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The Overflow Distribution for Constant Holding Time

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An infinite trunk group split into a finite first-choice group and an overflow group is studied. The equilibrium distribution, at an arbitrary instant, of the number of busy trunks in the overflow is obtained for the case of Poisson input and constant holding time. Some numerical comparisons of variances and distributions for exponential and constant holding time are given. The variance of the overflow was found to be always the greater for constant holding time, and in the case of one trunk in the first-choice group this inequality is proven to be true analytically. In some cases studied, the variances differ markedly—by as much as 50 percent. Implications of these results for the traffic engineering of overflow groups with nonexponential holding time are discussed.

I. INTRODUCTION

We consider an infinite trunk group which is split into a finite first-choice group and an overflow group. Calls that find all trunks busy in the first-choice group are placed on the overflow group. It is assumed that the input is Poisson and that the system is in equilibrium. Under these conditions the distributions of the number of calls in the total group and of that in the first-choice group are known and, for a given load, are independent of the holding-time distribution. For the case of exponential holding time the distribution of the number of calls in existence at an arbitrary instant in the overflow group is also known, having been found by Kosten in 1937.[1] The latter distribution, in particular its second moment, is basic to the method of engineering overflow groups, often called the "equivalent random" method, pioneered by Wilkinson and Bretschneider. (See Ref. 2 for a description of this method.) If this distribution were also independent of the holding-time distribution, then the equivalent random method could be applied with uniform validity regardless of the underlying holding-time distribution.

It is indeed tempting to speculate that the overflow distribution has this independence property, since the number of busy trunks in the overflow group is the difference between two random variables: the number of busy trunks in the total group and that in the first-choice group, each of which is independent of the holding-time distribution.

Unfortunately the simple example of constant holding time studied here proves that the overflow distribution does in fact depend on the holding-time distribution. This result is reminiscent of that of Tånge and Wikell,[3] who found that the blocking probability in a grading depends on the form of the holding-time distribution. In the case of the grading the differences between the blocking probabilities for constant and exponential holding times are too small to be of practical significance. In contrast, the differences between the respective overflow distributions, as measured by the relative differences between variances, can be quite large, running close to 50 percent in some cases.

Unpublished work by N. P. Archer indicates, nevertheless, that when the equivalent random method is based on constant holding time, the results of engineering a first-stage overflow group are essentially the same as when the method is based on exponential holding time. That is, for a given configuration of first-choice groups and Poisson offered loads the equivalent random method, consistently applied, results in the same size overflow group for a given loss probability whether the holding time is taken to be constant or exponential. Thus the computing aids—that is, algorithms, tables, and graphs used in the application of the equivalent random method—which are based on exponential-holding-time theory can be used without change, as a practical matter, for the purpose of engineering a single overflow group even when the holding time is constant. A word of caution, however, is in order. It should be obvious that the substitution of an exponential for a constant holding-time distribution must be done throughout the procedure, in the calculation both of the overflow variances and of the size of the equivalent group. If exponential theory is used for the overflow variances and constant theory for dimensioning the equivalent group, there will be a bias toward too few trunks in the overflow group and the service may be significantly worse than that aimed for. If the opposite error is made and the overflow variances are based on a constant holding-time assumption while the exponential charts are used for estimating the size of the equivalent group, the overflow group will be over-engineered. The latter erroneous result will occur also if constant-holding-time overflow variances are estimated by actual measurements of overflow traffic rather than, as is done presently, from theoretical considerations only, and if consideration is not given to the effect of the holding-time distribution on the overflow variance.

It is fortunate that the constant-holding-time case is tractable, since constant holding-time represents an extreme point in the set of holding-time distributions when these are ordered according to their coefficients of variation. It is reasonable to conjecture, for example, that an overflow variance for any holding-time distribution whose coefficient of variation is less than unity will differ less from that for exponential holding time than does the overflow variance for constant holding time. Thus, if the equivalent random method is applicable in the case of constant holding time, it is *a fortiori* so when the coefficient of variation of the holding time distribution is between zero and one. Furthermore, the fact that a constant-holding-time equivalent random procedure yields results which are almost indistinguishable from those for exponential holding time lends support to the conjecture that in the application of the procedure the form of the holding-time distribution may be ignored even when its coefficient of variation is greater than unity.

Although the chief purpose of this study was to gain information about the extent of the dependence of the parameters of overflow distributions on holding-time distributions, the original motivation was provided otherwise. In fact the present investigation was sparked by the observation that the formula for the decomposition of the variance of overflow traffic resulting from a superposition of independent Poisson input streams offered to the same first-choice group, derived rigorously by A. Descloux for exponential holding time in unpublished work, is valid for any holding-time distribution. This formula is

$$\mathrm{Var}\,(y_i) = p_i^2\,\mathrm{Var}\,(y) + p_i(1 - p_i)\mathrm{E}(y), \tag{1}$$

where

y_i = the number of calls in the overflow belonging to the ith stream,
p_i = the proportion of the offered load in the ith stream,

and

$$y = \textstyle\sum_i y_i\,.$$

The question immediately arises whether the observation that (1) is independent of the holding-time distribution has any application, since it involves Var (y), which was known heretofore only for the exponential case. To settle this question, a characterization of the overflow traffic in nonexponential cases is required. As it turns out, in the constant-holding-time case not only the variance but also the distribution itself may be found exactly.

Kosten's formula for the probability of y trunks busy in the overflow for an offered load of a erlangs offered to c trunks may be written, after some simplification, as

$$w_{.y}^{(E)} = \frac{a^y}{y!} \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \frac{1}{\sum_{i=0}^{c} \binom{y+j+i-1}{i} \frac{(c)_i}{a^i}} \tag{2}$$

where $(c)_0 = 1$ and $(c)_i = c(c-1) \cdots (c-i+1)$.

The corresponding formulas for the constant case are

$$w_{.y} = \frac{e^{-a}}{\sum_{i=0}^{c} \frac{a^i}{i!}} \frac{a^{c+y}}{(c+y+1)!} \sum_{h=0}^{c} \frac{a^h}{h!} (c+y-h+1)(c-h+1), \quad y > 0, \tag{3}$$

and

$$w_{.0} = \frac{e^{-a}}{\sum_{i=0}^{c} \frac{a^i}{i!}} \left\{ \sum_{h=0}^{c} \frac{(2a)^h}{h!} + \sum_{h=c+1}^{2c} \frac{a^h}{h!} \cdot \left[\sum_{i=h-c}^{c} \binom{h}{i} - \binom{h}{c+1} (2c-h+1) \right] \right\}. \tag{4}$$

(A brief outline of the derivation of (3) and (4) is given in Section II. Algebraic details of the derivations of these and subsequent formulas are given in the Appendix.)

It might be remarked that the state probability formulas in the constant case are the simpler for computational purposes, since they involve only finite sums of positive terms.

With respect to complexity, the formulas for the moments are another story. The means, of course, are equal and are given by

$$M_1 = aE_{1,c}(a) = \frac{a^{c+1}}{c!} \left[\sum_{i=0}^{c} \frac{a^i}{i!} \right]^{-1}. \tag{5}$$

Although the variance is the second-order moment of direct interest, the second factorial moment is equivalent for our purpose and will be given here since it is simpler. For exponential service time this moment may be written

$$F_2^{(E)} = a^2 \left[(c+1-a) \sum_{i=1}^{c} \frac{(c)_i}{a^i} + c + 1 \right]^{-1}. \tag{6}$$

(From this expression it is immediate that $F_2^{(E)} = c + 1$ for $a = c + 1$, a fact that was found useful in checking computer programs.) For constant holding time the second factorial moment is

$$F_2 = \frac{1}{\sum_{i=0}^{c} \frac{(c)_i}{a^i}} \left\{ a^2 - ac - \sum_{j=0}^{c} \frac{(c)_j}{a^j} [j(j+3) - 2c] + e^{-a} \sum_{k=0}^{2c-2} \frac{a^{k-c} c!}{k!} \cdot \sum_{h=[k-c]^+}^{\min[k,c]} \binom{k}{h} (c + 1 - h)(c - 2k + 3h) \right\}. \quad (7)$$

In all the numerical cases studied, it turns out that $F_2 > F_2^{(E)}$. In the case $c = 1$, the formulas are simple enough to allow an easy analytic proof that this relationship is true uniformly in a. The fact that $F_2 > F_2^{(E)}$ implies that the correlation between the number of busy trunks in the overflow group and that in the first-choice group is lower in the constant than in the exponential case. This fact is perhaps less surprising to the intuition than is the result that the overflow is more variable for constant than for exponential holding time.

II. OUTLINE OF THE DERIVATION

Owing to the constancy of the holding time, taken here and below to be of unit length, the calls present in the overflow group at an arbitrary instant, t_0, are precisely those that overflowed during the preceding time interval of length unity. The number of calls present in the first-choice group at the instant $t_0 - 1$ is known to have the truncated Poisson distribution (also known as Erlang's first distribution). We condition first on the number of calls present in the first-choice group at $t_0 - 1$. Next, we condition on the number of arrivals during the interval $[t_0 - 1, t_0)$, which, of course, has the ordinary Poisson distribution. We now observe that the hang-up or departure instants together with the arrival instants, as conditioned, are mutually-independently, uniformly distributed on the unit interval. This observation enables us to complete the calculation by an application of a ballot theorem.

Let

c = number of trunks in the first-choice group
a = offered load in erlangs
π_i = probability of i calls on the first-choice group at an arbitrary instant
p_j = probability of j arrivals during a unit of time; $p_j = e^{-a}a^j/j!$
w_{xy} = probability of x calls on the first-choice group and y calls on the overflow group at an arbitrary instant

$w_{\cdot y}$ = probability of y calls on the overflow group at an arbitrary instant

$f(y; i, j)$ = probability of y overflows during a unit interval at whose initial instant there are i calls on the first-choice group and during which j new calls arrive.

Then we may write

$$w_{x0} = p_x \sum_{i=0}^{c} \pi_i f(0; i, x) \tag{8}$$

and

$$w_{xy} = p_{x+y} \sum_{i=c-x}^{c} \pi_i f(y; i, x + y), \qquad y > 0; \tag{9}$$

and, with a reversal of the order of summation, we have

$$w_{\cdot 0} = \sum_{i=0}^{c} \pi_i \sum_{j=0}^{c} p_j f(0; i, j), \tag{10}$$

and

$$w_{\cdot y} = \sum_{i=0}^{c} \pi_i \sum_{j=c-i+y}^{c+y} p_j f(y; i, j), \qquad y > 0. \tag{11}$$

The distribution of the number of calls on the first-choice group at an arbitrary instant is independent of the holding-time distribution and is given by

$$\pi_i = (a^i/i!)\left[\sum_{h=0}^{c} (a^h/h!)\right]^{-1}, \qquad i = 0, \cdots, c, \tag{12}$$

while the remaining service times of these i calls are independently and identically distributed according to the equilibrium excess distribution; that is, the remaining service time of each call has the distribution function

$$F(t) = \int_0^t (1 - H(u))\, du \tag{13}$$

where $H(u)$ is the service-time distribution function (with unit mean). The last two results have been published by several authors; an elementary proof has been given by L. Takács,[4] whose paper includes a bibliography on the problem.

The implication of (13) in the present case is that the remaining service times of the calls initially on the first-choice trunks are independently uniformly distributed on the unit interval. Furthermore, it is well-known that when the number of Poisson arrivals during a fixed time interval is given, the individual arrival instants are independently uniformly distributed over this interval. Hence any specific sequence of the i departures and j arrivals during the interval $[t_0 - 1, t_0)$ has the same probability, namely $1/\binom{i+j}{i}$.

The quantity $f(0; i, j)$ is the probability that a sequence of arrivals and departures has the property that at all times the excess of the accumulated arrivals over the accumulated departures is strictly less than one more than the initial number of idle trunks, i.e., less than $c - i + 1$. Thus the problem of calculating $f(0; i, j)$ can be recognized as a "ballot" problem. Successive arrivals and departures are called "events," and we denote by α_r and β_r, respectively, the accumulated number of arrivals and departures at the rth event. With this notation,

$$f(0; i, j) = \Pr\{\alpha_r < \beta_r + c - i + 1, \quad r = 1, \cdots, i + j\}. \tag{14}$$

The required probability is given as a solution to exercise 3 of Chapter 1 of Ref. 5. We have, with the usual conventions concerning binomial coefficients,

$$\begin{aligned} f(0; i, j) &= 1 - \frac{\binom{j}{c-i+1}}{\binom{c+1}{c-i+1}}, \qquad j \leqq c, \\ &= 0, \qquad j > c. \end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned} f(y; i, j) &= \Pr\{\alpha_r < \beta_r + c - i + y + 1, r = 1, \cdots, i + j\} \\ &\quad - \Pr\{\alpha_r < \beta_r + c - i + y, r = 1, \cdots, i + j\}, \qquad y > 0, \end{aligned} \tag{16}$$

and thus for $j \leqq c + y$,

$$f(y; i, j) = \frac{\binom{j}{c-i+y}}{\binom{c+y}{c-i+y}} - \frac{\binom{j}{c-i+y+1}}{\binom{c+y+1}{c-i+y+1}}. \tag{17}$$

We are particularly interested in the marginal overflow state probabilities, namely the quantities denoted $w_{\cdot 0}$ and $w_{\cdot y}$. These may be put into a form suitable for numerical calculation by substituting the values of the expressions π_i, p_i, and $f(y; i, j)$ into (10) and (11). The final results, obtained after a small amount of manipulation, are shown as (3) and (4). (See Appendix.)

III. MOMENTS

The results (3) and (4) are considerably simpler than their analogs for exponential holding time, shown in (2), which was obtained from equation (38) of Ref. 1. It does not follow, however, that the moments of the distribution defined by (3) and (4) are simpler in form than those for exponential holding time. In fact, although in principle each of the moments can be written as a finite sum, it seems to be a rather tedious task to obtain closed-form expressions for them. Since only the first two moments of overflow distributions are presently of practical interest, we shall confine our attention to the two lowest-order moments for the case of constant holding time.

The mean of the distribution $\{w_{\cdot y}\,;\, y = 0, 1, \cdots\}$ is obviously the same as that for exponential holding time. Nevertheless, a direct calculation of the mean from the expression for the state probabilities is useful as a check on the accuracy of the algebraic manipulation indicated previously. That is, as a check, the equation

$$\sum_{y=1}^{\infty} y w_{\cdot y} = \frac{a^{c+1}}{c!}\left[\sum_{i=0}^{c} \frac{a^i}{i!}\right]^{-1} \tag{18}$$

should be shown to be an identity, as indeed it is. (See Appendix.)

The calculation of the second factorial moment was done by direct summation. The details are given in the Appendix.

It was remarked above that a simple proof that $F_2 > F_2^{(E)}$ can be given for $c = 1$. To do this, we observe that by substituting $c = 1$ into (7) we obtain

$$F_2 = \frac{1}{1+a}\{a^3 - a^2 + 2a - 2 + 2e^{-a}\}. \tag{19}$$

By substituting $c = 1$ into (6), we obtain

$$F_2^{(E)} = \frac{a^3}{2+a}. \tag{20}$$

Thus we must prove

$$\frac{a^3 - a^2 + 2a - 2 + 2e^{-a}}{1+a} > \frac{a^3}{2+a}. \tag{21}$$

After cross-multiplying and simplifying, there results, equivalently,

$$(2 + a)e^{-a} > 2 - a \tag{22}$$

or

$$e^{-a} > \frac{1 - \frac{a}{2}}{1 + \frac{a}{2}}, \tag{23}$$

which is true for a positive (take logarithms).

IV. NUMERICAL RESULTS

A comparison of the overflow variances from first-choice trunk group sizes ranging from 1 to 100 is shown in Table I. These numbers were calculated on the basis of (6) and (7). The starred entries, for low offered loads and high values of c, were calculated separately because of a loss of accuracy in (7) at these values. It should be noted that (3), having no subtractions, does not suffer from loss of accuracy from this cause and hence was used to obtain the variance by direct numerical summation.

The distributions of tails for several trunk-group sizes and offered loads are shown in Figs. 1 and 2. It should be observed that even when the variances differ noticeably, as for example for $c = a$, where in both cases shown the variance difference is greater than 20 percent, the tails distributions are very close at all probabilities of practical interest.

Since the (almost indistinguishable) curves for $c = 10$, $a = 20$ are truncated at a high probability level, it may be well to point out that they differ negligibly even at low values. Thus the probability of exceeding 25 busy trunks in the overflow is 0.0096 for exponential and 0.0109 for constant holding time. A negligible difference in the state probability distributions, as seen in the case of $c = 10$, $a = 20$, was also seen in other cases for which $a = 2c$ and $c > 10$. For $c = 20$ and $a = 40$, however, a comparison could not be made because of a complete loss of accuracy in (2). Since the relative difference in the variances is even less in this case than for c small, it is reasonable to conjecture that (3) provides an excellent approximation to (2) in this case. This suggests that (3) is useful as an approximation to (2) for $a \gg c$, precisely in the range where (2) is unsatisfactory for numerical computations.

TABLE I—COMPARISON OF VARIANCES OF OVERFLOW FOR CONSTANT HOLDING TIME AND EXPONENTIAL HOLDING TIME

c	a	Mean M1	Exp Var VE	Const Var VD	VE/M1	VD/M1	(VD/VE) − 1
1	0.40	0.1143	0.1279	0.1331	1.1190	1.1647	0.0408
	0.60	0.2250	0.2575	0.2704	1.1442	1.2017	0.0503
	0.80	0.3556	0.4120	0.4351	1.1587	1.2236	0.0560
	1.00	0.5000	0.5833	0.6179	1.1667	1.2358	0.0592
	1.20	0.6545	0.7661	0.8127	1.1705	1.2416	0.0607
	1.40	0.8167	0.9568	1.0152	1.1716	1.2431	0.0611
	1.60	0.9846	1.1529	1.2228	1.1709	1.2419	0.0606
	1.80	1.1571	1.3529	1.4334	1.1692	1.2387	0.0595
	2.00	1.3333	1.5556	1.6458	1.1667	1.2343	0.0580
2	0.80	0.1208	0.1478	0.1595	1.2240	1.3211	0.0794
	1.20	0.2959	0.3778	0.4132	1.2767	1.3966	0.0939
	1.60	0.5278	0.6873	0.7562	1.3021	1.4327	0.1003
	2.00	0.8000	1.0489	1.1555	1.3111	1.4443	0.1016
	2.40	1.1006	1.4425	1.5864	1.3106	1.4413	0.0997
	2.80	1.4218	1.8551	2.0330	1.3048	1.4299	0.0959
	3.20	1.7579	2.2784	2.4858	1.2961	1.4140	0.0910
	3.60	2.1054	2.7075	2.9392	1.2860	1.3960	0.0856
	4.00	2.4615	3.1392	3.3902	1.2753	1.3773	0.0800
5	2.00	0.0734	0.1040	0.1198	1.4176	1.6319	0.1512
	3.00	0.3302	0.5186	0.6078	1.5707	1.8409	0.1720
	4.00	0.7963	1.3013	1.5284	1.6342	1.9194	0.1745
	5.00	1.4243	2.3332	2.7210	1.6381	1.9104	0.1662
	6.00	2.1624	3.4864	4.0166	1.6123	1.8575	0.1521
	7.00	2.9730	4.6819	5.3164	1.5748	1.7882	0.1355
	8.00	3.8321	5.8806	6.5798	1.5346	1.7170	0.1189
	9.00	4.7242	7.0661	7.7970	1.4957	1.6504	0.1034
	10.00	5.6395	8.2327	8.9712	1.4598	1.5908	0.0897
10	4.00	0.0212	0.0329	0.0399	1.5485	1.8816	0.2151
	6.00	0.2589	0.4872	0.6044	1.8821	2.3349	0.2406
	8.00	0.9733	1.9857	2.4530	2.0402	2.5203	0.2354
	10.00	2.1458	4.3624	5.2907	2.0330	2.4656	0.2128
	12.00	3.6231	7.0710	8.3540	1.9516	2.3058	0.1815
	14.00	5.2820	9.7876	11.2465	1.8530	2.1292	0.1491
	16.00	7.0490	12.4046	13.8965	1.7598	1.9714	0.1203
	18.00	8.8826	14.9080	16.3499	1.6783	1.8407	0.0967
	20.00	10.7592	17.3130	18.6671	1.6091	1.7350	0.0782
20	8.00	0.0013	0.0021	0.0027	1.6141	2.1127	0.3089
	12.00	0.1175	0.2584	0.3401	2.1986	2.8935	0.3161
	16.00	1.0306	2.7028	3.5055	2.6226	3.4015	0.2970
	20.00	3.1778	8.2920	10.3872	2.6093	3.2686	0.2527
	24.00	6.1700	14.8143	17.7126	2.4010	2.8708	0.1956
	28.00	9.5977	20.9311	23.9282	2.1808	2.4931	0.1432
	32.00	13.2464	26.4675	29.2087	1.9981	2.2050	0.1036
	36.00	17.0146	31.5567	33.9567	1.8547	1.9957	0.0761
	40.00	20.8522	36.3397	38.4213	1.7427	1.8426	0.0573
50	20.00	0.0000	0.0000	0.0000*	1.6452	2.2408*	0.3403*
	30.00	0.0066	0.0161	0.0230	2.4215	3.4727	0.4341
	40.00	0.7476	2.7343	3.7810	3.6573	5.0574	0.3828
	50.00	5.2393	19.7746	25.5769	3.7743	4.8817	0.2934
	60.00	12.9671	40.9397	48.5749	3.1572	3.7460	0.1865
	70.00	21.9661	57.8384	64.1752	2.6331	2.9216	0.1096
	80.00	31.4446	71.6558	76.4680	2.2788	2.4318	0.0672
	90.00	41.1339	83.8760	87.5797	2.0391	2.1291	0.0442
	100.00	50.9303	95.2624	98.2036	1.8704	1.9282	0.0309
100	60.00	0.0000	0.0001	0.0001*	2.4634	3.6885*	0.4973*
	80.00	0.3194	1.4158	2.0681	4.4331	6.4758	0.4608
	100.00	7.5700	38.5958	50.7901	5.0985	6.7094	0.3160
	120.00	23.5523	89.6717	103.9455	3.8073	4.4134	0.1592
	140.00	42.1741	123.6750	133.0991	2.9325	3.1559	0.0762
	160.00	61.5406	149.8224	156.0855	2.4345	2.5363	0.0418

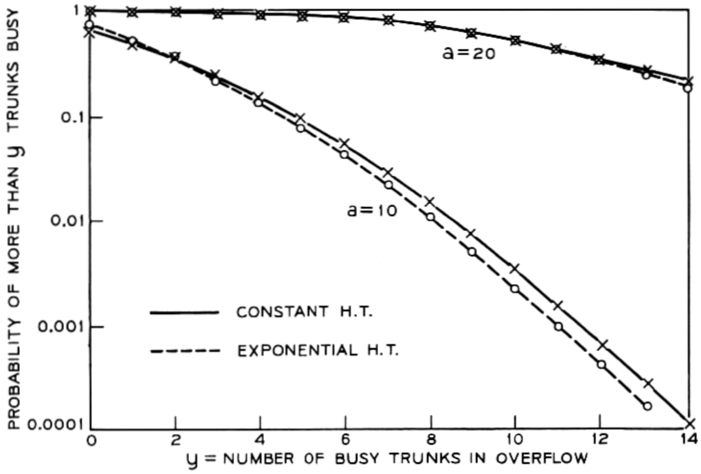

Fig. 1—Tails distributions, $c = 10$.

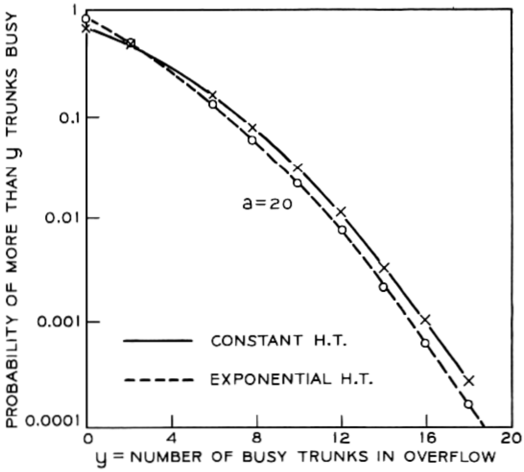

Fig. 2—Tails distributions, $c = 20$.

APPENDIX

A.1 *Formulas for the Overflow State Probabilities*

The probability of zero calls in the overflow group is given by (10), which, after substitution of the values of π_i and p_i, becomes

$$w_{.0} = \sum_{i=0}^{c} \frac{a^i/i!}{\sum_{h=0}^{c} \frac{a^h}{h!}} \sum_{j=0}^{c} e^{-a} \frac{a^j}{j!} f(0; i, j). \tag{24}$$

For brevity, let

$$f(a) = e^{-a}\left[\sum_{h=0}^{c} \frac{a^h}{h!}\right]^{-1}. \tag{25}$$

Then, replacing $f(0; i, j)$ by its value as given in (15), we may write

$$w_{.0} = f(a) \sum_{i,j=0}^{c} \frac{a^{i+j}}{i!\,j!}\left[1 - \frac{\binom{j}{c-i+1}}{\binom{c+1}{c-i+1}}\right]$$

$$= f(a) \sum_{i,j=0}^{c} \frac{a^{i+j}}{i!\,j!}\left[1 - \frac{i!\,j!}{(i+j-(c+1))!\,(c+1)!}\right]. \tag{26}$$

Setting $i + j = h$ and multiplying the numerator and denominator of each term by $h!$,

$$w_{.0} = f(a)\left[\sum_{h=0}^{c} \frac{a^h}{h!} \sum_{i=0}^{h} \frac{h!}{i!\,(h-i)!} + \sum_{h=c+1}^{2c} \frac{a^h}{h!} \sum_{i=h-c}^{c}\left(\frac{h!}{i!\,(h-i)!} - \frac{h!}{(h-(c+1))!\,(c+1)!}\right)\right] \tag{27}$$

$$w_{.0} = f(a) \sum_{h=0}^{c} \frac{a^h}{h!} \cdot 2^h + \sum_{h=c+1}^{2c} \frac{a^h}{h!}\left[\sum_{i=h-c}^{c} \binom{h}{i} - (2c-h+1)\binom{h}{c+1}\right]. \tag{28}$$

Use of (25) now yields (4).

The probability of y calls in the overflow group is given by (11). We first simplify $f(y; i, j)$, as given by (17). We have

$$f(y; i, j) = \frac{i!\,j!}{[i+j-(c+y)]!\,(c+y)!} - \frac{i!\,j!}{[i+j-(c+y+1)]!\,(c+y+1)!}. \tag{29}$$

Thus, from (11),

$$w_{.y} = \sum_{i=0}^{c} \frac{a^i/i!}{\sum_{h=0}^{c} \frac{a^h}{h!}} \sum_{j=c-i+y}^{c+y} e^{-a} \frac{a^j}{j!}\left[\frac{i!\,j!}{[i+j-(c+y)]!\,(c+y)!} - \frac{i!\,j!}{[i+j-(c+y+1)]!\,(c+y+1)!}\right] \tag{30}$$

$$w_{.y} = f(a) \sum_{i=0}^{c} \sum_{j=c-i+y}^{c+y} a^{i+j}\left[\frac{1}{(i+j-(c+y))!\,(c+y)!} - \frac{1}{(i+j-(c+y+1))!\,(c+y+1)!}\right]. \tag{31}$$

Setting $h = i + j - (c + y)$, reversing the order of summation, and factoring yields

$$w_{.y} = f(a) \sum_{h=0}^{c} \sum_{i=h}^{c} \frac{a^{h+c+y}}{h!\,(c+y+1)!}(c+y+1-h), \tag{32}$$

$$w_{.y} = f(a) \frac{a^{c+y}}{(c+y+1)!} \sum_{h=0}^{c} \frac{a^h}{h!} \sum_{i=h}^{c} (c+y+1-h), \tag{33}$$

and (3) follows.

A.2 *Proof That (18) Is an Identity*

Let the mean of the overflow distribution be denoted by M. From (3),

$$M/f(a) = \sum_{y=1}^{\infty} \frac{ya^{c+y}}{(c+y+1)!} \sum_{j=0}^{c} \frac{a^j}{j!}(c-j+1)(c+y+1-j) \tag{34}$$

$$M/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!}(c-j+1) \sum_{y=1}^{\infty} (c+y+1) \frac{ya^{c+y}}{(c+y+1)!} - \sum_{j=0}^{c} \frac{ja^j}{j!}(c-j+1) \sum_{y=1}^{\infty} \frac{ya^{c+y}}{(c+y+1)!} \tag{35}$$

$$M/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!}(c-j+1) \sum_{y=1}^{\infty} \frac{ya^{c+y}}{(c+y)!} - \sum_{j=1}^{c} \frac{a^{j-1}}{(j-1)!}(c-j+1) \sum_{y=1}^{\infty} \frac{ya^{c+y+1}}{(c+y+1)!}. \tag{36}$$

After changing the indices in the subtracted sums by replacing j by $j+1$ and y by $y-1$, we obtain

$$M/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!}(c-j+1) \sum_{y=1}^{\infty} \frac{ya^{c+y}}{(c+y)!} - \sum_{j=0}^{c} \frac{a^j}{j!}(c-j) \sum_{y=1}^{\infty} (y-1) \frac{a^{c+y}}{(c+y)!}. \tag{37}$$

Simplification yields

$$M/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!} \sum_{y=1}^{\infty} \frac{ya^{c+y}}{(c+y)!} + \sum_{j=0}^{c} \frac{a^j}{j!}(c-j) \sum_{y=1}^{\infty} \frac{a^{c+y}}{(c+y)!}. \tag{38}$$

Putting $h = y + c$ and adding and subtracting terms, we have

$$M/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!} \left[\sum_{h=0}^{\infty} (h-c) \frac{a^h}{h!} - \sum_{h=0}^{c} (h-c) \frac{a^h}{h!} \right] + \sum_{j=0}^{c} \frac{a^j}{j!}(c-j) \left[\sum_{h=0}^{\infty} \frac{a^h}{h!} - \sum_{h=0}^{c} \frac{a^h}{h!} \right]. \tag{39}$$

We notice that the finite series vanish and the infinite series combine to give

$$M/f(a) = \left[\sum_{j=0}^{c} \frac{a^{j+1}}{j!} - \sum_{j=0}^{c-1} \frac{a^{j+1}}{j!}\right] \sum_{h=0}^{\infty} \frac{a^h}{h!} = \frac{a^{c+1}}{c!} e^a. \tag{40}$$

Replacing $f(a)$ by its explicit expression, we have

$$M = \frac{a^{c+1}}{c!} \left(\sum_{i=1}^{c} \frac{a^i}{i!}\right)^{-1}. \tag{41}$$

A.3 *Derivation of the Formula for the Second Factorial Moment of the Constant-Holding-Time Overflow Distribution*

From (3),

$$F_2/f(a) = \sum_{y=2}^{\infty} y(y-1) \frac{a^{c+y}}{(c+y+1)!} \cdot \sum_{j=0}^{c} \frac{a^j}{j!} (c+y+1-j)(c-j+1) \tag{42}$$

$$F_2/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!} (c-j+1) \sum_{y=2}^{\infty} y(y-1)(c+y+1) \frac{a^{c+y}}{(c+y+1)!} - \sum_{j=0}^{c} \frac{ja^j}{j!} (c-j+1) \sum_{y=2}^{\infty} y(y-1) \frac{a^{c+y}}{(c+y+1)!}. \tag{43}$$

Again replacing j by $j+1$ and y by $y-1$ in the subtracted sums in (43), we obtain

$$F_2/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!} (c-j+1) \sum_{y=2}^{\infty} y(y-1) \frac{a^{c+y}}{(c+y)!} - \sum_{j=0}^{c} \frac{a^j}{j!} (c-j) \sum_{y=2}^{\infty} (y-1)(y-2) \frac{a^{c+y}}{(c+y)!} \tag{44}$$

$$F_2/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!} \sum_{y=2}^{\infty} y(y-1) \frac{a^{c+y}}{(c+y)!} + 2 \sum_{j=0}^{c} \frac{a^j}{j!} (c-j) \sum_{y=2}^{\infty} (y-1) \frac{a^{c+y}}{(c+y)!}. \tag{45}$$

Letting $h = c + y$ and adding and subtracting terms allows us to write

$$F_2/f(a) = \sum_{j=0}^{c} \frac{a^j}{j!} \left\{ \sum_{h=0}^{\infty} (h-c)(h-c-1) \frac{a^h}{h!} \right.$$

$$- \sum_{h=0}^{c} \frac{a^h}{h!} (h - c)(h - c - 1)\Bigg\} + 2 \sum_{j=0}^{c} \frac{a^j}{j!} (c - j)$$

$$\cdot \left\{ \sum_{h=0}^{\infty} (h - c - 1) \frac{a^h}{h!} - \sum_{h=0}^{c} \frac{a^h}{h!} (h - c - 1) \right\} \tag{46}$$

$$F_2/f(a) = \sum_{h=0}^{\infty} [h(h - 1) - 2hc + c(c + 1)] \frac{a^h}{h!} \sum_{j=0}^{c} \frac{a^j}{j!}$$

$$+ \sum_{h=0}^{\infty} 2[hc - c(c + 1)] \frac{a^h}{h!} \sum_{j=0}^{c} \frac{a^j}{j!}$$

$$- \sum_{h=0}^{\infty} [2ah - 2a(c + 1]] \frac{a^h}{h!} \sum_{j=0}^{c-1} \frac{a^j}{j!}$$

$$- \sum_{j=0}^{c} \frac{a^j}{j!} \sum_{h=0}^{c} \frac{a^h}{h!} (c - h)(c + 1 + h)$$

$$+ 2 \sum_{j=0}^{c} \frac{a^j}{j!} (c - j) \sum_{h=0}^{c} \frac{a^h}{h!} (c + 1 - h). \tag{47}$$

After substituting k for $h + j$ in the finite sums and substituting e^a for its series expansion, we obtain

$$F_2/f(a) = e^a \Bigg\{ [2a(c + 1) - a^2 - c(c + 1)]$$

$$\cdot \sum_{j=0}^{c} \frac{a^j}{j!} + [2a^2 - 2a(c + 1)] \frac{a^c}{c!} \Bigg\}$$

$$+ \sum_{k=0}^{2c-1} \frac{a^k}{k!} \sum_{h=[k-c]^+}^{\min[k,c]} \binom{k}{h} (c + 1 - h)(c - 2k + 3h). \tag{48}$$

It turns out that the second sum vanishes for $k = 2c - 1$, and thus after deleting the vanishing term and writing the coefficient of e^a as a polynomial in a, we have

$$F_2/f(a) = e^a \Bigg\{ \frac{a^{c+2}}{c!} = \frac{a^{c+1}}{(c - 1)!}$$

$$- \sum_{j=0}^{c} \frac{a^j}{j!} [j(j - 1) - 2j(c + 1) + c(c + 1)] \Bigg\}$$

$$+ \sum_{k=0}^{2c-2} \frac{a^k}{k!} \sum_{h=[k-c]^+}^{\min[k,c]} \binom{k}{h} (c + 1 - h)(c - 2k + 3h). \tag{49}$$

Finally, after substituting the value of $f(a)$, given by (25), transposing the limits of summation in the first sum on the right-hand side, and some minor simplification, we obtain (7).

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