On Beneš Rearrangeable Networks

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V. E. Beneš considered a class of multi-stage switching networks[1] *and proved that if the linkage pattern between two stages is chosen in a specific way, then the resulting networks are rearrangeable. We offer a simpler proof by pointing out the relation between Beneš class networks and the Slepian–Duguid Theorem on three-stage Clos networks.*

I. INTRODUCTION

V. E. Beneš[1] considered the class, denoted here by $B(n_1, n_2, \cdots n_{t+1})$, of all connecting networks ν with the following properties:

(*i*) ν is two sided, with N terminals on each side where $N = \prod_{i=1}^{t+1} n_i$.
(*ii*) ν is built of an odd number $s = 2t + 1$ of stages ζ_k, $k = 1, \cdots 2t + 1$ connected as specified by permutations $\varphi_1, \cdots \varphi_{2t}$. In the notation of Beneš,

$$\nu = \zeta_1 \varphi_1 \zeta_2 \cdots \varphi_{2t} \zeta_{2t+1}.$$

(*iii*) ζ_k consists of N/n_k identical square switches of size n_k.
(*iv*) $\zeta_k = \zeta_{2t+2-k}$ for $k = 1, \cdots t$.

Beneš proceeded to prescribe a specific way of choosing φ_k (See page 113 of Ref 1):

"Order the switches of each stage; to define φ_k for a given $1 \leqq k \leqq t$, take the first switch of ζ_k, say with n_k outlet and n_k a divisor of N, and connect these outlets one to each of the first n_k switch of ζ_{k+1}; go on to the second switch of ζ_k and connect its n_k outlets one to each of the next n_k switches of ζ_{k+1}; when all the switches of ζ_{k+1} have one link on the inlet side, start again with the first switch; proceed cyclically in this way till all the outlets of ζ_k are assigned."

Beneš also specified $\varphi_k = \varphi_{2t+1-k}$ for $k = t + 1, \cdots 2t$. We call a network $\nu \in B(n_1, n_2, \cdots n_{t+1})$ constructed in this manner a cyclic Beneš network.

Beneš proved that a cyclic Beneš network is rearrangeable. The proof given by Beneš in Ref. 1 is, however, quite lengthy and involved. In the present work, we offer an alternate proof by pointing out the links between cyclic Beneš networks and the Slepian–Duguid Theorem on three-stage Clos networks. It is hoped that the simpler proof will lead to new insights into the problem of constructing rearrangeable networks.

II. MULTISTAGE REARRANGEABLE NETWORKS

We show a way to construct $\nu \in B(n_1, n_2, \cdots n_{t+1})$ from $\nu' \in B(n_2, n_3, \cdots n_{t+1})$ such that if ν' is rearrangeable then ν is rearrangeable.

We construct a three-stage network by having the first stage and the third stage each consist of N/n_1 copies of $n_1 \times n_1$ square switches, $A_1 \cdots A_{N/n_1}$, $C_1 \cdots C_{N/n_1}$ say, where the second stage consists of n_1 copies of ν', say, $B_1, \cdots B_{n_1}$. Each switch in the first stage and the third stage is then linked to every B_j in the second stage. (It does not matter which inlet or outlet of B_j is linked to which A_i or which C_i.) This gives a network $\nu \in B(n_1, n_2, \cdots n_{t+1})$. Now if ν' is also constructed in this manner and so on down to the three-stage network, we then call ν a Beneš network. That a Beneš network is rearrangeable will follow from a multistage version of the Slepian–Duguid Theorem on three-stage Clos networks. To be complete, we state this multistage version and give a proof.

Theorem 1: Let $\nu \in B(n_1, n_2, \cdots n_{t+1})$ be a Beneš network. Then ν is rearrangeable.

Proof: For $t = 0$, we have a special case of a $n_1 \times n_1$ square switch which is clearly rearrangeable and no construction is needed. Supposing that Theorem 1 is true for $t' = t - 1 \geqq 0$, we prove Theorem 1 for $t' = t$.

A maximal assignment between each inlet terminal and each outlet terminal is a permutation φ of the set of numbers $\{i : i = 1, 2, \cdots N\}$ where $N = \prod_{j=1}^{t+1} n_j$. (Following Beneš, we need only consider maximal assignments.) A given maximal assignment can also be viewed as a set of defining relationships between each switch in the first stage and each switch in the third stage (treating the set of ν' as the second stage). Set $N_2 = n_2 \cdot n_3 \cdot \cdots n_{t+1}$, note that $N = n_1 N_2$. Let A_i be the ith switch in the first stage, $A = \{A_i : i = 1, 2, \cdots N_2\}$ and C_j be the jth switch in the third stage, $C = \{C_j : j = 1, 2, \cdots N_2\}$. Consider a particular first stage switch A_i. Suppose the n_1 inlet terminals of

A_i are assigned by φ to third-stage outlet terminals, $y_{i1}, y_{i2}, \cdots y_{in}$. Denote by $Y_{ik}(Y_{ik} \varepsilon C)$ the third-stage switch that contains y_{ik}, $k = 1, 2, \cdots n_1$, and let $S_i = \{Y_{ik} : k = 1, 2, \cdots n_1\}$. Since $\bigcup_{i=1}^{x} S_i$ has a total of $x \cdot n_1$ elements and since each distinct element has only n_1 repetitions, there are at least x distinct elements in $\bigcup_{i=1}^{x} S_i$ for each $x = 1, 2, \cdots N_2$. Hence the condition of P. Hall's Theorem[2] on distinct representation of subsets is satisfied and there exists a set $Z = \{Z_i : i = 1, 2, \cdots N_2\}$ such that $Z_i \varepsilon S_i$ and $Z = C$. Then x_i, $i = 1, \cdots N_2$ can be chosen such that $x_i \varepsilon A_i$, $\varphi x_i \varepsilon Z_i$. Now we may choose to route the n_1 calls $x_i \to \varphi x_i$ through the first second-stage switch B_1, which by induction hypothesis is rearrangeable. The problem is then reduced to that of a maximal assignment in a network of type $B(n_1 - 1, n_2, \cdots n_{t+1})$. By repeatedly applying the same argument, we obtain sub-assignments on each B_k.

III. CYCLIC BENEŠ NETWORKS

Theorem 2: Every cyclic Beneš network is a Beneš network, hence is rearrangeable.

Proof: A single stage cyclic Beneš network is a $N \times N$ square matrix, which is a Beneš network. Suppose Theorem 2 is true for $(2t - 1)$ stage network where $t \geqq 1$. We prove a $(2t + 1)$ stage cyclic Beneš network is a Beneš network.

Let $\nu = \zeta_1\varphi_1\zeta_2 \cdots \varphi_{2t}\zeta_{2t+1} \varepsilon B(n_1, n_2, \cdots n_{t+1})$ be a cyclic Beneš network. We shall show that the complex $p = \zeta_2\varphi_2\zeta_3 \cdots \varphi_{2t-1}\zeta_{2t}$ can be decomposed into n_1 copies of ν' where $\nu' \varepsilon B(n_2, n_3 \cdots n_{t+1})$ is also a cyclic Beneš network. Furthermore φ_1 and φ_{2t} are such that each switch in ζ_1 and ζ_{2t+1} is linked to each ν' as is required by the definition of a Beneš network. By our inductive hypothesis, ν' is a Beneš network. This is enough to prove that ν is a Beneš network.

Let the notation $\{A/\zeta_i(\sigma)\}$ denote a set of switches $A \varepsilon \zeta_i$ in the network σ. Similarly, let $\{B/\zeta_{i+1}(\sigma)\}$ denote a set of switches $B \varepsilon \zeta_{i+1}$ in σ. We define a relation R on the two sets A and B and write

$$\{A/\zeta_i(\sigma)\}R\{B/\zeta_{i+1}(\sigma)\}$$

if every switch of A is linked to every switch of B in the network σ.

For the cyclic Beneš network $\nu = \zeta_1\varphi_1\zeta_2 \cdots \varphi_{2t}\zeta_{2t+1}$, let the switches in each stage be ordered. Note that the φ_i, for $i = 1, 2, \cdots t$, in ν can be described by,

$$\{x(\text{mod } f_i)/\zeta_i(\nu)\}R\{n_i(x - 1) + l_i : l_i = 1, 2, \cdots n_i/\zeta_{i+1}(\nu)\}$$

for

$$x = 1, 2, \cdots f_i \quad \text{where} \quad f_i = N/n_i n_{i+1} ,$$

and

$$\varphi_i = \varphi_{2t+1-i} \quad \text{for} \quad i = t + 1, \cdots 2t.$$

Decompose the compex $P = \zeta_2 \varphi_2 \zeta_3 \cdots \varphi_{2t-1} \zeta_{2t}$ into $\{\nu_1 , \nu_2 , \cdots \nu_{n_1}\}$ where ν_h , $h = 1, 2, \cdots n_1$, consists of those sets of switches:

$$\{h(\text{mod } n_1)/\zeta_2(\nu)\}$$

$$\left\{(h - 1) \prod_{j=2}^{k-1} n_j + m_k \cdot m_k = 1, 2, \cdots \prod_{j=2}^{k-1} n_j \Big/ \zeta_k(\nu)\right\}$$

$$\text{for} \quad k = 3, 4, \cdots 2t - 1$$

and

$$\{h(\text{mod } n_1)/\zeta_{2t}(\nu)\}.$$

Then clearly $\nu_h \in B(n_2 , n_3 \cdots n_{t+1})$, $h = 1, 2, \cdots n_1$.

In each ν_h , let the switches in each stage be ordered as is consistent with their orderings in ν. Suppose s_i is a switch $\in \zeta_i(\nu_h)$ and g'_i its coordinate in ν_h (i.e., s_i is the g'^{th}_i switch in $\zeta_i(\nu_h)$). Then $s_i \in \zeta_i(\nu)$ and has the coordinate g_i in ν.

If we write g'_i uniquely as

$$g'_i = C \prod_{j=2}^{i-1} n_j + d'_i \qquad \text{for} \quad C \geqq 0, \qquad 1 \geqq d'_i \geqq \prod_{j=2}^{i-1} n_j \tag{1}$$

$$\left(\text{using the convention} \prod_{j=2}^{i-1} n_j = 1 \quad \text{if} \quad i = 2\right)$$

then g_i can be uniquely written as

$$g_i = C \prod_{j=1}^{i-1} n_j + d'_i + (h - 1) \prod_{j=2}^{i-1} n_j . \tag{2}$$

Vice versa, if $s_i \in \zeta_i(\nu)$, $i \neq 1, 2t + 1$, has coordinate g_i as expressed in equation (2), then s_i is also a switch in $\zeta_i(\nu_h)$ with coordinate g'_i as expressed in equation (1).

Next we show that ν_h is a cyclic Beneš network, i.e., let $\nu_h = \zeta'_2 \varphi'_2 \zeta'_3 \cdots \varphi'_{2t-1} \zeta'_{2t}$, then φ_i , $i = 2, 3 \cdots t$ can be described by

$$\{x'(\text{mod } f'_i)/\zeta'_i(\nu_h)\} R \{n_i(x' - 1) + l_i : l_i = 1, 2, \cdots n_i/\zeta'_{i+1}(\nu_h)\} \tag{3}$$

for

$$x' = 1, 2, \cdots f'_i$$

where

$$f' = N'/n_i n_{i+1} \quad \text{and} \quad N' = N/n_1 ,$$

and

$$\varphi_k = \varphi_{2t+1-k} \quad \text{for} \quad k = t+1, \cdots 2t-1.$$

(Note that in equation (3), the two sets of switches are written in coordinates g' of ν_h.) From equations (1) and (2),

$$g_i = n_1 g_i' - (n_1 - 1)d_i' + (h-1) \prod_{j=2}^{i-1} n_j . \tag{4}$$

Let $s_i \varepsilon \zeta_i'(\nu_h)$ be a switch having coordinate $g_i' \equiv x' \pmod{f_i'}$. Then we have the unique expression

$$g_i' = q \cdot f_i' + x' \quad \text{for some} \quad q \geqq 0. \tag{5}$$

Since $\prod_{j=1}^{i-1} n_j$ divides f_i', from equations (1) and (5), there exists a unique u, $0 \leqq u \leqq \prod_{j=i+1}^{t+1} n_j$, such that

$$x' = u \prod_{j=2}^{i-1} n_j + d_i' . \tag{6}$$

From equation (4), the corresponding g_i is

$$\begin{aligned} g_i &= n_1(qf_i' + x') - (n_1 - 1)d_i' + (h-1) \prod_{j=2}^{i-1} n_j \\ &= qn_1 f_i' + n_1 x' - (n_1 - 1)d_i' + (h-1) \prod_{j=2}^{i-1} n_j \\ &= qf_i + x \end{aligned} \tag{7}$$

where $f_i = n_1 f_i'$ and

$$\begin{aligned} x &= n_1 x' - (n_1 - 1)d_i' + (h-1) \prod_{j=2}^{i-1} n_j \\ &= n_1\left(u \cdot \prod_{j=2}^{i-1} n_j + d_i'\right) - (n_1 - 1)d_i' + (h-1) \prod_{j=2}^{i-1} n_j , \qquad \text{by} \quad (5) \\ &= u \cdot \prod_{j=1}^{i-1} n_j + d_i' + (h-1) \prod_{j=2}^{i-1} n_j . \end{aligned} \tag{8}$$

It can be easily verified that $0 \leqq x < f_i$.

Equation (7) says that a switch $s_i \varepsilon \zeta_i(\nu_h)$ which has coordinate $g_i' \equiv x_i' \pmod{f_i'}$ has coordinate $g \equiv x \pmod{f_i}$ in ν.

Next we show that if the two sets $\{G_{i+1}'/\zeta_{i+1}'(\nu_h)\}$ and $\{G_{i+1}/\zeta_{i+1}(\nu)\}$

are such that

$$\{x'(\text{mod } f'_i)/\zeta'_i(\nu_h)\}R\{G'_{i+1}/\zeta'_{i+1}(\nu_h)\} \tag{9}$$

and

$$\{x(\text{mod } f_i)/\zeta_i(\nu)\}R\{G_{i+1}/\zeta_{i+1}(\nu)\} \tag{10}$$

hold, then G'_{i+1} and G_{i+1} are coordinates of the same set of switches.

Equation (9) implies

$$\begin{aligned}\{G'_{i+1}/\zeta'_{i+1}(\nu_h)\} &= \{n_i(x' - 1) + l_i : l_i = 1, 2, \cdots n_i/\zeta'_{i+1}(\nu_h)\},\\ &= \left\{n_i\left(u\cdot\prod_{j=2}^{i-1} n_j + d'_i - 1\right) + l_i : l_i = 1, 2, \cdots n_i/\zeta'_{i+1}(\nu_h)\right\},\\ &= \left\{u\prod_{j=2}^{i} n_j + (d'_i - 1)n_i + l_i : l_i = 1, 2, \cdots n_i/\zeta'_{i+1}(\nu_h)\right\}.\end{aligned}$$

Equation (10) implies

$$\begin{aligned}\{G_{i+1}/\zeta_{i+1}(\nu)\} &= \{n_i(x - 1) + l_i : l_i = 1, 2, \cdots n_i/\zeta_{i+1}(\nu)\},\\ &= \left\{n_i\left(u\cdot\prod_{j=1}^{i-1} n_j + d'_i + (h-1)\prod_{j=2}^{i-1} n_j - 1\right)\right.\\ &\qquad \left. + l_i : l_i = 1, 2, \cdots n_i/\zeta_{i+1}(\nu)\right\},\\ &= \left\{u\prod_{j=1}^{i} n_j + (d'_i - 1)n_i + (h-1)\prod_{j=2}^{i} n_j\right.\\ &\qquad \left. + l_i : l_i = 1, 2, \cdots n_i/\zeta_{i+1}(\nu)\right\}.\end{aligned}$$

But from equation (2), if $g'_i = u\prod_{j=2}^{i} n_j + (d'_i - 1)n_i + l_i$, $1 \leqq l_i \leqq n_i$, then its corresponding g_i is

$$g_i = u\cdot\prod_{j=1}^{i} n_j + (d' - 1)n_i + l_i + (h-1)\prod_{j=2}^{i} n_j\ .$$

Therefore $\{G'_{i+1}/\zeta'_{i+1}(\nu_h)\}$ and $\{G_{i+1}/\zeta_{i+1}(\nu)\}$ are clearly coordinates of the same set of switches.

That each switch in $\zeta_1(\nu)$ is linked to each ν'_h by φ_1 is a direct result of

$$\{x(\text{mod } f_1)/\zeta_1(\nu)\}R\{n_1(x - 1) + l_1 : l_1 = 1, 2, \cdots n_1/\zeta_2(\nu)\}$$

since

$$| \{h(\text{mod } f_1)\} \cap \{n_1(x - 1) + l_1 : l_1 = 1, 2, \cdots n_1\} | = 1$$

for each $h = 1, 2, \cdots n_1$.

Since $\nu = \zeta_1\varphi_1\zeta_2 \cdots \varphi_{2t}\zeta_{2t+1}$ is symmetric with respect to its middle stage, and $\nu_h \varepsilon B(n_2, n_3, \cdots n_{t+1})$, clearly $\varphi_k' = \varphi_{2t+1-k}'$ for $k = t + 1, \cdots 2t - 1$. And finally, again by an argument of symmetry, each switch in ζ_{2t+1} is linked to each ν_h' by φ_{2t}.

REFERENCES

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