random variable $G_\infty$ which is the limit of the $G_n$ as $n$ goes to infinity and which is assumed to exist. (We shall soon see that for $G_\infty$ to have finite mean, the average number of direct secondaries per primary must be less than 1.)

As mentioned, we wish to determine some statistics defined upon the partial sums $G_n$. For reasons which will soon be clear, we define $f_{n,\,n-1}(x, y)$ as the discrete joint density on $G_n$ and $G_{n-1}$ (i.e., the probability that simultaneously $G_n$ equals $x$ and $G_{n-1}$ equals $y$). Now we ask the question: How can we have $G_n = x$ given $G_{n-1} = y$ assuming $x$ greater than $y$? Well, by definition $S_n$ must equal the difference $x - y$. That is, the number of $n$th-order secondaries must be $x - y$. How can this happen? We know $G_{n-1} = y$. Suppose $G_{n-2} = z$ of course less than $y$ (or equal). Therefore $S_{n-1} = y - z$. It is these secondaries (i.e., $S_{n-1}$) which must contribute $x - y$ $n$th-order secondaries. The probability that $S_{n-1} = u$ and that $G_{n-1} = y$ is just $f_{n-1,\,n-2}(y, y - u)$. The probability that the number of $n$th-order secondaries created by these $u(n - 1)$th-order secondaries is $x - y$ is the probability that $u$ independent random variables whose distributions are all $f(k)$ add up to $x - y$. This is simply $f^{*u}(x - y)$ where $f^{*u}(x - y)$ is the convolution of $f(z)$ with itself $u$ times, the result evaluated at $z = (x - y)$. (A well-known result about sums of independent random variables.) Thus the probability that $G_n$ equals $x$ and $G_{n-1} = y$ is equal to the probability that $G_{n-1} = y$ and $G_{n-2} = y - u$ and that $u$ secondaries of order $n - 1$ result in $x - y$ secondaries of order $n$—the preceding averaged over all possible values of $u$. In symbols

$$f_{n,n-1}(x, y) = \sum_z f^{*(y-z)}(x - y) f_{n-1,n-2}(y, z) \tag{1}$$

where $f^{*(x)}$ is the convolution of $f$ with itself $x$ times.

The next steps we shall take are not intuitive. They come from experience with dealing with equations like equation (1).

First to change convolutions to products, Laplace transform equation (1) on both variables $x$ and $y$ using the corresponding transform variables $s$ and $t$.

$$\begin{aligned} M_{k,k-1}(s, t) &= \sum_{x,y} f_{n,n-1}(x, y) \exp(sx + ty) = \sum_x \sum_y \sum_z f^{*(y-z)}(x - y)\,\cdot \\ &\quad \cdot f_{n-1,n-2}(y, z) \exp(sx + ty) \\ &= \sum_y \sum_z \exp[(\psi_0(s))(y - z)] \exp(sy) \exp(ty) f_{n-1,n-2}(y, z), \\ &= M_{n-1,n-2}(\psi_0(s) + s + t, -\psi_0(s)), \end{aligned} \tag{2}$$

where $\psi_0(s)$ is given by*

$$\psi_0(s) = \ln \sum_{k=0}^{\infty} f(k) \exp(sk). \tag{2a}$$

* $\ln x$ = natural logarithm of $x$.

*Lemma: The iterative relationship in equation (2) implies the iterative relation*

$$M_{n+1,n}(s, t) = \exp [s + t + \psi_0(\ln (M_{n,n-1}(s, t)))]. \tag{3}$$

*Proof:* Using induction. We know the joint moment generating function of $G_1$ and $G_0$ since $G_0 = 1$ with probability 1 and $G_1 = 1 +$ a random variable with density $f(k)$.

$$M_{1,0}(s, t) = \exp [s + t + \psi_0(s)]. \tag{4}$$

Using equation (2) we obtain

$$\begin{aligned} M_{2,1}(s, t) &= \exp [\psi_0(s) + s + t - \psi_0(s) + \psi_0(s + t + \psi_0(s))] \\ &= \exp [s + t + \psi_0(s + t + \psi_0(s))] \end{aligned} \tag{5}$$

which is also the result predicted by equation (3).

Now assume equation (3) holds for some arbitrary $n$. Then using equation (2), we obtain

$$\begin{aligned} M_{n+2,n+1}(s, t) &= M_{n+1,n}(\psi_0(s) + s + t, - \psi_0(s)) \\ &= \exp [s + t + \psi_0(s) - \psi_0(s) \\ &\quad + \psi_0(\ln (M_{n,n-1}(s + t + \psi_0(s), - \psi_0(s))))] \end{aligned} \tag{6}$$

which using our result (2) immediately yields

$$M_{n+2,n+1}(s, t) = \exp [s + t + \psi_0(\ln (M_{n+1,n}(s, t)))]. \tag{7}$$

Thus if equation (3) holds for $n = j$, it holds for $n = j + 1$. Furthermore equation (3) holds for $n = 1$. Thus the lemma is proven true.

If we set $t = 0$, then we have the moment generating function of $G_n$ in terms of the moment generating function of $G_{n-1}$.

$$\begin{aligned} p_n(m) &= \text{density of } G_n , \\ M_n(s) &= \Sigma p_n(m) \, exp \, (sm) = \Sigma f_{n,n-1}(m, r) \exp (sm) \exp (0r), \\ &= M_{n,n-1}(s, 0). \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} M_n(s) = M_{n,n-1}(s, 0) &= \exp [s + \psi_0(\ln (M_{n-1,n-2}(s, 0)))], \\ &= \exp [s + \psi_0(\ln (M_{n-1}(s)))]. \end{aligned} \tag{9}$$

Now assume that as order gets higher and higher, the probability of a secondary of that order goes down fast enough so that $M_k(s)$ converges (i.e., we get a finite number of electron-hole pairs out in response to the initial pair). Then we obtain

$$M_\infty(s) = \exp\,[s + \psi_0(\ln\,(M_\infty(s)))]. \tag{10}$$

We obtain the differential equation*

$$M'_\infty(s) = M_\infty(s)\left[1 + \psi'_0(\ln\,(M_\infty(s)))\,\frac{M'_\infty(s)}{M_\infty(s)}\right],$$

$$= \frac{M_\infty(s)}{1 - \psi'_0(\ln\,(M_\infty(s)))}; \qquad M_\infty(0) = 1. \tag{11}$$

In principle we can solve for $M_\infty(s)$ using equation (11) and we can take the inverse Laplace transform to find $p_\infty(m)$ the probability of the sum of all the secondaries. In addition we can solve for all the moments of $G_\infty$ without inverse transforming by repeated differentiation of equation (11) as will be shown in the next section.

## V. APPLICATION TO AVALANCHE DIODE

Getting back to the avalanche diode with equal carrier collision ionization probabilities, it is clear that each secondary (pair) created in the multiplication process discussed in Section III produces higher-order secondaries as discussed in Section IV with $f(k)$ a Poisson density with mean equal to the integral of the ionization probability per unit length over the path of an electron pair in the high field region of the bulk. That is

$$f(k) = Z^k \exp\,(-Z)/k! \tag{12}$$

where $Z$ is the mean of the density $f(k)$.

From equation (2a) we obtain

$$\psi_0(s) = Z[\exp\,(s) - 1]. \tag{13}$$

Plugging into equation (11), we get

$$M'_\infty(s) = \frac{M_\infty(s)}{1 - ZM_\infty(s)}. \tag{14}$$

If we call the total number of pairs in the response $G$ (for Gain) we have ($G$ is a random variable equal to the total number of electron-

* Throughout the text $x'(\cdot)$ means differentiation with respect to the total argument in parenthesis.

hole pairs at the output resulting from an input pair)

$$\bar{G} = E(G) = M'_\infty(s)\Big|_{s=0} = (1 - Z)^{-1} = \text{mean secondary pairs in response.}$$

$$\overline{G^2} = E(G^2) = M''_\infty(s)\Big|_{s=0} = \frac{M'_\infty(s)}{1 - ZM_\infty(s)}\Big|_{s=0} + \frac{ZM_\infty(s)}{(1 - ZM_\infty(s))^2}\Big|_{s=0}$$

$$= \left(\frac{1}{1 - Z}\right)^2 + \frac{Z}{(1 - Z)^3} = \left(\frac{1}{1 - Z}\right)^3$$

$$= \text{mean square number of output pairs.} \tag{15}$$

This corresponds to results by previous authors[1,2] which were limited to calculating these first two moments. We can calculate all order moments, and in addition we can numerically integrate equation (14) to obtain $M_\infty(s)$.

## VI. CASE 2: SINGLE CARRIER AVALANCHE DIODE

Returning to the discussion of Section III, suppose that only one type of carrier causes ionizing collisions. Further, assume that the probability per unit length of an ionizing collision as a carrier of that type travels through the high field region of the bulk is constant-$\beta$. Also assume that the initial carrier of the ionizing type starting the multiplication process will travel a total distance $W$ on its way through the multiplication region. We shall now set up bookkeeping.* Divide the multiplication region into increments of length $\Delta$. Model the multiplication process as follows (see Fig. 2).

Let $p_k(n)$ be the probability density of the number of carriers $n$ present at the beginning of the $k$th increment when the carriers pass that point. Assume each of these carriers generates a Poisson distributed number of new carriers in the $k$th increment with mean $\beta\Delta$. The total number of carriers at the beginning of the $(k + 1)$th increment given $N$ are present at the beginning of the $k$th increment is $N$ plus a random variable whose density is the above Poisson density convolved with itself $N$ times. Therefore we have (by argument similar to those in Section IV)

$$p_{k+1}(n) = \sum_j p_k(j) P_\Delta^{*j}(n - j) \tag{16}$$

* The reader is cautioned that our bookkeeping scheme for case 1 is quite different from that for case 2.

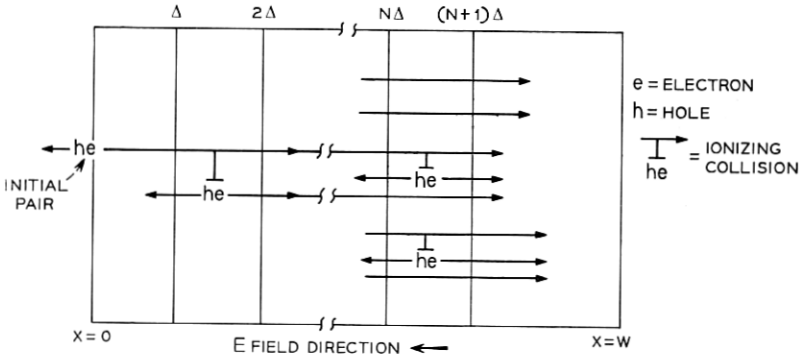


Fig. 2—Model of unilateral gain device.

where $P_\Delta^{*j}(n)$ is the $j$-fold convolution of

$$P_\Delta(n) = \frac{(\beta\Delta)^n \exp(-\beta\Delta)}{n!}$$

the result evaluated at the argument $n$. Taking the Laplace transform of equation (16) we obtain

$$\begin{aligned} M_{k+1}(s) &= \sum_n p_{k+1}(n) \exp(sn) = \sum_j p_k(j) M_\Delta^j(s) \exp(sj) \\ &= M_k(\psi_\Delta(s) + s) \end{aligned} \tag{17}$$

where

$$M_\Delta(s) = \sum_n \frac{(\beta\Delta)^n}{n!} \exp(-\beta\Delta) \exp(sn) = \exp[(\beta\Delta)(\exp(s) - 1)]$$

and

$$\psi_\Delta(s) = \beta\Delta(\exp(s) - 1).$$

*Lemma: The solution of equation (17) satisfies the iterative relation*

$$M_k(s) = M_{k-1}(s) \exp(\beta\Delta[M_{k-1}(s) - 1]). \tag{18}$$

*Proof:* At the input, we have one count, therefore

$$M_1(s) = \exp(s).$$

By equation (17) we have

$$M_2(s) = \exp(\beta\Delta(\exp(s) - 1) + s). \tag{19}$$

But equation (19) is the result predicted by equation (18). Thus (18) is true for $k = 1$. Next, assume equation (18) is true for $k = \ell$, by (18)

$$M_\ell(s) = M_{\ell-1}(s) \exp (\beta\Delta[M_{\ell-1}(s) - 1]). \tag{20}$$

Using equation (17), we obtain

$$\begin{aligned} M_{\ell+1}(s) &= M_{\ell-1}(\psi_\Delta(s) + s) \exp (\beta\Delta[M_{\ell-1}(\psi_\Delta(s) + s) - 1]) \\ &= M_\ell(s) \exp (\beta\Delta[M_\ell(s) - 1]). \end{aligned}$$

Thus by induction, the lemma is true.

We now wish to let $\Delta$ approach zero. Defining

$$\psi_k(s) = \ln M_k(s) \tag{21}$$

we obtain

$$\psi_k(s) = \psi_{k-1}(s) + \beta\Delta[\exp (\psi_{k-1}(s)) - 1].$$

Taking the limit as $\Delta \rightarrow 0$ ($\Delta$ is the width of the increment in position $x$ along the bulk)

$$\frac{\partial}{\partial x} \psi(x, s) = \beta[\exp (\psi(x, s)) - 1]. \tag{22}$$

(Where $x$ replaces $k\Delta$ as the variable of position in the multiplication region.) And where

$$\psi(0, s) = s$$

(i.e., $M(0, s) = \exp(s)$ since there is one pair at $x = 0$). Solving equation (22) we obtain

$$M(x, s) = \frac{1}{1 - \exp (\beta x)(1 - \exp (-s))} \tag{23}$$

which can be checked by differentiation.

If we evaluate equation (23) at $x = W$, we obtain

$$M(W, s) = \frac{1}{1 - \bar{G}(1 - \exp (-s))} \tag{24}$$

where $\bar{G} = \exp (\beta W) = (\partial/\partial s)M(W, s) |_{s=0}$ = mean multiplication = average number of pairs at the output in response to an initial pair.

The corresponding probability density of the total carriers leaving the multiplication region is the inverse Laplace transform of $M(W, s)$ as given by equation (24).

$$P_W(n) = \frac{1}{\bar{G}} \left( \frac{\bar{G} - 1}{\bar{G}} \right)^{n-1} \qquad n = 1, 2, 3, \cdots . \tag{25}$$

## VII. APPLICATION TO OPTICAL COMMUNICATION

In the following sections, we shall obtain bounds to the received energy per pulse required to achieve a desired error rate in a simple optical binary receiver. That is, there is some optimal adjustment of system parameters which requires a minimum of signal energy to achieve exactly a desired error rate. We cannot exactly calculate the minimum energy but we can find a range which we prove that energy falls into. We call upon the Chernoff bound[3] to upper bound the energy requirements. We obtain a lower bound by studying a receiver which we prove performs better than the receiver we wish to obtain the bounds for, but which is easy to analyze. Hopefully, the upper and lower bounds will be close enough to each other to be useful for determining the actual energy requirements. The assumed receiver is shown in Fig. 3.

## VIII. THE CHERNOFF BOUND

The Chernoff bound is useful for studying the tails of distributions. It is simply derived as follows.

Let $p(x)$ be a probability density. Suppose we wish to know $Q(\gamma)$ defined as

$$Q(\gamma) = \int_{\gamma}^{\infty} p(x)\, dx. \tag{26}$$

Let $s$ be a real number. Clearly

$$\frac{\exp(sx)}{\exp(s\gamma)} > 1 \qquad \text{for} \qquad s > 0,\, x > \gamma. \tag{27}$$

Therefore we have

$$Q(\gamma) \leqq \int_{\gamma}^{\infty} \frac{\exp(sx)}{\exp(s\gamma)}\, p(x)\, dx \leqq \int_{-\infty}^{\infty} \frac{\exp(sx)}{\exp(s\gamma)}\, p(x)\, dx \qquad \text{for} \quad s > 0. \tag{28}$$

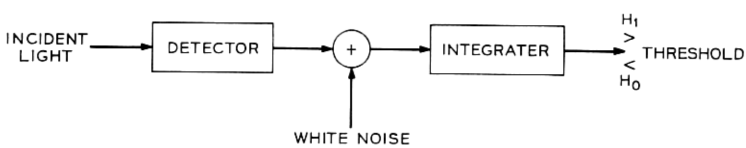


Fig. 3—Receiver.

Define the moment generating function of $x$, $M_x(s)$ as

$$M_x(s) = \int_{-\infty}^{\infty} \exp(sx)p(x)\, dx = \exp(\psi_x(s)), \tag{29}$$

where $\psi_x(s)$ is a natural log of $M_x(s)$ and is called the semi-invariant moment generating function (S.I.M.G.F.) of the random variable $x$.

Examining the bound of equation (28), we can minimize the right side (obtain the tightest bound) by choosing an optimal $s$ within the constraint region. Differentiating we obtain

$$\psi_x'(s) = \gamma \text{ as the optimal } s \text{ provided } s > 0 \tag{30}$$

(prime denotes derivative). It can be shown that a unique $s$ satisfying equation (30) exists and is greater than zero if $\gamma$ is greater than the mean of the distribution of the random variable $x$. This situation will always be satisfied for cases of interest here. We obtain the bound

$$Q(\gamma) \leqq \exp(\psi_x(s) - s\psi_x'(s)) \,|_{\psi'_x(s)=\gamma} \,. \tag{31}$$

Similarly, if we define

$$P(\gamma) = \int_{-\infty}^{\gamma} p(x)\, dx \tag{32}$$

where $\gamma$ is less than the mean of $x$, then we obtain the bound

$$P(\gamma) \leqq \exp(\psi_x(s) - s\psi_x'(s)) \,|_{\psi'_x(s)=\gamma} \tag{33}$$

(where $s$ will turn out to be negative).

Equations (31) and (33) constitute the Chernoff bounds.

## IX. APPLICATION OF THE CHERNOFF BOUND TO A SIMPLE ADDITIVE GAUSSIAN NOISE PROBLEM

Suppose an observer must distinguish between two hypotheses $H(1)$ and $H(0)$ based upon his received value of a random variable $X$. He desires to minimize his probability of error. For the case of interest here the best (minimal error test) technique simply compares the received value of the random variable to a threshold—and a decision $H(1)$ or $H(0)$ is made.

Define $p_m$, the miss probability, as the probability that the decision is $H(0)$ when $H(1)$ was actually the true hypothesis. Define $p_f$, the false alarm probability, as the probability that the decision is $H(1)$ when $H(0)$ is the true hypothesis. For a communication system where $H(1)$ and $H(0)$ are *a priori* equally probable, the error probability $p_e$

equals $\frac{1}{2}$ the miss probability $p_m$ plus $\frac{1}{2}$ the false alarm probability $p_f$. (An error occurs if either a miss or a false alarm occurs.) In this paper we shall assume that we desire $p_m \cong p_f \cong p_e$.

Suppose we have chosen a threshold, and we decide $H(1)$ if the received random variable is above threshold and $H(0)$ if the received random variable is below threshold. Let $p_1(x)$ be the probability density of the received random variable $X$ given $H(1)$ is true. Let $p_0(x)$ be the density given $H(0)$ is true. Then clearly if the threshold is $\gamma$, we have

$$p_m = \int_{-\infty}^{\gamma} p_1(x)\, dx, \qquad p_f = \int_{\gamma}^{\infty} p_0(x)\, dx. \tag{34}$$

We can bound $p_m$ and $p_f$ using the Chernoff bound if we know the semi-invariant moment generating function of the received statistic $X$ on both hypotheses.

For cases of interest here, $X$ will consist of the sum of two independent random variables. There will be a gaussian random variable of zero mean and known variance $\sigma^2$ representing thermal noise contributions. This gaussian component will be independent of which hypothesis is true. In addition there will be added a random variable $Y$, whose statistics depend upon the hypothesis. From the definition in equation (29) it is easy to see that the semi-invariant moment generating function of the sum of two independent random variables is simply the sum of the two separate semi-invariant moment generating functions. The S.I.M.G.F. of a gaussian zero mean random variable is $s^2\sigma^2/2$. Thus to apply the Chernoff bound, we must know the S.I.M.G.F. of $Y$ under both hypothesis. As we shall see, we can use equation (34) to pick the threshold and the required received energy to achieve at least a desired performance.

## X. DETAILS OF THE RECEIVER

Figure 3 is a block diagram of the system. Figure 4 schematically depicts the system in somewhat more detail. The current $i_s(t)$ is generated when incident light is present. The presence of incident light is from now on called hypothesis one. If the incident light is from a coherent or highly incoherent source (the required bandwidth depends upon the intensity), then $i_s(t)$ consists of electrons arriving as in a Poisson process with rate $\eta P/hf$ per second. $\eta$ is the quantum efficiency

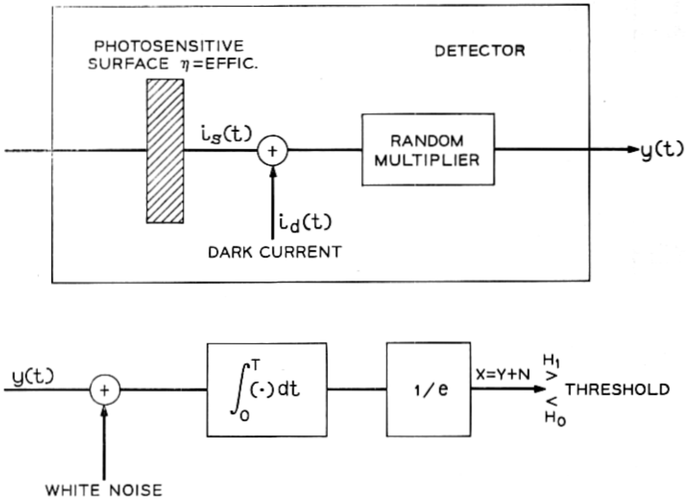


**Fig. 4—Schematic.**

of the light sensitive surface; $P$ is the incident light intensity and $hf$ is Planck's constant multiplied by the incident light center frequency. If the light is partially coherent, then the statistics of the current $i_s(t)$ are considerably more complicated. We shall not study this case here, but shall assume the Poisson statistics for both incoherent and coherent light sources.

The current $i_d(t)$ arises when electrons are spontaneously emitted in the detection device due to thermal effects. In the literature it is usually called dark current. It shall be modelled as consisting of electrons arriving as a Poisson process with rate $\lambda_d$.

The devices we are interested in employ internal current gain. That is, every electron in the processes $i_s(t)$ and $i_d(t)$ enters a random multiplication mechanism whereby it generates a random number of secondary electrons. We shall ignore the time dispersion of the secondaries around the arrival time of a primary. We are only interested here in the statistics of the number of secondaries per primary. (The electrons in $i_s(t)$ and $i_d(t)$ make up the primary process.)

We call the current at the output of the random multiplier $y(t)$. To $y(t)$ we add white gaussian noise representing the thermal noises of the receiver following the detector. The sum is integrated over the pulse duration and the resulting random variable normalized by the electron charge, $e$, is compared to a decision threshold. The random variable $X$ consists of a gaussian random variable whose mean is zero and whose variance is the thermal noise spectral height $N_0$ times the pulse duration $T^*$ and an added random variable $Y$ dependent upon the hypothesis, equal to the number of secondary electrons leaving the random multiplier during the pulse duration interval. As discussed in Section III, we need the S.I.M.G.F. of $Y$ under both hypothesis. (Note the electron charge $e$ has been absorbed by normalization into $N_0$.)

## XI. OBTAINING THE S.I.M.G.F.

As discussed above, the random variable $Y$ is generated when two Poisson processes—one due to signal if present and the other due to dark current—drive a random multiplier. In Sections II–VI, statistics of two avalanche multiplier processes are discussed. These results will be called upon soon.

Suppose we wish to know the S.I.M.G.F. of the total number of secondary electrons leaving the random multiplier in an interval. We know that the number of electrons incident upon the multiplier has Poisson statistics with mean proportional to the interval length $T$. Let $\Lambda$ equal this mean. Let $P_\Lambda(n)$ equal the Poisson density with mean $\Lambda$. Let $P_m(x)$ equal the density of the number of secondaries resulting from a given primary. The density of the total secondaries is given by

$$P_{\text{total sec}}(x) = \sum_{n=0}^{\infty} P_\Lambda(n) P_m(x)^{*n} \tag{35}$$

where the notation $P_m(x)^{*n}$ denotes convolution of $P_m(x)$ with itself $n$ times.

If $M_Y(s)$ is the moment generating function of the total secondaries, then transforming equation (35) we obtain

$$\begin{aligned} M_Y(s) &= \sum_0^{\infty} P_\Lambda(n) M_m^n(s) \\ &= \sum_0^{\infty} P_\Lambda(n) \exp(n\psi_m(s)) \\ &= M_\Lambda(\psi_m(s)) \end{aligned} \tag{36}$$

where

$M_m(s)$ is the moment generating function of the multiplication statistics, $\psi_m(s)$ is the corresponding S.I.M.G.F. [natural log of

* If $n(t)$ = thermal noise and $N = \int_0^T n(t)\,dt$ where $E[n(t)n(u)] = N_0\delta(t-u)$, then we have $E[N^2] = E[\int_0^T\int n(t)n(u)\,dt\,du] = N_0T$.

$M_m(s)$] and $M_\Lambda(s)$ is the moment generating function associated with the Poisson density.

But

$$M_\Lambda(s) = \sum_0^\infty [\Lambda^n \exp(-\Lambda)/n!] \quad \exp(sn) = \exp(\Lambda(\exp(s) - 1)). \quad (37)$$

Thus

$$\begin{aligned} M_Y(s) &= \exp(\Lambda(M_m(s) - 1)), \\ \psi_Y(s) &= \Lambda(M_m(s) - 1). \end{aligned} \quad (38)$$

Thus, if we know $\lambda_d T$ the component of $\Lambda$ due to dark current, and if we know $M_m(s)$, then we have the S.I.M.G.F. of $Y$ under both hypotheses.

## XII. APPLICATION OF CHERNOFF BOUND

Using equation (38), we know that the S.I.M.G.F. of the statistic $X$ under hypothesis zero is

$$\psi_{X0}(s) = \lambda_d T(M_m(s) - 1) + N_0 T s^2/2. \quad (39)$$

Under hypothesis one, the S.I.M.G.F. is

$$\psi_{X1}(s) = (\lambda_d + P\eta/hf)T(M_m(s) - 1) + N_0 T s^2/2. \quad (40)$$

Since equation (30) is independent of the signal power $P$, we can use the following Chernoff bound

$$p_f \leqq \exp[\psi_{X0}(s) - s\psi'_{X0}(s)] \,|_{\psi'_{x0}(s)=\gamma} \quad (41)$$

to determine an upper bound to the required $\gamma$ to achieve the desired false alarm probability. That is, we evaluate (41) at equality to determine $\gamma$ given the desired $p_f$.

Having obtained $\gamma$ in this way, we use the bound

$$p_m \leqq \exp[\psi_{X1}(s) - s\psi'_{X1}(s)] \,|_{\psi'_{x1}(s)=\gamma} \quad (42)$$

to determine the upper bound on the power $P$. That is, using the value of $\gamma$ found above and using (42) we determine an upper bound to the required power $P$. It is an upper bound since $\gamma$ is larger than is actually required, and (42) itself is a bound given $\gamma$.

## XIII. OBTAINING A LOWER BOUND

Examine the system shown in Fig. 5. Essentially we have there the system of Fig. 4, except that the random multiplier has been replaced

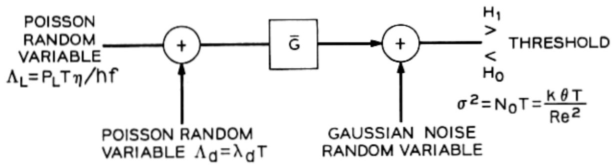


Fig. 5—Lower bounding receiver.

by a deterministic gain. It is intuitive and straightforward to prove that the system of Fig. 5 performs better than the system of Fig. 4, for a given energy per pulse and a given dark current. Therefore, if we determine the energy per pulse required for the system of Fig. 5 to achieve a given error rate, that value is a lower bound to the corresponding quantity for the system of Fig. 4.

The technique we use to analyze the system in Fig. 5 depends upon the parameter values. If under hypothesis one or under both one and zero, the mean number of electrons per pulse is large enough (greater than 250), we can simply replace the Poisson random variable by its mean. This results in a slightly looser lower bound, since we are eliminating another random quantity. Calculations indicate that the difference between the lower bounds for such large mean count numbers is small. On the other hand, for small mean count numbers on either hypothesis zero or zero and one, we should retain the Poisson statistics and use a computer to calculate the required threshold and light energy. It should be noted that the system is indifferent to a change of noise standard deviation $\sigma$ and gain $G$ provided the ratio of these two quantities is fixed. A calculation of the required mean number of counts per pulse $\Lambda_L = P_L T\eta/hf_0$ in the presence of light, assuming no dark current and a $10^{-9}$ error rate is displayed in Fig. 6.

## XIV. APPLICATION TO AVALANCHE DIODE DETECTORS

In Sections II through VI, the moment generating function we call $M_m(s)$ (of the multiplication probability density) was derived for two types of avalanche diodes. The first is a two-carrier device with equal hole and electron ionization probabilities (e.g., a Germanium diode). For such a device $M_m(s)$ must satisfy the differential equation

$$d/ds[M_m(s)] = M_m(s) \Big/ \left(1 - \left(\frac{\bar{G}-1}{\bar{G}}\right) M_m(s)\right); \qquad M_m(0) = 1 \tag{43}$$

where $\bar{G}$ is the mean multiplication.

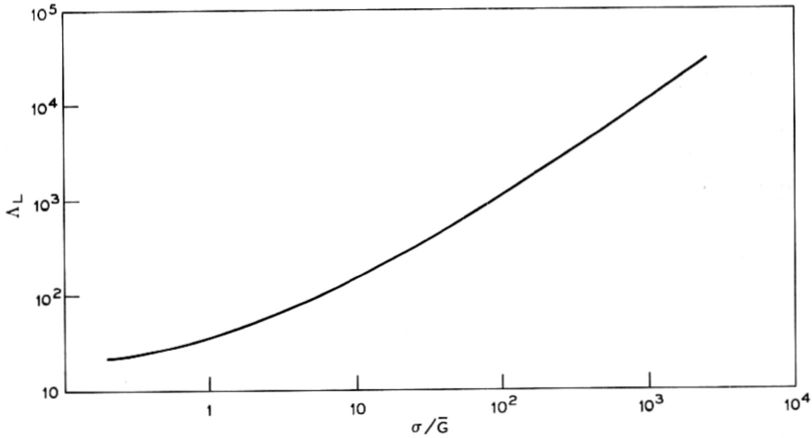


Fig. 6—Lower bound to required mean detected photons per pulse vs $\sigma/\bar{G}$. $\sigma = \sqrt{N_0 T}$.

The second type of diode is a unilateral gain device (single carrier ionization). The corresponding moment generating function is

$$M_m(s) = 1/[1 + \bar{G}\,(\exp(-s) - 1)]. \tag{44}$$

For $\bar{G}$ larger than about 10 the probability density (25) of the multiplication for this device is approximately exponential (it is exactly a shifted Bose–Einstein) and would have moment generating function

$$M_m(s) \underset{\text{expon. approx.}}{=} 1/(1 - s\bar{G}). \tag{45}$$

Suppose we wish to apply the bounds to these devices. Let us take the following special case for our calculations. We shall assume zero dark current. We will set the desired error rate to $10^{-9}$. Under these assumptions the probability of a false alarm is the probability that the thermal noise of variance $N_0T$ drives the statistic $X$ above threshold. Since the noise is gaussian, it is straightforward to see that the threshold is very nearly (to two decimal places) six standard deviations, i.e., $6(N_0T)^{\frac{1}{2}}$, for a $10^{-9}$ error rate. Using this threshold, we calculate the value of $P$, the light power needed to achieve a $10^{-9}$ miss probability.

We have

$$10^{-9} = p_m \leqq \exp\left((PT\eta/hf)(M_m(s) - 1 - M_m'(s)) - N_0Ts^2/2\right) \tag{46}$$

where we must have

$$(PT\eta/hf)M_m'(s) + N_0Ts = 6(N_0T)^{\frac{1}{2}}.$$

Solving equation (45) with equality for $P$ yields an upper bound.

The lower bound, calculated as described in Section VII as a function of the noise standard deviation over the mean gain is plotted in Fig. 6.

Without loss of generality, we solve for the upper and lower bounds to $\Lambda_s = TP_s\eta/hf$, thus eliminating the quantum efficiency and carrier frequency as parameters.

Now we must specify $N_0$ and $T$. The thermal noise has been normalized by the electron charge squared so that $Y$ could represent the total number of received electrons rather than the total charge per pulse. Therefore

$$N_0 = k\theta/(Re^2) \tag{47}$$

where

$k$ is the Boltzman constant,
$R$ is the equivalent receiver input resistance,
$e$ is the electron charge,
$\theta$ is the absolute receiver temperature (assumed 300°K).

Assuming $T$ roughly in the range of $10^{-8}$ to $10^{-11}$ seconds, and $R$ roughly 10 to 100 ohms we find that the square root of $N_0T$ the noise standard deviation is in the range of 100 to 10,000.

If we apply these parameter values to equation (46) using the multiplication statistics of (43), (44) and (45), we obtain the following results.

For the single carrier diode, and mean gains of 20 and 100 in the multiplier, the actual moment generating function (44) and exponential approximation (45) yield equivalent results within desired computational accuracy. This leads to an interesting consequence. The upper bound is independent of changes in the noise standard deviation and the mean gain, provided the ratio of these two quantities is the same. (This is a consequence of the exponential multiplication and is not a general result.) Thus, a single curve can be plotted where the abscissa represents this ratio, and the ordinate is the percent by which the upper bound $\Lambda_U$ to the required counts $\Lambda_s$ exceeds the lower bound $\Lambda_L$. In fact, we can also plot the same ordinate against the lower bound $\Lambda_L$ as abscissa. Such plots are shown in Figs. 7 and 8 respectively.

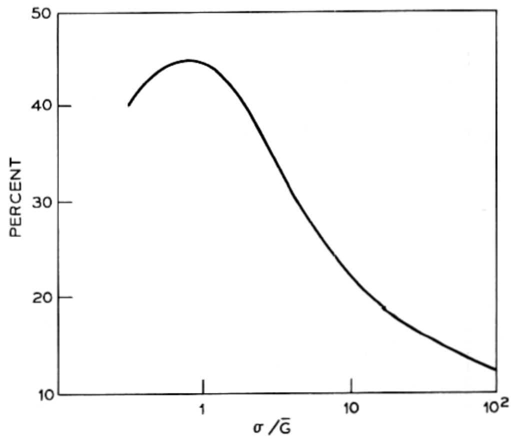


Fig. 7—Percent by which upper bound exceeds lower bound vs $\sigma/\bar{G}$ for unilateral gain diode. Percent $= ((\Lambda_U - \Lambda_L)/\Lambda_L) \times 100$.

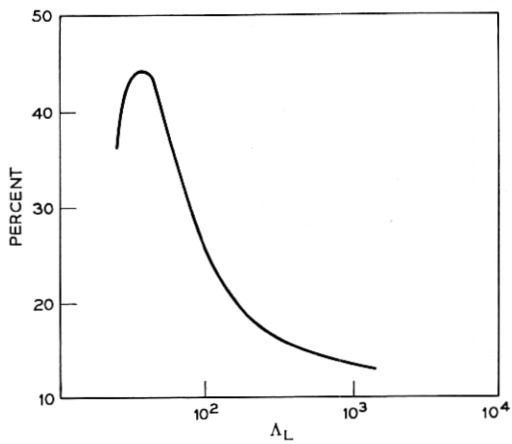


Fig. 8—Percent by which upper bound exceeds lower bound vs lower bound for unilateral gain diode.

For the two carrier diode, the above simplification does not occur. A family of curves similar to the ones above must be plotted for different mean multiplications. Two such curves (for $\bar{G} = 100$ and $\bar{G} = 20$) are plotted in Fig. 9.

14.1 *Example:*

To use the curves, assume for instance the following parameter values

mean avalanche gain $\bar{G} = 100$,
light pulse duration $T = 10^{-10}$,
dark current–negligible,
equivalent noise resistance of circuitry following avalanche detector $R = 100\Omega$,

then we have

$$N_0 = \frac{k\theta}{Re^2} = \frac{4.14 \times 10^{-21}}{2.56 \times 10^{-36}} = 1.6 \times 10^{15} ,$$

$$\sigma = (N_0 T)^{\frac{1}{2}} = 4 \times 10^2 ,$$

$$\sigma/\bar{G} = 4.$$

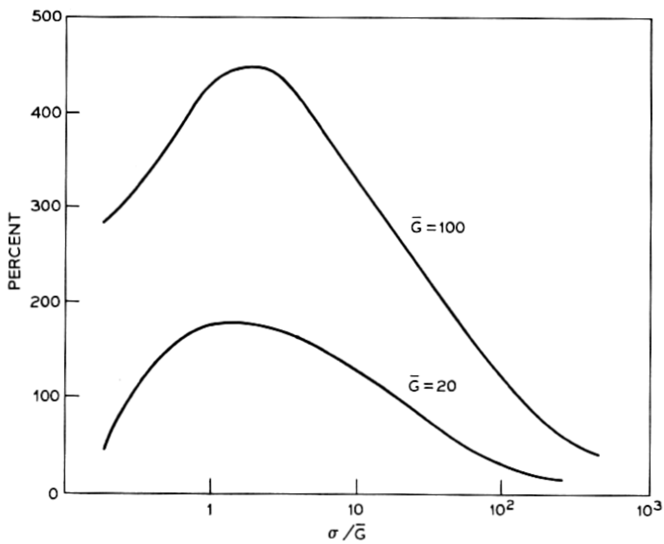


Fig. 9—Percent by which upper bound exceeds lower bound vs $\sigma/\bar{G}$ for equal ionization diode.

From Fig. 6, the lower bound $\Lambda_L = \eta P_L T/hf_0$ to the required mean number of detected photons for a $10^{-9}$ error rate is roughly 100. From Fig. 7, the upper bound for a unilateral gain device is 30 percent more or 130 photons.

From Fig. 9, the upper bound for an equal ionization avalanche diode is 500 percent more than the lower bound or 600 photons. The upper bound gives an indication of the degradation associated with random gain rather than deterministic gain.

### 14.2 *Comments*

The curves of Figs. 6–9 are intended as an example of the use of equations (39) through (42) with devices having statistics given by (43) or (44) and (45) assuming no dark current and an error rate of $10^{-9}$.

For other error rates and nonnegligible dark current, we must return to equations (39) through (45) and generate new curves.

For other types of random gain we can still use equations (39) through (42). The cases of equal electron-hole ionization and unilateral gain are probably bounds to the practical case of unequal nonzero ionizations. That is, if results on mean square gain[1,2] are any indication, the degradation for an unequal ionization device should lie between the degradations (compared to deterministic gain) of the devices studied above. To test this conjecture, we need the statistics (moment generating functions) of such unequal ionization devices.

## XV. CONCLUSIONS

We have derived the moment generating functions for two special case avalanche diodes. The important intermediate cases of unequal ionization coefficients is an important area for further study.

Application to the direct detection receiver indicates that for important ranges of parameter values we can tightly bound the energy required for a desired error rate. Further, the nature of the lower bound indicates that there may not be much degradation, for certain parameter values, due to the use of random rather than deterministic gain.

## APPENDIX

### *Glossary of Terms*

$\bar{G}$ = mean avalanche gain
$T$ = input pulse duration

$e$ = electron charge
$R$ = post detector circuit equivalent noise resistance
$\theta$ = 300°K
$k$ = Boltzmann constant
$N_0$ = $(k\theta/Re^2)$ = post detection circuit thermal noise spectral height
$\sigma$ = $\sqrt{N_0 T}$
$\Lambda_s$ = required mean number of detected photons to achieve a desired error rate
$\Lambda_U$ = upper bound to $\Lambda_s$
$\Lambda_L$ = lower bound to $\Lambda_s$

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