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Some Considerations of Error Bounds in Digital Systems

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Simple upper and lower bounds on the distribution function of the sum of two random variables are presented in terms of the marginal distribution functions of the variables. These bounds are then used to obtain upper and lower bounds to the error probability of a coherent digital system in the presence of intersymbol interference and additive gaussian noise. The bounds are expressed in terms of the error probability obtained with a finite pulse train, and the bounds to the marginal distribution function of the residual pulse train. Since the difference between the upper and lower bounds can be shown to be a monotonically decreasing function of the number of pulses in the finite pulse train, the bounds can be used to compute the error probability of the system with arbitrarily small error. Also when the system performance is evaluated by simulation techniques, the methods presented in our paper can be utilized to estimate the error caused by using a finite pulse train approximation.

I. INTRODUCTION

In digital transmission systems the transfer characteristics of the transmitting and receiving filters are far from ideal, and the real transmission channel usually exhibits some form of time dispersion.[1,2] When an ideal digital signal is passed through such filters or is transmitted through such a channel, the successive pulses overlap; this form of distortion is usually known as intersymbol interference. Intersymbol interference may also result from the choice of nonoptimum sampling instants, imperfect demodulating-carrier phase, improper pulse design, etc. In addition the signal may be corrupted by thermal noise, co-channel and adjacent channel interference, and other forms of noise that may be present in the channel or in the system used to transmit the information.

In digital transmission systems, one of the main performance characteristics is the probability of error; this probability of error can often be expressed as a finite weighted sum of one or more distribution functions.

Various authors have tried to evaluate this probability of error by a variety of methods,[2–14] but this highly complex probability distribution can seldom be exactly computed.

Simulation techniques that may be used to solve this and other similar problems are never exact since one is constrained to use only a finite number of pulses and no bounds to the truncation error have been derived.*

Another method is an analysis by means of a worst-case or "eye pattern" analysis. Since the probability of occurrence of a worst sequence may be very small, this analysis usually leads to very pessimistic results and suboptimum system design.

Recently, some authors have derived[7–9] several different upper bounds on the probability of error when the system is subject to both intersymbol interference and additive gaussian noise. Some of these bounds make use of the Chernoff inequality in their derivation, and hence are often more useful than the worst-case bound.[8] However, since these bounds, in certain cases, can be shown to be loose,[11] and since no useful lower bounds have been derived, they are not as useful in system design as the evaluation of the exact error rate of the system.

The third method consists in using the finite pulse train approximation and calculating the error probability either by the direct enumeration of all possible sequences[2] or by the series expansion method.[10–11] The series expansion method, which involves the computation of the moments of the intersymbol interference, is a convenient method but is still inexact as no truncation error bounds due to the residual pulse train have been derived. Note that in this method the number of terms in the finite pulse train is gradually increased until the change in probability of error is less than a given number ϵ.[11]

In this paper we first present simple upper and lower bounds to the distribution function of the sum of two random variables z_N and z_R in terms of their marginal distribution functions. If the spread or dispersion[15] of the random variable z_R is smaller than the spread of the

* In simulation techniques the number N of pulses are usually chosen so that the computed probability of error stops changing by less than ϵ when the number N is increased by 1. Noting that the series $\sum_1^\infty 1/n$ diverges, and that the difference between two successive partial sums of this series can be made less than any given number ϵ, one concludes that this technique of choosing N is mathematically unsound.

random variable z_N, one can show that these two bounds are fairly close to each other and that one can evaluate the distribution function of the sum of the variables in terms of the distribution function of z_N and the bounds on the distribution function of z_R.

We then use these bounds to obtain upper and lower bounds on the error probability of a binary coherent digital system in the presence of intersymbol interference and additive gaussian noise. Since the difference between the upper and lower bounds can be shown to be a monotone decreasing function of the number N of pulses in the finite pulse train, the bounds can be used to compute the error probability of the system with arbitrarily small error.

Also when the system performance is evaluated by simulation techniques, the methods presented in our paper can be utilized to estimate the error caused by using a finite pulse train approximation.

If the symbols are equally likely, we also show that another set of upper and lower bounds can be derived for the probability of error of a system subject to intersymbol interference and additive gaussian noise.

The usefulness of the bounds is illustrated by two examples.

II. DISTRIBUTION FUNCTION AND ITS EVALUATION

Let us assume that a random variable z is the sum of two random variables z_N and z_R,

$$z = z_N + z_R , \tag{1}$$

and that we are interested in the distribution function of z

$$F_z(a) = \Pr [z \leqq a] = \Pr [z_N + z_R \leqq a]. \tag{2}$$

In this section we shall also assume that z_N and z_R are statistically independent random variables.

The probability of error of a large number of digital systems subject to various forms of noise can often be expressed as a weighted sum of $F_z(a)$'s. If z is the sum of an infinite number of random variables, and if z_N represents its partial sum of the first N terms, we sometimes can evaluate $F_{z_N}(a)$, but $F_z(a)$ can seldom be computed exactly. In such a case it is often advantageous to obtain upper and lower bounds to $F_z(a)$ in terms of $F_{z_N}(a)$ and some known parameters associated with the random variable z_R, the sum of the remaining terms in z. If the difference between the two bounds is a strictly monotone-decreasing function of N, we can then calculate $F_z(a)$ with arbitrarily small error.

Without loss of generality we shall assume that the mean of z_R is zero. From (2) we can write (see Fig. 1)

$$F_z(a) = \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f_{z_N, z_R}(x, y)\, dx\, dy \tag{3}$$

if the joint probability density function $f_{z_N, z_R}(x, y)$ exists; and

$$F_z(a) = \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} dF_{z_N}(x)\, dF_{z_R}(y) = \int_{-\infty}^{\infty} F_{z_N}(a - y)\, dF_{z_R}(y), \tag{4}$$

or

$$F_z(a) = \langle F_{z_N}(a - y_{z_R}) \rangle_{z_R} . \tag{5}$$

Let us now select an interval $(-\Delta l, \Delta u)$ from the range of the random variable z_R . From (4) we can write

$$F_z(a) = I_1 + I_2 + I_3 \tag{6}$$

where

$$I_1 \equiv \int_{-\infty}^{-\Delta l} F_{z_N}(a - y)\, dF_{z_R}(y), \tag{7}$$

$$I_2 \equiv \int_{-\Delta l}^{\Delta u} F_{z_N}(a - y)\, dF_{z_R}(y), \tag{8}$$

and

$$I_3 \equiv \int_{\Delta u}^{\infty} F_{z_N}(a - y)\, dF_{z_R}(y). \tag{9}$$

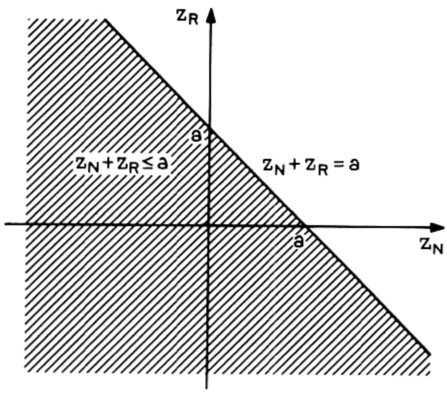

Fig. 1—Distribution function of $z = z_N + z_R$.

One can show (see Fig. 2) that

$$0 \leqq I_1 \leqq \int_{-\infty}^{-\Delta l} dF_{z_R}(y) = F_{z_R}(-\Delta l), \tag{10}$$

$$\begin{aligned} 0 \leqq I_3 \leqq F_{z_N}(a - \Delta u) \int_{\Delta u}^{\infty} dF_{z_R}(y) &= F_{z_N}(a - \Delta u)\{1 - F_{z_R}(\Delta u)\} \\ &\leqq F_{z_N}(a + \Delta l)\{1 - F_{z_R}(\Delta u)\}, \end{aligned} \tag{11}$$

$$I_2 \geqq F_{z_N}(a - \Delta u) \int_{-\Delta l}^{\Delta u} dF_{z_R}(y) = F_{z_N}(a - \Delta u)\{F_{z_R}(\Delta u) - F_{z_R}(-\Delta l)\}, \tag{12}$$

and

$$I_2 \leqq F_{z_N}(a + \Delta l) \int_{-\Delta l}^{\Delta u} dF_{z_R}(y) = F_{z_N}(a + \Delta l)\{F_{z_R}(\Delta u) - F_{z_R}(-\Delta l)\}. \tag{13}$$

Combining (6) with (10)–(13), we have

$$\begin{aligned} F_{z_N}(a - \Delta u)[F_{z_R}(\Delta u) - F_{z_R}(-\Delta l)] \\ \leqq F_z(a) \leqq F_{z_R}(-\Delta l) + F_{z_N}(a + \Delta l)[1 - F_{z_R}(-\Delta l)] \\ \leqq F_{z_R}(-\Delta l) + F_{z_N}(a + \Delta l). \end{aligned} \tag{14}$$

In general it is not easy to compute $F_{z_R}(y)$. However we may be able to bound $F_{z_R}(y)$ so that

$$0 \leqq F_{z_R}(-\Delta l) = \Pr[z_R \leqq -\Delta l] \leqq L_{z_R}(-\Delta l) \leqq 1, \tag{15}$$

$$0 \leqq 1 - F_{z_R}(\Delta u) = \Pr[z_R > \Delta u] \leqq U_{z_R}(\Delta u) \leqq 1, \tag{16}$$

and

$$\begin{aligned} 1 \geqq F_{z_R}(\Delta u) - F_{z_R}(-\Delta l) &= \Pr[-\Delta l < z_R \leqq \Delta u] \\ &\geqq 1 - L_{z_R}(-\Delta l) - U_{z_R}(\Delta u) \geqq 0. \end{aligned} \tag{17}$$

If these bounds can be found, (14)–(17) can be made to yield

$$\begin{aligned} F_{z_N}(a - \Delta u)[1 - L_{z_R}(-\Delta l) - U_{z_R}(\Delta u)] \leqq F_z(a) \\ \leqq F_{z_N}(a + \Delta l) + L_{z_R}(-\Delta l). \end{aligned} \tag{18}$$

These are the basic bounds that we shall use in the rest of this paper.

If the mass of the distribution of z_R is very much concentrated around $y = 0$, our technique of computing $F_z(a)$ from (18) relies on the assumption that we can find two numbers Δu and Δl such that

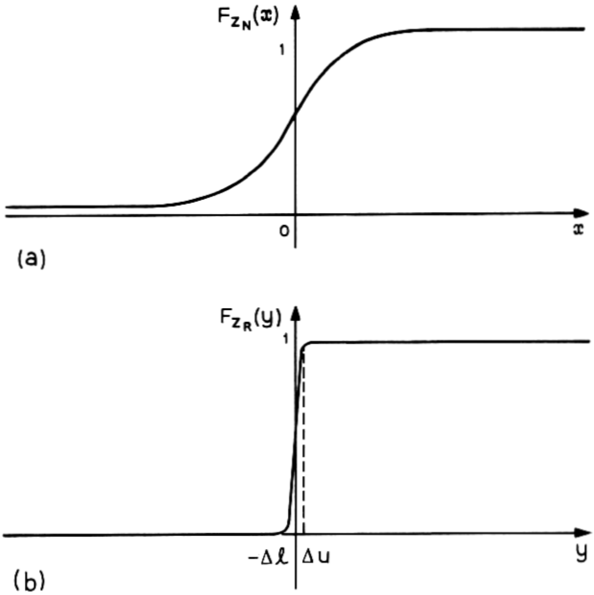

Fig. 2a—Distribution function $F_{z_N}(x)$.
Fig. 2b—Distribution function $F_{z_R}(y)$. The interval $(-\Delta l, \Delta u)$ is contained in the range of z_R, and for all practical purposes the mass of z_R is contained in $(-\Delta l, \Delta u)$.

$\Delta u \ll |a|$, $\Delta l \ll |a|$, $L_{z_R}(-\Delta l) \ll F_{z_N}(a)$, $U_{z_R}(\Delta u) \ll F_{z_N}(a)$, and $F_{z_N}(a - \Delta u) \approx F_{z_N}(a + \Delta l)$.

The difference $D(\Delta u, \Delta l)$ between the upper and lower bounds can be written as

$$D(\Delta u, \Delta l) = \Pr[a - \Delta u < z_N \leqq a + \Delta l] + F_{z_N}(a - \Delta u)[L_{z_R}(-\Delta l) + U_{z_R}(\Delta u)] + L_{z_R}(-\Delta l). \quad (19)$$

If Δu and Δl can be so chosen that they are strictly monotone-decreasing functions of N, $\Delta u \to 0$, $\Delta l \to 0$, as $N \to \infty$, and if the bounds on the distribution of z_R are such that, for sufficiently large N, $L_{z_R}(-\Delta l)$ and $U_{z_R}(\Delta u)$ can be made smaller than any given number ϵ_1, we can estimate $F_z(a)$ from (18) with arbitrarily small error.*

For any given N even though Δu and Δl can be chosen by optimizing the bounds in (18), this optimization leads to very complex equations. Hence we think that an algorithm should be developed to choose Δu and Δl for any given z_N and z_R. The development of this algorithm will be illustrated by an example in Section IV.

* We assume that $\Pr[a - 0 < a_N \leqq a + 0] = 0$.

2.1 *Lower Bound Evaluation with Convex* $F_{z_N}(a)$

We shall now derive a simpler lower (upper) bound to $F_z(a)$ if $F_{z_N}(a)$ is a convex (concave) function and if z_R is an even random variable, or

$$F_z(-t) = 1 - F_z(t). \tag{20}$$

From (20) one can show that the mean m of z is zero, and that its probability density $f_z(t)$, if it exists, satisfies the equation

$$f_z(-t) = f_z(t). \tag{21}$$

If z_R is an even random variable, we shall set $\Delta u = \Delta l$ in (18).

Let us now assume that z_R is an even random variable and that $F_{z_N}(a)$ is convex over the range $(a - \Delta u, a + \Delta u)$ where $(-\Delta u, \Delta u)$ is the range of z_R. Since z_R is an even random variable

$$F_z(a) = \langle F_{z_N}(a - y_{z_R}) \rangle_{z_R} = \langle F_{z_N}(a + y_{z_R}) \rangle_{z_R} \tag{22}$$

or

$$F_z(a) = \langle \tfrac{1}{2}[F_{z_N}(a - y_{z_R}) + F_{z_N}(a + y_{z_R})] \rangle_{z_R} . \tag{23}$$

Since $F_{z_N}(a)$ is convex over the range $(a - \Delta u, a + \Delta u)$,[16]

$$\tfrac{1}{2}[F_{z_N}(a - y_{z_R}) + F_{z_N}(a + y_{z_R})] \geqq F_{z_N}(a). \tag{24}$$

From (23) and (24) we have

$$F_z(a) \geqq F_{z_N}(a). \tag{25}$$

Since this bound does not contain Δu and Δl, it is simpler to calculate than that given in (18). It is also tighter than the lower bound in (18). In this case we then have

$$F_{z_N}(a) \leqq F_z(a) \leqq F_{z_N}(a + \Delta l) + L_{z_R}(-\Delta l). \tag{26}$$

If $F_{z_N}(a)$ is concave over the domain $(a - \Delta l, a + \Delta u)$ and if z_R is an even random variable, we can similarly show that

$$F_{z_N}(a) \geqq F_z(a) \geqq F_{z_N}(a - \Delta u)[1 - L_{z_R}(-\Delta l) - U_{z_R}(\Delta u)]. \tag{27}$$

2.2 *Evaluation of Another Upper Bound to* $F_z(a)$

Often we find that z contains a gaussian random variable n and can be written as

$$z = n + w_N + z_R = z_N + z_R , \qquad z_N = n + w_N , \tag{28}$$

where n, w_N, and z_R are statistically independent random variables.

We have already assumed that the mean of z_R is zero. Without loss of generality we shall now assume that the mean of n is zero, and its variance is σ^2.

From (28) one can show that

$$F_{z_N}(a) = \frac{1}{2}\left\langle \operatorname{erfc}\left(\frac{-a + x_{w_N}}{\sigma\sqrt{2}}\right)\right\rangle_{w_N}, \tag{29}$$

where

$$\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2)\, dt. \tag{30}$$

Hence we have[17,18]

$$F_{z_N}(a) = \tfrac{1}{2} \operatorname{erfc}\left(\frac{-a}{\sigma\sqrt{2}}\right) + \frac{1}{\sqrt{\pi}} \exp[-a^2/2\sigma^2] \cdot \sum_{k=1}^{\infty} (-1)^k H_{k-1}(-a/\sigma\sqrt{2})(1/\sigma\sqrt{2})^k \mu_k/k!, \tag{31}$$

where $H_k(x)$ is the kth order Hermite polynomial and μ_k is the kth moment of w_N,

$$\mu_k = \int_{-\infty}^{\infty} x^k\, dF_{w_N}(x). \tag{32}$$

If the range $(-\Omega_l, \Omega_u)$ of w_N is finite and if Ω denotes the maximum absolute value that can be attained by w_N, we can show that

$$|\mu_{k+s}| \leqq m_k \Omega^s, \qquad k \geqq 0,\ s \geqq 0, \tag{33}$$

$$m_k = \int_{-\infty}^{\infty} |x|^k\, dF_{w_N}(x). \tag{34}$$

m_k is called the kth absolute moment of w_N.

If the first K moments are used in estimating $F_{z_N}(a)$ from (31), the truncation error T_K is given by

$$T_K = \frac{1}{\sqrt{\pi}} \exp(-a^2/2\sigma^2) \sum_{k=K+1}^{\infty} (-1)^k H_{k-1}(-a/\sigma\sqrt{2})(1/\sigma\sqrt{2})^k \mu_k/k!. \tag{35}$$

Since it can be shown[19] that

$$|H_n(t)| < b 2^{n/2} \sqrt{n!} \exp(t^2/2), \qquad b \approx 1.086435, \tag{36}$$

one can show from (35)–(36) that

$$| T_K | < (b/\sqrt{2\pi}) \exp(-a^2/4\sigma^2) \frac{m_K}{\sigma^K} \frac{(\Omega/\sigma)}{(K+1)\sqrt{K!}} \cdot \left[1 - \frac{\Omega}{\sigma\sqrt{K+1}}\right]^{-1}, \quad \frac{\Omega}{\sigma\sqrt{K+1}} < 1. \tag{37}$$

From (31) and (37) one may observe that $F_{z_N}(a)$ may be estimated with as great an accuracy as desired if the range of w_N is finite and if the moments of w_N are known.

If w_N is an even random variable, we can also show that[10,18]

$$\mu_{2k-1} = 0, \qquad k \geqq 1 \tag{38}$$

$$F_{z_N}(a) = \tfrac{1}{2} \operatorname{erfc}(-a/\sigma\sqrt{2}) + \frac{1}{\sqrt{\pi}} \exp(-a^2/2\sigma^2) \cdot \sum_{k=1}^{\infty} H_{2k-1}(-a/\sigma\sqrt{2})(1/\sigma\sqrt{2})^{2k}\mu_{2k}/(2k)! \tag{39}$$

$$| T_{2K} | < \frac{b}{\sqrt{2\pi}} \exp(-a^2/4\sigma^2) \frac{\mu_{2K}}{\sigma^{2K}} \frac{(\Omega/\sigma)^2}{(2K+2)\sqrt{(2K+1)!}} \cdot [1 - (\Omega/\sigma)^2/\sqrt{(2K+2)(2K+3)}]^{-1}, \quad (\Omega/\sigma)^2/[(2K+2)(2K+3)]^{1/2} < 1. \tag{40}$$

By using the inequality[19]

$$| H_{2k+1}(t) | \leqq | t | \exp(t^2/2)(2k+2)!/(k+1)!, \tag{41}$$

we can also show that

$$| T_{2K} | \leqq \frac{| a |}{\sigma\sqrt{2\pi}} \exp(-a^2/4\sigma^2) \frac{\mu_{2K}}{\sigma^{2K}} \frac{(\Omega/\sigma)^2}{(K+1)!} [1 - (\Omega/\sigma)^2/(K+2)]^{-1}, \quad (\Omega/\sigma)^2/(K+2) < 1. \tag{42}$$

If z_R is an even random variable, we have

$$F_z(a) = \tfrac{1}{4}\langle \operatorname{erfc}[(-a + x_{w_N} + y_{z_R})/\sigma\sqrt{2}]\rangle_{w_N, z_R} + \tfrac{1}{4}\langle \operatorname{erfc}[(-a + x_{w_N} - y_{z_R})/\sigma\sqrt{2}]\rangle_{w_N, z_R}. \tag{43}$$

Since one can show (see Appendix A) that

$$\tfrac{1}{2} \operatorname{erfc}(x + \lambda) + \tfrac{1}{2} \operatorname{erfc}(x - \lambda) \geqq \operatorname{erfc}(x), \qquad x \geqq 0, \tag{44}$$

we can write

$$F_z(a) \geqq \tfrac{1}{2}\langle \operatorname{erfc}[(-a + x_{w_N})/\sigma\sqrt{2}]\rangle_{w_N}, \qquad -a + x_{w_N} \geqq 0,$$
$$= F_{z_N}(a), \qquad -a + x_{w_N} \geqq 0, \qquad \forall\, x_{w_N}. \tag{45}$$

Now from (28) we can write

$$F_z(a) = \tfrac{1}{2}\langle \eta_{w_N, z_R} \rangle_{w_N, z_R}, \tag{46}$$

where

$$\eta_{w_N, z_R} = \operatorname{erfc}(x_1),\ x_1 = (-a + x_{w_N} + y_{z_R})/\sigma\sqrt{2},$$

$$= \frac{2}{\sqrt{\pi}} \int_{x_2}^{\infty} \exp\{-(s + y_{z_R}/\sigma\sqrt{2})^2\}\, ds,\ x_2 = (-a + x_{w_N})/\sigma\sqrt{2}. \tag{47}$$

Since

$$\exp[-(y_{z_R}/\sigma\sqrt{2})^2] \leqq 1, \quad \forall\ y_{z_R}, \tag{48}$$

we have

$$\eta_{w_N, z_R} \leqq \frac{2}{\sqrt{\pi}} \int_{x_2}^{\infty} \exp[-s^2 - s(\sqrt{2}/\sigma)y_{z_R}]\, ds, \tag{49}$$

$$F_z(a) \leqq \frac{1}{\sqrt{\pi}} \left\langle \int_{x_2}^{\infty} \exp[-s^2 - s(\sqrt{2}/\sigma)y_{z_R}]\, ds \right\rangle_{w_N, z_R} \tag{50}$$

$$= \frac{1}{\sqrt{\pi}} \left\langle \int_{x_2}^{\infty} \exp(-s^2)\Phi_{z_R}(-s\sqrt{2}/\sigma)\, ds \right\rangle_{w_N}, \tag{51}$$

where

$$\Phi_{z_R}(t) = \int_{-\infty}^{\infty} \exp(ty)\, dF_{z_R}(y) \tag{52}$$

is the moment-generating function of the random variable z_R.

If we can find two numbers m_R and σ_R^2 such that

$$\Phi_{z_R}(t) \leqq \exp[tm_R + \sigma_R^2 t^2/2], \quad \forall\ t, \tag{53}$$

one can show from (51) that

$$F_z(a) \leqq B_{z_N}(a, m_R, \sigma_R^2) = (1 - \sigma_R^2/\sigma^2)^{-1/2} \exp[m_R^2/\{2\sigma^2(1 - \sigma_R^2/\sigma^2)\}]$$
$$\cdot \frac{1}{2} \left\langle \operatorname{erfc}\left[\frac{-a + m_R/(1 - \sigma_R^2/\sigma^2) + x_{w_N}}{\sigma 2^{1/2}(1 - \sigma_R^2/\sigma^2)^{-1/2}}\right] \right\rangle_{w_N}, \quad \sigma_R^2/\sigma^2 < 1. \tag{54}$$

The derivation of the upper bound in (54) is based on results given in Ref. 20.

In this case we then have

$$F_{z_N}(a) \leqq F_z(a) \leqq B_{z_N}(a, m_R, \sigma_R^2),$$
$$-a + x_{w_N} \geqq 0, \qquad \sigma_R^2/\sigma^2 < 1, \qquad \forall\ x_{w_N}. \tag{55}$$

Since the lower bound in (55) may not be valid if $-a + x_{wN}$ can be nonpositive for some value of x_{wN}, and if the maximum absolute value of x_{wN} is a monotone-increasing function of N, we note that there is an upper bound $N_{\max}$ to N that can be used in estimating the lower bound in (55). If this upper bound $N_{\max} < \infty$, we may not be able to estimate $F_z(a)$ from (55) with arbitrarily small error. However if there is no finite upper bound to N such that $-a + x_{wN}$ is nonpositive (system with an "open eye pattern") and if $|m_R|$ and σ_R^2 are strictly monotone-decreasing functions of N, it is clear that we can estimate $F_z(a)$ from (55) with any desired accuracy.

III. BOUNDS ON THE TAILS OF PROBABILITY DISTRIBUTIONS

To use the bounds given in (18), it is necessary to determine $L_{zR}(\Delta l)$ and $U_{zR}(\Delta u)$. There are several methods (including numerical methods) of determining these parameters, and here we shall discuss two of them.

From Chebyshev-Bienayme bounds[15,21] we have

$$\Pr[z_R \leqq -\Delta l] \leqq \frac{(\mu_{2n})_{zR}}{(\Delta l)^{2n}}, \tag{56}$$

$$\Pr[z_R > \Delta u] \leqq \frac{(\mu_{2n})_{zR}}{(\Delta u)^{2n}}, \tag{57}$$

where

$$(\mu_{2n})_{zR} = \langle y_{zR}^{2n} \rangle. \tag{58}$$

Hence we can set

$$L_{zR}(-\alpha) = U_{zR}(\alpha) = \frac{(\mu_{2n})_{zR}}{\alpha^{2n}}. \tag{59}$$

Also in communication problems, bounds of the Chernoff type have been used on the tails of the probability distributions, and these Chernoff bounds are often tighter than the Chebyshev-Bienayme bounds.[7-9,21-23]

One can show[23] that

$$\begin{aligned}\Pr[z_R \leqq -\Delta l] &\leqq \exp(-\lambda\,\Delta l)\langle \exp(-\lambda y_{zR})\rangle \\ &= \exp(-\lambda\,\Delta l)\Phi_{zR}(-\lambda), \qquad \lambda \geqq 0,\end{aligned} \tag{60}$$

$$\Pr[z_R > \Delta u] \leqq \exp(-\lambda\,\Delta u)\Phi_{zR}(\lambda), \qquad \lambda \geqq 0. \tag{61}$$

The parameter λ is arbitrary and is chosen so as to optimize the bounds in (60) and (61).

If we can find two functions $\psi_{zR}(-\lambda)$ and $\Psi_{zR}(\lambda)$ such that

$$0 \leqq \Phi_{zR}(-\lambda) \leqq \psi_{zR}(-\lambda), \qquad \lambda \geqq 0, \tag{62}$$

and

$$0 \leqq \Phi_{z_R}(\lambda) \leqq \Psi_{z_R}(\lambda), \qquad \lambda \geqq 0, \tag{63}$$

and then optimize $\exp(-\lambda\ \Delta l)\psi_{z_R}(-\lambda)$ and $\exp(-\lambda\ \Delta u)\Psi_{z_R}(\lambda)$, we can make

$$L_{z_R}(-\Delta l) = \exp(-\lambda_{\text{opt}}\ \Delta l)\psi_{z_R}(-\lambda_{\text{opt}}), \tag{64}$$

and

$$U_{z_R}(\Delta u) = \exp(-\lambda_{\text{opt}}\ \Delta u)\Psi_{z_R}(\lambda_{\text{opt}}). \tag{65}$$

The functions $\psi_{z_R}(-\lambda)$ and $\Psi_{z_R}(\lambda)$ are often chosen so that (64) and (65) have the desired functional forms for optimization.[7,9,24] From (52) one may note that it is not necessary to determine (explicitly) $\Phi_{z_R}(\lambda)$ to get $\psi_{z_R}(-\lambda)$ and $\Psi_{z_R}(\lambda)$. Bounds can be used to determine these functions. Also one may make use of the semi-invariant moment-generating function of z_R in determining $\Psi_{z_R}(-\lambda)$ and $\psi_{z_R}(\lambda)$.

If z_R is an even random variable, note also that

$$\Phi_{z_R}(-\lambda) = \Phi_{z_R}(\lambda), \qquad \lambda \geqq 0, \tag{66}$$

and we can make

$$\psi_{z_R}(-\lambda) = \Psi_{z_R}(\lambda), \qquad \lambda \geqq 0, \tag{67}$$

$$L_{z_R}(-\alpha) = U_{z_R}(\alpha) = \exp(-\alpha\lambda_{\text{opt}})\Psi_{z_R}(\lambda_{\text{opt}}). \tag{68}$$

IV. ERROR BOUNDS WITH INTERSYMBOL INTERFERENCE AND ADDITIVE GAUSSIAN NOISE

The methods presented in Section II are now applied to the analysis of a binary coherent digital system subject to intersymbol interference and additive gaussian noise. Various methods have been proposed to evaluate this error probability.[1–11] They provide either an upper bound to the error rate or error rate with a finite pulse train approximation.

Let us now assume that the signal at the input to the receiver detector (see Fig. 3) can be represented as

$$y(t) = \sum_{l=-\infty}^{\infty} a_l p(t - lT) + n(t), \tag{69}$$

where $n(t)$ is a gaussian random variable with mean zero and variance σ^2. We shall also assume that $\{a_l\}$ is a sequence of independent random variables, and $a_l = \pm 1$ with equal probability.

If the zeroth transmitted symbol is $a_0 = 1$ and if it is detected by

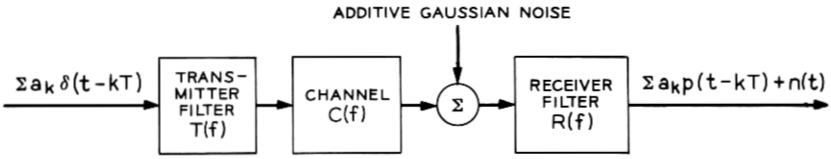

Fig. 3—Simplified block diagram of a coherent digital communication system. C(f), T(f), and R(f) denote respectively the transfer functions of the channel, and transmitting and receiving filters. T is the signaling interval.

sampling $y(t)$ at $t = t_0$, we can show that

$$y(t_0) = p(t_0) + \sum{}' a_l p(t_0 - lT) + n(t_0), \tag{70}$$

where $\sum{}'$ does not include the term $l = 0$. Assuming that the slicing level of the system is zero, and that there are no other imperfections in the system, we can show that the probability of error P_2 can be written as

$$P_2 = \Pr [n + \sum{}' a_l p_l < -p_0], \tag{71}$$

where

$$p_l = | p(t_0 - lT) |, \tag{72}$$

and

$$n = n(t_0). \tag{73}$$

Without loss of generality we shall now reorder sequence $\{p_l\}$ in such a way that the terms of the sequence are nonincreasing with increasing l, and let us denote this new sequence by $\{r_k\}$. Hence we can write

$$P_2 = \Pr \left[n + \sum_1^\infty a_k r_k < -p_0 \right] \tag{74}$$

or

$$P_2 = \Pr [z < -p_0] = F_z(-p_0), \tag{75}$$

$$z \equiv n + \sum_1^\infty a_k r_k = z_N + z_R , \tag{76}$$

$$z_N \equiv n + \sum_{k \epsilon \zeta_N} a_k r_k , \qquad \zeta_N = \{1, 2, 3, \cdots, N\}, \tag{77}$$

$$w_N = z_N - n, \tag{78}$$

$$z_R = \sum_{k \epsilon \zeta_N{}^c} a_k r_k . \tag{79}$$

Since z_N and z_R are statistically independent random variables, (18) gives bounds to $F_z(-p_0)$. Let us first determine $F_{z_N}(a)$ where $a = -p_0 + \Delta u$ or $a = -p_0 - \Delta l$. Methods given in Section 2.2 can be used in determining $F_{z_N}(a)$.* We would like to note here that (31) must be used in determining $F_{z_N}(a)$ when $+1$ and -1 do not occur with equal probability.

The recurrence relation given in Ref. 10 to calculate the even order moments μ_{2n}'s is to be used with care since the summation in the recurrence relation contains both positive and negative terms. In Appendix B we give another recurrence relation to compute μ_{2l}'s (and μ_{2l+1}'s, $l \geqq 0$). Since the new recurrence relation for μ_{2n}'s contains the summation of positive terms only, we consider this method of computing μ_{2n}'s preferable to that given in Ref. 10.

We used (31) and our new method for computing μ_{2n}'s to calculate $F_{z_N}(a)$.

We shall now determine $L_{z_R}(-\Delta l)$ and $U_{z_R}(\Delta u)$. Since z_R is an even random variable, we will set $\Delta u = \Delta l$, $L_{z_R}(-\Delta u) = U_{z_R}(\Delta u)$. Also one can show[7] that

$$\begin{aligned}\Phi_{z_R}(\lambda) &= \prod_{k \in \zeta_N^c} \cosh \lambda r_k \\ &\leqq \exp\left[\lambda \sum_{l \in \Lambda} r_l + \frac{\lambda^2}{2} \sum_{l \in \Lambda^c} r_l^2\right], \qquad \Lambda + \Lambda^c = \zeta_N . \end{aligned} \tag{80}$$

From (68) we have

$$U_{z_R}(\Delta u) = \exp\left[-\frac{\{\Delta u - \sum_{\Lambda} r_l\}^2}{2 \sum_{\Lambda^c} r_l^2}\right], \qquad \Delta u - \sum_{\Lambda} r_l \geqq 0,\ \Lambda \subset \zeta_N^c. \tag{81}$$

Equation (18) now yields

$$\begin{aligned}F_{z_N}(-p_0 - \Delta u)&\left[1 - 2\exp\left\{-\frac{[\Delta u - \sum_{\Lambda} r_l]^2}{2 \sum_{\Lambda^c} r_l^2}\right\}\right] \\ &\leqq F_z(-p_0) \leqq F_{z_N}(-p_0 + \Delta u) + \exp\left[-\frac{[\Delta u - \sum_{\Lambda} r_l]^2}{2 \sum_{\Lambda^c} r_l^2}\right]. \end{aligned} \tag{82}$$

For any given N, an optimum Δu can be chosen to minimize the difference between the upper and lower bounds in (82). This is often

* Other methods (including simulation) can also be used in determining $F_{z_N}(a)$.

found to be difficult and tedious and relies heavily on the search methods given in Ref. 7.

Here we assume that

$$\Lambda = \zeta_N^c , \tag{83}$$

and we write

$$F_{z_N}(-p_0 - \Delta u)[1 - 2 \exp \{-(\Delta u)^2/2\beta_R^2\}]$$
$$\leqq F_z(-p_0) \leqq F_{z_N}(-p_0 + \Delta u) + \exp [-(\Delta u)^2/2\beta_R^2], \tag{84}$$

$$\beta_R^2 = \sum_{\zeta_N^c} r_l^2 . \tag{85}$$

Note that any number $\beta_R^2 \geqq \sum_\Lambda r_l^2$ can be used in computing the bounds in (84). This may be done to simplify computing $\sum_\Lambda r_l^2$.

The difference $D_N(\Delta u, \Delta u)$ between the upper and lower bounds can be written as

$$D_N(\Delta u, \Delta u) = \Pr [-p_0 - \Delta u < z_N \leqq -p_0 + \Delta u]$$
$$+ \exp [-(\Delta u)^2/2\beta_R^2]\{1 + 2F_{z_N}(-p_0 - \Delta u)\}. \tag{86}$$

Since β_R^2 is a strictly monotone-decreasing function of N, $D_N(\Delta u, \Delta u)$ can be made smaller than any given number ϵ. Hence we can calculate $F_z(-p_0)$ from (84).

Several different algorithms can be developed to compute $F_z(-p_0)$. One of our algorithms is as follows. Let us assume that we have to calculate $F_z(-p_0)$ with a fractional error less than ϵ_1 .

Since $F_z(-p_0) \leqq 1$, we assume that there exists an N such that

$$| F_{z_N}(-p_0) - F_{z_{N+1}}(-p_0) | < \epsilon_2 , \tag{87}$$

where

$$\epsilon_2 \leqq \tfrac{1}{2}\epsilon_1 \min \{F_{z_N}(-p_0), F_{z_{N+1}}(-p_0)\}. \tag{88}$$

For this N we calculate β_R^2 and choose Δu so that

$$\exp [-(\Delta u)^2/2\beta_R^2] = \epsilon_2/3. \tag{89}$$

We then calculate $D_N(\Delta u, \Delta u)$ and compare it with

$$\chi_N = \epsilon_1 F_{z_N}(-p_0 - \Delta u)[1 - 2 \exp \{-(\Delta u)^2/2\beta_R^2\}]. \tag{90}$$

We increase N so that

$$D_{N'}(\Delta u, \Delta u) \leqq \chi_{N'} , \qquad N' \geqq N. \tag{91}$$

It is not necessary to increase N in steps of one. The step size can be chosen to suit particular examples.

From (18) and (91) we can write

$$A_{N'}(-p_0) \leqq F_z(-p_0) \leqq B_{N'}(-p_0), \tag{92}$$

$$A_{N'}(-p_0) \equiv F_{zN'}(-p_0 - \Delta u)[1 - 2 \exp \{-(\Delta u)^2/2\beta_R^2\}], \tag{93}$$

$$B_{N'}(-p_0) \equiv F_{zN'}(-p_0 + \Delta u) + \exp \{-(\Delta u)^2/2\beta_R^2\}, \tag{94}$$

$$B_{N'}(-p_0) - A_{N'}(-p_0) \leqq \epsilon_1 A_{N'}(-p_0). \tag{95}$$

It is evident from (92) and (95) that $F_z(-p_0)$ is equal to $A_{N'}(-p_0)$ or $B_{N'}(-p_0)$ with an error less ϵ_1 .

We have programmed this algorithm on a digital computer and we have been very successful in evaluating $F_z(-p_0)$ from this algorithm.

4.1 *Applications*

Let us now assume that $p(t)$ is obtained by passing a square pulse through a single-pole RC-filter or that

$$p(t) = 0, \qquad t < 0 \tag{96}$$

$$p(t) = 1 - \exp(-2\pi W t), \qquad 0 \leqq t \leqq T, \tag{97}$$

$$p(t) = \exp[-2\pi W(t - T)] - \exp[-2\pi W t], \qquad t \geqq T. \tag{98}$$

For this pulse we can write

$$p_0 = 1 - \exp(-2\pi W t_0), \qquad 0 \leqq t_0 \leqq T, \tag{99}$$

and

$$r_k = [1 - \exp(-2\pi W T)] \exp[-2\pi W\{t_0 + (k - 1)T\}], \qquad k \geqq 1. \tag{100}$$

For $2WT = 0.5$, and $t_0 = T$, we plot in Fig. 4 $F_z(-p_0)$ with an error less than 0.2 percent. In this figure we also plot N', the number of terms required in estimating $F_z(-p_0)$. $F_{zN}(-p_0)$ is calculated from (31) with a truncation error of less than 0.01 percent.

Let us now consider the ideal bandlimited pulse $p(t)$ where

$$p(t) = \frac{\sin \pi t/T}{\pi t/T}, \tag{101}$$

$$p_0 = \frac{\sin \pi\delta}{\pi\delta}, \ \delta = t_0 T < 1, \ t_0 \geqq 0, \tag{102}$$

$$r_{2k-1} = \frac{\sin \pi\delta}{\pi[k - \delta]}, \qquad k \geqq 1, \tag{103}$$

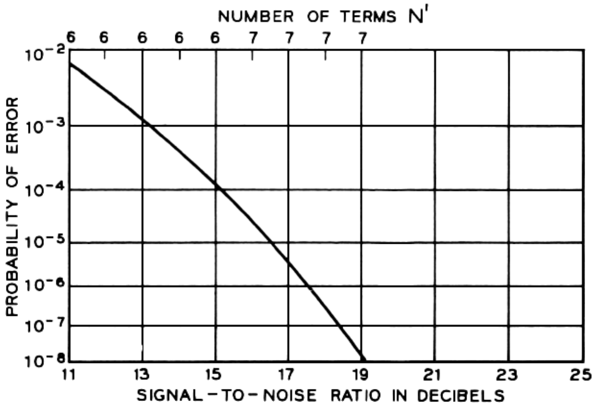

Fig. 4—Probability of error of binary coherent digital system with intersymbol interference and additive gaussian noise. The received pulse is an exponential pulse, and $2WT = 0.5$. The upper bound $B_{N'}(-p_0)$ is plotted in this figure and N was increased in steps of one. $[B_{N'}(-p_0)-F_z(-p_0)]/F_z(-p_0) < 0.002$. The truncation error is less than 0.01 percent.

$$r_{2k} = \frac{\sin \pi\delta}{\pi[k + \delta]}, \qquad k \geqq 1. \tag{104}$$

We shall assume that we take an even number of terms in w_N in estimating $F_{zN}(-p_0)$.

We have

$$\begin{aligned} \beta_R^2 &= \sum_{l=2N+1}^{\infty} r_l^2 \\ &= \sum_{k=N+1}^{\infty} \frac{\sin^2 \pi\delta}{\pi^2} \left[\frac{1}{(k - \delta)^2} + \frac{1}{(k + \delta)^2}\right] \\ &\leqq 2\,\frac{(1 + \delta^2)}{(1 - \delta^2)^2}\,\frac{\sin^2 \pi\delta}{\pi^2} \left[\frac{\pi^2}{6} - \sum_{l=1}^{N} 1/l^2\right] = \alpha_R^2\,. \end{aligned} \tag{105}$$

Since α_R^2 is more easily computed than β_R^2, we shall use α_R^2 in (84).

For $\delta = 0.05$ we plot in Fig. 5, $F_z(-p_0)$ with an error less than 50 percent when $F_z(-p_0) \geqq 2 \times 10^{-6}$ and less than 100 percent when $F_z(-p_0) < 2 \times 10^{-6}$. In this figure we also plot N' the number of terms required in estimating $F_z(-p_0)$. Since α_R^2 is a slowly decreasing function of N, the number of terms required for estimating $F_z(-p_0)$ is much larger than that in the earlier example.

Since z_N contains a gaussian random variable and since z_R is an even random variable, (55) can also be used to obtain upper and lower bounds to $F_z(-p_0)$. Equations (53) and (80) can be shown to yield

$$m_R = \sum_{\Lambda} r_l , \tag{106}$$

$$\sigma_R^2 = \sum_{\Lambda^c} r_l^2 . \tag{107}$$

By choosing $\Lambda^c = \zeta_N^c$, we obtain the bounds given in Ref. 20.

Here we would like to note that the relative merits of the two sets of bounds cannot be compared as the bounds in (55) may not be applicable when the system has a closed eye pattern. The lower bound in (55) can be shown to be tighter than that in (18) but is not applicable to a system with a noneven z_R. The random variable z_R is noneven if $+1$ and -1 do not occur with equal probability. From the point of view of computation, tightness, and applicability, we think that specific problems should determine the set of bounds best suited to them.

The extension of this analysis to m-ary coherent digital systems, $m > 2$, and binary coherent phase-shift keyed systems is obvious from

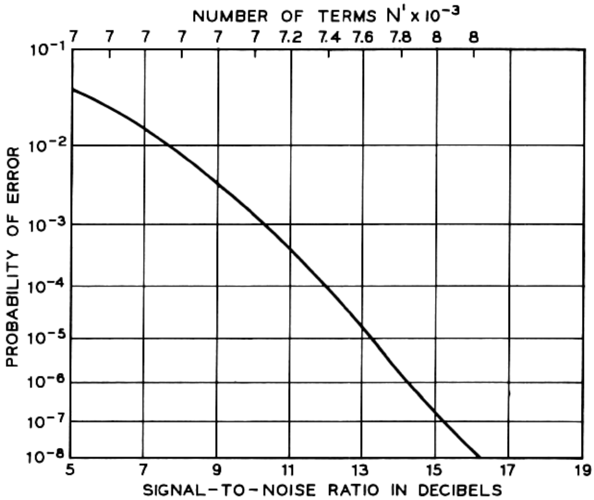

Fig. 5—Probability of error of binary coherent digital system with intersymbol interference and additive gaussian noise. The received pulse is an ideal bandlimited pulse, and it is sampled at t_0, $t_0T = 0.05$. The upper bound $B_{N'}(-p_0)$ is plotted in this figure and N was increased in steps of 100. $[B_{N'}(-p_0)-F_z(-p_0)]/F_z(-p_0) < 0.5$, $F_z(-p_0) \geq 2 \times 10^{-6}$, $[B_{N'}(-p_0)-F_z(-p_0)]/F_z(-p_0) < 1$, $F_z(-p_0) < 2 \times 10^{-6}$. The truncation error is less than 0.1 percent.

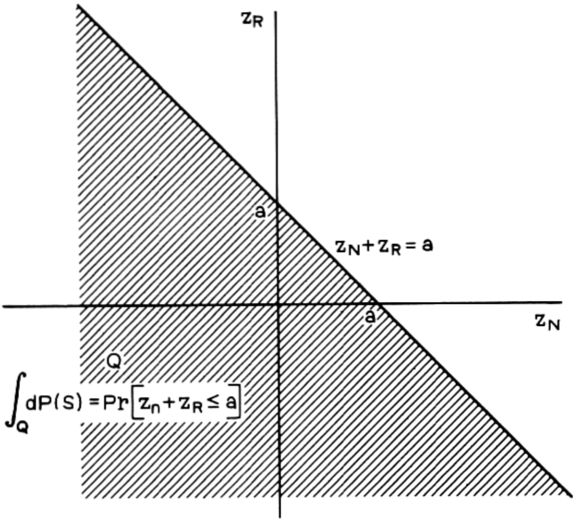

Fig. 6—Distribution function $F_z(a) = \Pr[z_N + z_R \leq a]$.

Refs. 7 and 9. The analysis for higher-order phase-shift keyed systems needs extensive modification and will be treated in a future publication.

V. DISTRIBUTION FUNCTION $F_z(a)$ WITH ARBITRARY z_N AND z_R

Consider two one-dimensional random variables z_N and z_R. The joint probability distribution of z_N and z_R is a distribution in $\mathcal{R}^2$, or a two-dimensional distribution.

Now the probability distribution of $z = z_N + z_R$ is given by (see Fig. 6)

$$F_z(a) = \Pr[z_N + z_R \leqq a]$$

$$= \int_Q dP(S), \qquad (x, y) \in Q \quad \text{if} \quad x + y \leqq a \tag{108}$$

and $P(S)$ is the probability function of z_N and z_R.[15] $P(S)$ represents the probability of the relation $(x, y) \subset S$.

Since $dP(S) \geqq 0$, note that (see Fig. 7)

$$\int_Q dP(S) \leqq \int_{Q_1} dP(S) + \int_{Q_2} dP(S), \quad (x, y) \in Q_1 \quad \text{if} \quad x \leqq a + \Delta l,$$

$$(x, y) \in Q_2 \quad \text{if} \quad y \leqq -\Delta l. \tag{109}$$

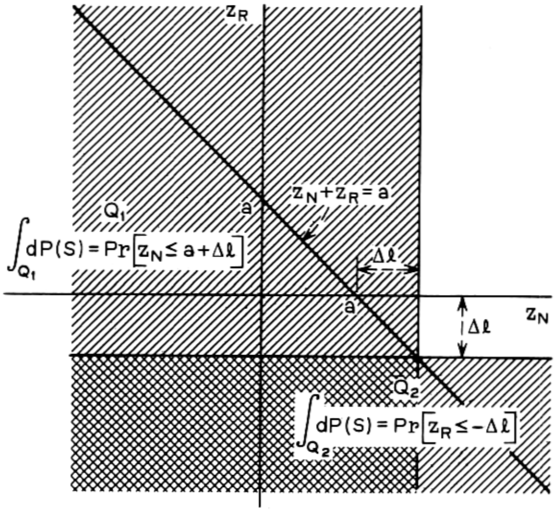

Fig. 7—Upper bound on distribution function $F_z(a)$.

Since

$$\int_{Q_1} dP(S) = \Pr\,[z_N \leqq a + \Delta l] = F_{z_N}(a + \Delta l), \tag{110}$$

$$\int_{Q_2} dP(S) = \Pr\,[z_R \leqq -\Delta l], \tag{111}$$

$$F_z(a) \leqq F_{z_N}(a + \Delta l) + \Pr\,[z_R \leqq -\Delta l]. \tag{112}$$

Also we have (see Fig. 8)

$$\int_Q dP(S) \geqq \int_{Q_3} dP(S) - \int_{Q_4} dP(S), \quad (x, y) \in Q_3 \quad \text{if} \quad x \leqq a - \Delta u,$$
$$(x, y) \in Q_4 \quad \text{if} \quad y \geqq \Delta u. \tag{113}$$

Since

$$\int_{Q_3} dP(S) = \Pr\,[z_N \leqq a - \Delta u] = F_{z_N}(a - \Delta u), \tag{114}$$

$$\int_{Q_4} dP(S) = \Pr\,[z_R \geqq \Delta u], \tag{115}$$

$$F_z(a) \geqq F_{z_N}(a - \Delta u) - \Pr\,[z_R \geqq \Delta u]. \tag{116}$$

From (112) and (116) we can write

$$F_{z_N}(a - \Delta u) - \Pr[z_R \geqq \Delta u] \leqq F_z(a) \leqq F_{z_N}(a + \Delta l) + \Pr[z_R \leqq -\Delta l]. \tag{117}$$

Equation (117) is valid even when z_N and z_R are statistically dependent random variables.

If the distribution of z_R is very much concentrated around some point $y = y_0$, it was shown in Sections II and IV that $F_z(a)$ can be evaluated with arbitrarily small error if z_N and z_R are statistically independent random variables and if we can bound $F_{z_R}(\lambda)$. If z_N and z_R are statistically dependent random variables, equation (117) shows that the same techniques can be used to compute $F_z(a)$ if the distribution of z_R is very much concentrated around some point $y = y_0$.

VI. CONCLUSIONS

We have presented simple upper and lower bounds on the distribution function of the sum of two random variables in terms of the marginal distribution functions of the variables.

We have also derived several other bounds when one of the random variables is a gaussian random variable or when one of the distribution functions is convex or concave.

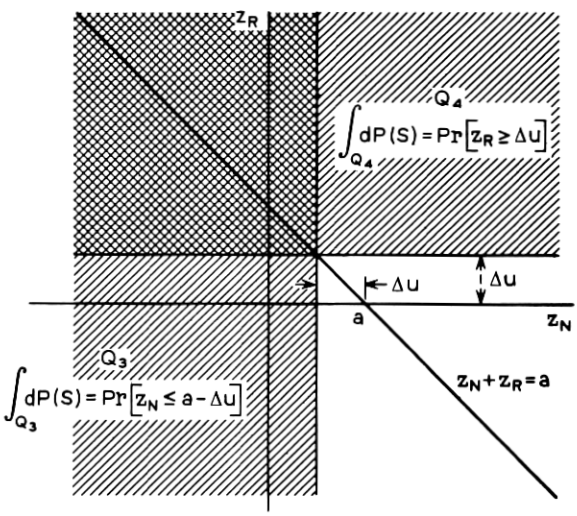

Fig. 8—Lower bound on distribution function $F_z(a)$.

These bounds are then applied to the error rate analysis of a binary coherent digital system subject to intersymbol interference and additive gaussian noise. Since the difference between the upper and lower bounds is a monotone decreasing function of the number of pulses in the finite pulse train, the bounds can be used to compute the error probability with arbitrarily small error. Application of these bounds is illustrated by two examples. Relative merits of the bounds are also briefly discussed.

Many other applications including the analysis of co-channel and adjacent channel interference in communication systems will be evident to the reader. Some such novel applications will be given in a future publication.

APPENDIX A

Let us write

$$G(\alpha) = \tfrac{1}{2} \operatorname{erfc} (x + \alpha) + \tfrac{1}{2} \operatorname{erfc} (x - \alpha). \tag{118}$$

If $x = 0$, one can easily show that

$$\tfrac{1}{2} \operatorname{erfc} (\alpha) + \tfrac{1}{2} \operatorname{erfc} (-\alpha) = 1 = \operatorname{erfc} (x). \tag{119}$$

We shall now assume that $x \neq 0$. Also since $G(\alpha)$ is an even function of α, we shall consider $\alpha \geqq 0$. From (31) and (118) we can write

$$G'(\alpha) = \frac{1}{\sqrt{\pi}} [\exp \{-(x - \alpha)^2\} - \exp \{-(x + \alpha)^2\}]. \tag{120}$$

Note that $G'(0) = 0$ and that there are no other finite stationary points of $G(\alpha)$, $x \neq 0$. Further one can show that

$$G'(\alpha) > 0, \qquad x > 0, \qquad \alpha > 0. \tag{121}$$

$$G'(\alpha) < 0, \qquad x < 0, \qquad \alpha > 0. \tag{122}$$

From (118), (121), and (122) we then have

$$\tfrac{1}{2} \operatorname{erfc} (x + \alpha) + \tfrac{1}{2} \operatorname{erfc} (x - \alpha) \geqq \operatorname{erfc} (x), \qquad x \geqq 0, \tag{123}$$

$$\tfrac{1}{2} \operatorname{erfc} (x + \alpha) + \tfrac{1}{2} \operatorname{erfc} (x - \alpha) \leqq \operatorname{erfc} (x), \qquad x \leqq 0. \tag{124}$$

For the sake of completeness we would like to note here that $\operatorname{erfc} (x)$ is a convex function for $x \geqq 0$ and is concave for $x \leqq 0$. Hence we can also show that

$$p \operatorname{erfc} (x + \alpha) + (1 - p) \operatorname{erfc} (x - \alpha)$$
$$\geqq \operatorname{erfc} (x), \quad x + \alpha \geqq 0, \quad x - \alpha \geqq 0, \quad 0 \leqq p \leqq 1, \tag{125}$$

and

$$p \operatorname{erfc}(x+\alpha) + (1-p)\operatorname{erfc}(x-\alpha) \leqq \operatorname{erfc}(x), \quad x+\alpha \leqq 0, \quad x-\alpha \leqq 0, \quad 0 \leqq p \leqq 1. \tag{126}$$

Observe that (125) is not sufficient to prove (123).

APPENDIX B

Let η_k denote the partial sum $\sum_{i=1}^{k} \xi_i$ where

$$w_N = \sum_{i=1}^{N} \xi_i , \tag{127}$$

and ξ_i's are statistically independent random variables. From (32) and (127) we can write

$$\mu_n = \langle x_{w_N}^n \rangle = \theta_n(N) \tag{128}$$

where

$$\theta_n(i) \equiv \langle \eta_i^n \rangle, \qquad n \geqq 1, \qquad \theta_0(i) = 1. \tag{129}$$

Now

$$\theta_n(k) = \langle [\eta_{k-1} + \xi_k]^n \rangle, \qquad k > 1, \tag{130}$$

or

$$\theta_n(k) = \sum_{p=0}^{n} \binom{n}{p} \theta_p(k-1)\alpha_{n-p}(k), \qquad k > 1, \tag{131}$$

where

$$\alpha_{n-p}(k) = \langle \xi_k^{n-p} \rangle, \qquad \alpha_0(k) = 1, \qquad k \geqq 1. \tag{132}$$

Since

$$\theta_n(1) = \langle \eta_1^n \rangle = \langle \xi_1^n \rangle = \alpha_n(1), \tag{133}$$

and since we shall assume that all $\alpha_{n-p}(k)$'s are known or can be evaluated, we have a recurrence relation in (131) to compute μ_n.

Often ξ_k's are even random variables, and in this case we can show that

$$\mu_{2l+1} = 0, \qquad l \geqq 0, \tag{134}$$

$$\mu_{2n} = \theta_{2n}(N), \tag{135}$$

$$\theta_{2n}(k) = \sum_{p=0}^{2n} \binom{2n}{2p} \theta_{2p}(k-1)\alpha_{2n-2p}(k). \tag{136}$$

The recurrence relation (136) contains only the sum of positive terms, and hence can easily be used to compute μ_{2n}'s.

In Section IV, $\xi_k = a_k r_k$, $\mu_{2l+1} = 0$, $l \geqq 0$ and $\alpha_{2i}(k) = r_k^{2i}$, $i \geqq 0$. All even order moments of w_N can therefore be easily calculated from (136).

In Refs. 18 and 25 methods have been developed to calculate μ_{2n} of the random variable θ where

$$\theta = \sum_{j=1}^{K} R_j \cos \theta_j \tag{137}$$

and θ_j's are independently distributed random variables uniformly distributed over the range $[0, 2\pi)$. Most of these methods use an infinite series expansion, and often the accuracy obtained from these methods is questionable.[25]

Noting that we can set $\xi_j = R_j \cos \theta_j$, $\mu_{2l+1} = 0$, $l \geqq 0$, and

$$\alpha_{2i}(j) = \langle (R_j \cos \theta_j)^{2i} \rangle$$

or

$$\alpha_{2i}(j) = R_j^{2i} \frac{(2i)!}{2^{2i}(i!)^2}, \tag{138}$$

all even order moments μ_{2n}'s can be calculated by using (136) and (138).

This method of calculating μ_{2n}'s can be shown to be analogous to that given in Ref. 26 and is preferable to that in Refs. 18 and 25.

REFERENCES

1. Lucky, R. W., Salz, J., and Weldon, E. J., Jr., *Principles of Date Communication,* New York: McGraw-Hill Book Co., 1968, pp. 59–63.
2. Aaron, M. R., and Tufts, D. W., "Intersymbol Interference and Error Probability," IEEE Trans. Inform. Theory, *IT-12,* January 1966, pp. 26–34.
3. Aein, J. M., and Hancock, J. C., "Reducing the Effects of Intersymbol Interference with Correlation Receivers," IEEE Trans. Inform. Theory, *IT-9,* July 1963, pp. 167–175.
4. Calandruno, L., Crippa, G., and Immovilli, G., "Intersymbol Interference in Binary and Quaternary PSK and DCPSK Systems," Alta Frequenza, *38,* May 1969, pp. 337–344.
5. Tufts, D. W., "Summary of Certain Intersymbol Interference Results," IEEE Trans. Inform. Theory, *IT-10,* October 1964, p. 380.
6. Saltzberg, B. R., "Error Probabilities for Binary Signal Perturbed by Intersymbol Interference and Gaussian Noise," IEEE Trans. Com. Sys., *CS-12,* March 1964, pp. 17–120.
7. Saltzberg, B. R., "Intersymbol Interference Error Bounds with Application to Ideal Bandlimited Signaling," IEEE Trans. Inform. Theory, *IT-14,* July 1968, pp. 563–568.
8. Lugannani, R., "Intersymbol Interference and Probability of Error in Digital Systems," IEEE Trans. Inform. Theory, *IT-15,* November 1970, pp. 686–688.

9. Prabhu, V. K., "Performance of Coherent Phase-Shift Keyed Systems with Intersymbol Interference," IEEE Trans. Inform. Theory, *IT-17*, No. 4 (July 1971), pp. 418–431.
10. Ho, E. Y., and Yeh, Y. S., "A New Approach for Evaluating the Error Probability in the Presence of Intersymbol Interference and Additive Gaussian Noise," B.S.T.J., *49*, No. 9 (November 1970), pp. 2249–2266.
11. Celebiler, M. I., and Shimbo, O., "Intersymbol Interference Considerations in Digital Communications," ICC Record, June 1970, pp. 8.1–8.10.
12. Shimbo, O., and Celebiler, M. I., "The Probability of Error Due to Intersymbol Interference and Gaussian Noise in Digital Communication Systems," IEEE Trans. on Com. Tech., *COM-19*, No. 2 (April 1971), pp. 113–119.
13. Mullins, J. H., "A Computer Program Simulating the Two-Level Millimeter Wave Waveguide Communication System," unpublished work.
14. Tu, P. J., "A Study of the System Impairments for the Two-Level FM-DCPSK WT-4 Transmission System," unpublished work.
15. Cramer, H., *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N. J., 1957.
16. Wagner, H. M., *Principles of Operations Research*, Englewood Cliffs, N. J.: Prentice-Hall, 1969, pp. 592–596.
17. Pagones, M. J., "Error Probability Upper Bound of a Coherently Detected PSK Signal Corrupted by Interference and Gaussian Noise," unpublished work.
18. V. K. Prabhu, "Error Rate Considerations for Coherent Phase-Shift Keyed Systems with Co-Channel Interference," B.S.T.J., *48*, No. 3 (March 1969), pp. 743–767.
19. Abramowitz, M., and Stegun, I. A., *Handbook of Mathematical Functions*, Washington, D. C.: National Bureau of Standards, 1967, p. 787.
20. Yeh, Y. S., and Ho, E. Y., "Improved Intersymbol Interference Error Bounds in Digital Systems," B.S.T.J., *50*, No. 8 (October 1971), pp. 2585–2598.
21. Algazi, V. R., "Bounds on the Spectra of Angle-Modulated Waves," IEEE Trans. Com. Tech., *COM-16*, No. 4 (August 1968), pp. 561–566.
22. Chernoff, H., "A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on a Sum of Observations," Ann. Math. Stat., *23*, April 1952, pp. 493–507.
23. Bennett, G., "On the Probability of Large Deviations from the Expectation for Sums of Bounded, Independent, Random Variables," Biometrika, *50*, 1963, pp. 528–535.
24. Prabhu, V. K., "Error-Probability Upper Bound for Coherently Detected PSK Signals with Co-Channel Interference," Elec. Letters, *5*, August 1969, pp. 383–385.
25. Rosenbaum, A. S., "Binary PSK Error Probabilities with Multiple Co-Channel Interferences," IEEE Trans. Com. Tech., *COM-18*, No. 3 (June 1970), pp. 241–253.
26. Goldman, J., "Moments of the Sum of Circularly Symmetric Random Variables," to be published.