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Statistical Behavior of a Fading Signal

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A general analysis of the statistical behavior of the envelope of a fading signal $V(t)e^{i\phi(t)}$ is presented in this paper. The statistics include the probability $P(V \leqq L)$ that the amplitude $V(t)$ will fade below a specified signal level L; the expected number $N(L)$ of fades of $V(t)$ below L per unit time; and the average duration $\bar{t}(L)$ of fades below L. The model for the fading signal is a constant vector plus a random interfering vector which represents the resultant of all the received extraneous signals and noise. The theoretical results agree with three empirically observed power relationships obtained in deep fades of nondiversity signals: $P(V \leqq L) \propto L^2$, $N(L) \propto L$ and $\bar{t}(L) \propto L$. The theoretical results are applicable to a wide class of fading problems. The analysis includes the previous works of Rice, Nakagami, Norton, Vogler, Mansfield, and Short as special cases.

I. INTRODUCTION

A general analysis of the statistical behavior of the envelope $V(t)e^{i\phi(t)}$ of a fading signal is presented in this paper. Our principal interests are the probability, $P(V \leqq L)$, that the amplitude $V(t)$ will fade below a specified signal level L* ; the expected number, $N(L)$, of fades per unit time below the specified level L; and the average duration, $\bar{t}(L)$, of fades below L. These statistics are all functions of the signal level L.

The theory presented herein has been developed to complement and extend the empirical results developed by my colleagues at Bell Telephone Laboratories[1-6] from their extensive experimental experience. Published data of other workers have also been considered.

The previous theoretical works on the statistics of a fading signal often assume a complex Gaussian model for the fading signal. The theoretical support for this assumption is that, by the central limit theorem, the real and the imaginary parts of the sum of a large number of independent interfering signals will be approximately Gaussian.

* More precisely speaking, in a long time period containing a large number of fades, the distribution $P(V \leqq L)$ represents the expected fraction of this long time period that the signal amplitude V will fade below L.

For tropospheric radio links, this model seems to be satisfactory. However, for line-of-sight radio links, the results of a short pulse experiment[7] and the angle-of-arrival measurements[8,9,10] indicate that the number of interfering signals is usually fairly small. Ray tracing theory also indicates that for typical line-of-sight radio links the number of paths contributing to multipath propagation is unlikely to be large. Furthermore, the theoretical results of the complex Gaussian model do not agree well with the experimental data on the statistics of fading signals of line-of-sight radio links, especially for certain overwater paths with severe fading.

In this paper, we do not impose the restrictive assumptions of the complex Gaussian model. Rather, we simply model the fading signal $Ve^{i\phi}$ as a constant vector plus an interfering random vector; i.e.,

$$Ve^{i\phi} = 1 + Re^{i\phi} = 1 + \alpha + j\beta, \tag{1}$$

where R, θ, α, and β are the amplitude, phase, real part, and imaginary part respectively of the interfering vector. The interfering vector is described by the joint probability density function $f(\alpha, \beta)$ and represents the resultant of all the received extraneous signals, echoes, rays, and noise. The analysis applies for R and θ either dependent or independent; θ uniformly or nonuniformly distributed; α and β either Gaussian or nonGaussian. Thus, the results of this analysis may be applied to a wide class of fading problems.

This paper treats the problem in three parts: The first is concerned with the amplitude distribution of V. The second considers the number of fades $N(L)$ and the average fade duration $\bar{t}(L)$. The final section investigates several special topics including m-distributions, chi-distributions, the Rayleigh distribution, Rice distribution, log-normal distribution and the sum of n unit random vectors.

Appendix A is a list of symbols and their definitions.

II. SUMMARY OF RESULTS

(*i*) In spite of great variations in fading environment and test conditions, the experimental data[1-6,11-14,15-17] on $P(V \leqq L)$, $N(L)$, and $\bar{t}(L)$ of most nondiversity* fading signals obey the following three prevailing power laws of deep fades:

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* The diversity signal is the output signal of a diversity combining system with two or more input signals. A "nondiversity fading signal" is a fading signal that is not a diversity signal.

$$\left.\begin{aligned} P(V \leqq L) &\propto L^2 \\ N(L) &\propto L \\ \bar{t}(L) &\propto L \end{aligned}\right\} \quad \text{for small} \quad L. \qquad \begin{aligned}(2)\\(3)\\(4)\end{aligned}$$

The theoretical analysis shows that if the probability density function $f(\alpha, \beta)$ of the resultant interfering vector, $Re^{j\theta} = \alpha + j\beta$, is a smooth function which is neither singular nor zero at the deep fade point ($\alpha = -1, \beta = 0$), then the statistics $P(V \leqq L)$, $N(L)$, and $\bar{t}(L)$ of deep fades follow the three prevailing power laws (2), (3), and (4). The easily satisfied condition, $\infty > f(-1, 0) > 0$, is sufficient to obtain these functional relationships.

(*ii*) The set of power laws (2), (3), and (4) apply for R and θ either independent or dependent, θ either uniformly or nonuniformly distributed, (or, α and β either Gaussian or nonGaussian, either independent or dependent) as long as $f(\alpha, \beta)$ is smooth.

(*iii*) If $f(\alpha, \beta)$ is singular at ($\alpha = -1, \beta = 0$), then the theory predicts that for small L

$$\left.\begin{aligned} P(V \leqq L) &\propto L^{2\mu} \\ N(L) &\propto L^{2\mu-1} \\ \bar{t}(L) &\propto L \end{aligned}\right\} \quad 1 > \mu > \tfrac{1}{2}. \qquad \begin{aligned}(5)\\(6)\\(7)\end{aligned}$$

The exceptional behavior (5) consistent with $\mu = 1/2$ has been observed experimentally on certain overwater radio links with severe fading. In this case, the resultant interfering vector contains the strong water-reflected ray as a dominant component. Therefore, the probability density function $f(\alpha, \beta)$ has a singularity at the position of the dominant component vector.*

(*iv*) If $f(\alpha, \beta)$ has a zero at ($\alpha = -1, \beta = 0$) or is negligibly small at ($\alpha = -1, \beta = 0$), the theory predicts that for small L:

$$\left.\begin{aligned} P(V \leqq L) &\propto L^{2\mu} \\ N(L) &\propto L^{2\mu-1} \\ \bar{t}(L) &\propto L \end{aligned}\right\} \quad \mu > 1 \qquad \begin{aligned}(8)\\(9)\\(10)\end{aligned}$$

The composite fading signals of diversity combining systems obey the set of power laws (8), (9), and (10). For overland radio

* The complex Gaussian model, which assumes that $f(\alpha, \beta)$ is a two-dimensional normal density function, is unable to explain the exceptional behavior (5) and (6).

links, the parameter μ is equal to the order of diversity. In our experiments, the nondiversity fading signal of a relatively short* radio link with path length 15.87 miles has also shown the exceptional behavior described by (8), (9), and (10).

(*v*) The theoretical results (4), (7), and (10) indicate that the power law, $\bar{t}(L) \propto L$, for the average fade duration is more universal than those of $P(V \leqq L)$ and $N(L)$. This prediction agrees with available experimental data.

(*vi*) In general, the relationship between $f(\alpha, \beta)$ and the amplitude distribution $P(V \leqq L)$ is not unique. As an example, this non-uniqueness shows that specifying the Rayleigh distribution for the amplitude of a fading signal does not necessarily imply that α and β are Gaussian, nor does it necessarily imply a large number of interfering signals.

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* At 4-GHz operating frequency, the average path length of line-of-sight radio links is about 27 miles.

Part 1. Amplitude Distribution

Contents

I. INTRODUCTION AND SUMMARY

In the study of fading signals due to multipath interference, the experimental data on the cumulative amplitude distribution, $P(V \leqq L)$, of deep fades are often plotted on a graph paper where the fade depth is expressed in dB and the probability of fade is expressed on a log scale as shown in Fig. 1. The consensus based on large amounts of experimental data[1–6,11–14,15–17] is that the cumulative amplitude distribution of most nondiversity fading signals in the deep-fade region can well be represented by a straight line with a prevailing inverse slope of 10 dB per decade of probability.

The equation which describes this typical distribution on Fig. 1 is

$$P(V \leqq L) = \epsilon \cdot L^2, \quad \text{for} \quad L_{\text{up}} \geqq L \geqq 0 \tag{11}$$

where V is the envelope voltage of the random fading signal normalized to its nonfaded signal level, L is any specified signal level, ϵ is a parameter depending on fading environment, and L_{up} is the upper bound of signal level below which the straight-line representation of $P(V \leqq L)$ on Fig. 1 is valid.

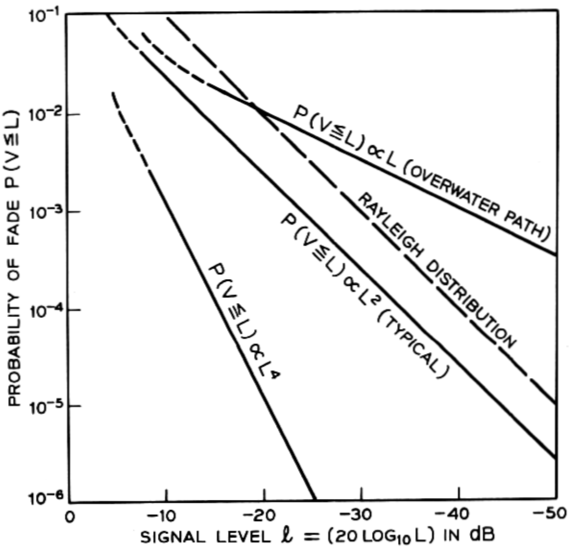

Fig. 1—Cumulative amplitude distributions of fading signals.

The empirical result (11) means that the amplitude distributions of most nondiversity fading signals obey the following square law of deep fades

$$P(V \leqq L) \propto L^2, \qquad L_{up} \geqq L \geqq 0 \tag{12}$$

in spite of the great variations of fading environment and test condition. However, there are some exceptional cases. The most profound exception occurs on certain overwater radio links with severe fading. In these instances, the probability of fades, $P(V \leqq L)$, decreases very slowly as the signal level L decreases and is characterized by an inverse slope of 20 dB per decade of probability as shown in Fig. 1, implying a power law

$$P(V \leqq L) \propto L \tag{13}$$

in the deep-fade region.

Another kind of exception occurs on certain radio links with relatively little multipath fading. The probability of fades, $P(V \leqq L)$, decreases very rapidly with L and is characterized by an inverse slope of 5 dB per decade of probability as shown in Fig. 1. This kind of distribution follows the power law

$$P(V \leqq L) \propto L^4 \tag{14}$$

in the deep-fade region. For example, at 4-GHz operating frequency, the average path length of line-of-sight radio links is about 27 miles. The behavior (14) has been observed to occur on a relatively short path with path length of 15.87 miles.

The theoretical amplitude distributions previously derived were based on a complex Gaussian model and predict a square-law dependence, $P(V \leqq L) \propto L^2$, in the deep-fade region. For example, the Nakagami distributions,[18] which include Hoyt distribution,[18,19] Rice distribution,[18] and Rayleigh distribution as special cases, all are square law in the deep-fade region. The explicit expression of Nakagami distribution can be found in Equation (4.6–28) of Reference 18. Figure 1 also includes Rayleigh distribution (dashed line) for comparison with the experimental data.

The small number of interfering signals in line-of-sight radio links suggests that the assumption of a complex Gaussian model may be unjustified. One of the main objectives of this paper is to determine the weakest set of assumptions under which the square law (12) is obtained, and the condition for which the exceptional case such as (13) or (14) will occur.

In summary:

(*i*) The theoretical model for the fading signal, Ve_i^{ϕ}, is a constant unit vector plus a resultant interfering vector

$$Ve^{i\phi} = 1 + Re^{i\theta} = 1 + \alpha + j\beta.$$

The resultant interfering vector $Re^{i\theta} = \alpha + j\beta$, with joint probability density function $f(\alpha, \beta)$, represents the sum of all the received extraneous signals, echoes, rays, and noise.

(*ii*) An infinite fade (i.e., $V = 0$) occurs whenever $\alpha = -1$ and $\beta = 0$. At this point $R = 1$ and $\theta = \pi$. Therefore, the behavior of $f(\alpha, \beta)$ near the infinite fade point ($\alpha = -1$, $\beta = 0$) is closely related to the power law of amplitude distribution $P(V \leqq L)$ in the deep-fade region.

(*iii*) For most radio links, the interfering signals and noise may be considered random, so that the joint probability density function $f(\alpha, \beta)$ of the resultant interfering vector is a smooth function near the infinite fade point ($\alpha = -1$, $\beta = 0$). The analysis shows that if $f(\alpha, \beta)$ is a smooth function which is neither singular nor zero at ($\alpha = -1$, $\beta = 0$), then $P(V \leqq L) \propto L^2$ for small L. The simple condition $\infty > f(-1, 0) > 0$ is easily satisfied by most radio links. The validity of this square law does not require that $f(\alpha, \beta)$ be a normal density function. Therefore, the number of interfering signals does not *have* to be large.

(*iv*) The analysis shows that if $f(\alpha, \beta)$ is not smooth, but is singular at $(\alpha = -1, \beta = 0)$, then for small L,

$$P(V \leqq L) \propto L^{2\mu}, \qquad 1 > \mu \geqq \tfrac{1}{2}.$$

A physical example for this case is the overwater radio link where the water-reflected ray is almost as stable as the direct ray. The resultant interfering vector in this case contains the water-reflected ray which is "not very random." In other words, the joint probability density function $f(\alpha, \beta)$ has a high peak at the position of the dominant, stable component, and may be considered singular at that point.

(*v*) If $f(\alpha, \beta)$ is zero or is negligibly small at the infinite fade point $(\alpha = -1, \beta = 0)$, then the analysis shows that for small L

$$P(V \leqq L) \propto L^{2\mu}, \qquad \mu > 1.$$

A physical example for this case is the short radio link where the phase differences among the multipath propagations are all small. Then the value of $f(-1, 0)$ is negligibly small because the phase, θ, of the resultant interfering vector is generally small. Another example for this case is the composite signal of the output of a diversity combining system where the artificial active combining device serves to create a zero of $f(\alpha, \beta)$ at $(\alpha = -1, \beta = 0)$.

II. FADING SIGNAL MODEL

The received fading signal is modeled as a constant vector plus an interfering random vector as shown in Fig. 2. The latter represents the resultant of all the received extraneous signals, echoes, rays, and noise. The received fading signal normalized to the magnitude of the constant vector can be written as

$$V(t)e^{i\phi(t)} = 1 + R(t)e^{i\theta(t)}, \tag{15}$$

where $R(t)$ and $\theta(t)$ are the normalized magnitude and the phase of the interfering random vector respectively; $V(t)$ and $\phi(t)$ are the normalized magnitude and the phase of the received fading signal respectively.

Let $x(t)$ and $y(t)$ be the real part and the imaginary part of the complex fading signal $Ve^{i\phi}$, i.e.,

$$V(t)e^{i\phi(t)} = x(t) + jy(t); \tag{16}$$

and let $\alpha(t)$ and $\beta(t)$ be the real part and the imaginary part of the complex interfering random vector $Re^{i\theta}$, i.e.,

$$R(t)e^{i\theta(t)} = \alpha(t) + j\beta(t). \tag{17}$$

V, ϕ, R, θ, x, y, α and β are all real random variables. The normalized output power is

$$V^2 = 1 + 2R \cos \theta + R^2 = (1 + \alpha)^2 + \beta^2. \tag{18}$$

The relative phase, $\theta(t)$, between the interfering vector and the constant vector can be taken to have values from 0 to 2π because θ and $(\theta \pm 2n\pi)$ for any integer n are indistinguishable to the received signal at an operating frequency.

A geometrical interpretation of equations (15) to (18) shows that deep fades (i.e., small V) occur when R and θ are near the infinite fade point, $(1, \pi)$, in the (R, θ) plane or equivalently when α and β are near the infinite fade point, $(-1, 0)$, in the (α, β) plane.

For line-of-sight radio links, notice that as far as the received signal $V(t)e^{i\phi(t)}$ is concerned, scintillation, atmospheric divergence, and earth bulge effects may also be replaced by a mathematically equivalent interfering signal $R(t)e^{i\theta(t)}$ which when combined with the constant vector gives the received fluctuating signal.

Therefore, the model described by equations (15) to (18) includes many possible fading mechanisms which may occur individually or simultaneously on a fading environment.

Going one step further, this model also includes the situations where no physical constant vector* exists because the mathematical decomposition (15) is applicable to any arbitrary fading signal. For such situations, the constant vector may represent the average signal level. The fluctuation of the signal is considered to be caused by an equivalent resultant interfering vector $Re^{i\theta}$.

For radio links subjected to multipath interference, the number of incoming component waves is usually more than two. Thus the resultant interfering vector $Re^{i\theta}$ consists of more than one echo and should not be interpreted as a simple physical echo. The main distinction is that the magnitude and the time delay of a physical echo are not functions of operating frequency whereas the magnitude R and the equivalent time delay, $\zeta_e = \theta/\omega$, of a resultant interfering vector consisting of more than one echo, are functions of operating frequency (i.e., are dispersive).

* For beyond-the-horizon radio links, there is no direct radio path between the transmitter and the receiver.

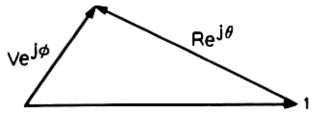

Fig. 2—Fading signal model.

III. GENERAL FORMULATION OF AMPLITUDE DISTRIBUTION

Equation (18) shows that the probability that the random signal V be faded below a specified signal level L is equal to the probability that α and β fall within the circular region

$$(1 + \alpha)^2 + \beta^2 \leqq L^2 \tag{19}$$

in the (α, β) plane as shown in Fig. 3. Let $f(\alpha, \beta)$ be the joint probability function of α and β. Then $P(V \leqq L)$ is the integral of $f(\alpha, \beta)$ over the circular region (19); i.e.,

$$P(V \leqq L) = \int_{\beta=-L}^{\beta=L} \int_{\alpha=-1-\sqrt{L^2-\beta^2}}^{\alpha=-1+\sqrt{L^2-\beta^2}} f(\alpha, \beta)\, d\alpha\, d\beta. \tag{20}$$

The statistical behavior of the interfering vector, $Re^{i\theta} = \alpha + j\beta$, is sometimes described by the joint probability density function $q(R, \theta)$ of the magnitude R and the phase θ of the interfering vector. A similar derivation in terms of R and θ yields

$$P(V \leqq L) = \int_{R=1-L}^{R=1+L} \int_{\theta=\pi-\theta_L}^{\theta=\pi+\theta_L} q(R, \theta)\, d\theta\, dR, \tag{21}$$

where

$$\theta_L = \cos^{-1}\left(\frac{1 + R^2 - L^2}{2R}\right). \tag{22}$$

Most of the following analysis is in terms of α and β. An equivalent result in terms of R and θ is given in Appendix B.

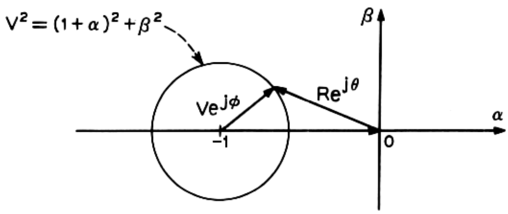

Fig. 3—Fading signal model on (α, β) plane.

IV. PROBABILITY DENSITY FUNCTION OF INTERFERING VECTOR

Equation (18) indicates that the infinite fade (i.e., $V = 0$) occurs when $\alpha = -1$, $\beta = 0$. For most overland radio links, the interfering signals and the noise vary continuously in a random manner, so that the joint probability density function $f(\alpha, \beta)$ of the resultant vector is a smooth function. On the other hand, there are some paths for which $f(\alpha, \beta)$ may not be smooth, but singular. For example, for overwater radio links, the water-reflected ray near the grazing angle is comparable in magnitude and stability to the direct ray. Therefore, the resultant interfering vector contains a dominant component and the probability density function $f(\alpha, \beta)$ has a sharp peak (i.e., singular) at this point. If the heights of the antennas and the path length are such that the average phase of the water-reflected ray is equal to π, then the singularity of $f(\alpha, \beta)$ occurs at the infinite fade point ($\alpha = -1$, $\beta = 0$).

These discussions suggest that in the general analysis of deep fades, one should consider not only the case with smooth $f(\alpha, \beta)$ but also the case where $f(\alpha, \beta)$ is singular at ($\alpha = -1$, $\beta = 0$). A general probability density function $f(\alpha, \beta)$, which is useful in our study, is*†

$$f(\alpha, \beta) = [(1 + \alpha)^2 + \beta^2]^{(\mu-1)} \cdot H(\alpha, \beta) \tag{23}$$

$$= V^{2(\mu-1)} \cdot H(\alpha, \beta). \tag{24}$$

where $H(\alpha, \beta)$ is an arbitrary smooth function. In the range $1 > \mu \geqq 1/2$, the density function $f(\alpha, \beta)$ has a singularity of order $2 | \mu - 1 |$ at ($\alpha = -1$, $\beta = 0$).

On the other hand, in the range $\infty > \mu > 1$, the density function has a zero of order $2(\mu - 1)$ at ($\alpha = -1$, $\beta = 0$). For convenience, we shall call the parameter, μ, the smoothness index of $f(\alpha, \beta)$.

Since the possible singularity or zero of $f(\alpha, \beta)$ at ($\alpha = -1$, $\beta = 0$) is taken care of by the factor $[(1 + \alpha)^2 + \beta^2]^{\mu-1}$, we shall assume that

$$\infty > H(-1, 0) > 0. \tag{25}$$

Thus, the density function $f(\alpha, \beta)$ given by equation (23) is neither singular nor zero at ($\alpha = -1$, $\beta = 0$) if, and only if, $\mu = 1$ because the condition $\mu = 1$ implies $f(\alpha, \beta) = H(\alpha, \beta)$, and vice versa.

In equations (23) and (24), the smoothness index μ can be either an

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* The reason for the use of the factor $(\mu - 1)$ instead of a simple power index in equations (23) and (24) is for the convenience of notation in Section IV of Part 3 when we investigate the m-distributions.

† The cases where the singularity of $f(\alpha, \beta)$ occurs at positions other than the infinite fade point ($\alpha = -1$, $\beta = 0$) will not be analyzed in this paper. A brief discussion is included in Section VII of this part.

integer or a noninteger. The only restriction on μ is that

$$\mu \geqq \tfrac{1}{2}. \tag{26}$$

The reason for this constraint on μ is that in Section V of Part 2 we find that if $\mu < 1/2$, then the expected number of fades $N(L)$ approaches infinity as the fade depth L approaches zero. This seems to be nonphysical. Therefore, we require that $\mu \geqq 1/2$.*

Since $H(\alpha, \beta)$ is an arbitrary smooth function and μ can range from 1/2 to ∞, then the probability density function $f(\alpha, \beta)$, as given by equation (23), includes a large variety of fading environments.

V. POWER SERIES REPRESENTATION OF AMPLITUDE DISTRIBUTION

We shall assume that $H(\alpha, \beta)$ is sufficiently smooth so that the two-dimensional Taylor series[20] expansion of $H(\alpha, \beta)$ is applicable in the neighborhood of $\alpha = -1$, and $\beta = 0$. Several situations, where the Taylor series expansion of $H(\alpha, \beta)$ is not applicable, will be discussed in Section IX and Appendix C.

The Taylor series[20] expansion of $H(\alpha, \beta)$ gives

$$H(\alpha, \beta) = \sum_{n=0}^{n=\infty} \left[\frac{1}{n!} \sum_{r=0}^{r=n} C_r^n H_{n-r,r}(-1, 0)(1 + \alpha)^{n-r} \beta^r \right]$$

$$= H(-1, 0) + H_{1,0}(-1, 0)(1 + \alpha) + H_{0,1}(-1, 0)\beta \tag{27}$$

$$+ \frac{1}{2!} [H_{2,0}(-1, 0)(1 + \alpha)^2 + 2H_{2,0}(-1, 0)(1 + \alpha)\beta + H_{0,2}(-1, 0)\beta^2]$$

$$+ \cdots, \tag{28}$$

where

$$H_{n-r,r}(-1, 0) = \frac{\partial^n}{\partial\alpha^{n-r}\, \partial\beta^r} H(\alpha, \beta) \bigg|_{\substack{\alpha=-1 \\ \beta=0}} \tag{29}$$

$$C_r^n = \frac{n!}{r!\,(n - r)!}. \tag{30}$$

Substituting (27) and (23) into (20) for $P(V \leqq L)$ and carrying out the integration (Appendix D) yields

$$P(V \leqq L) = \sum_{S=0}^{\infty} d_{2S+2} L^{2S+2\mu} \tag{31}$$

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* Notice that the unity total probability requires that the singularity of $f(\alpha, \beta)$ be integrable (i.e., $\mu > 0$). The constraint $\mu \geqq \frac{1}{2}$ does not violate this condition.

$$= d_2 L^{2\mu} + \sum_{S=1}^{\infty} d_{2S+2} L^{2S+2\mu}, \tag{32}$$

where

$$d_2 = \frac{\pi H(-1, 0)}{\mu}, \tag{33}$$

$$d_{2S+2} = \frac{\pi}{S!\,(S + \mu) 2^{2S}} \sum_{\nu=0}^{\nu=S} \frac{H_{2S-2\nu,2\nu}(-1, 0)}{(\nu!)(S - \nu)!}, \tag{34}$$

$$S! = S(S - 1)(S - 2)\ldots 3\cdot 2\cdot 1.$$

The corresponding amplitude probability density function $p(L)$ is

$$p(L) = \frac{\partial}{\partial L} P(V \leqq L) = \sum_{S=0}^{S=\infty} (2S + 2\mu)\, d_{2S+2} L^{2S+2\mu-1}. \tag{35}$$

In the deep-fade region where L is small, the limiting forms of $P(V \leqq L)$ and $p(L)$ are

$$P(V \leqq L) \underset{L\to 0}{\longrightarrow} \frac{\pi H(-1, 0)}{\mu} L^{2\mu} \tag{36}$$

and

$$p(L) \underset{L\to 0}{\longrightarrow} 2\pi H(-1, 0) L^{2\mu-1}, \tag{37}$$

where $\mu \geqq 1/2$. The power law of deep fades for the three different cases (*i*) $\mu = 1$, (*ii*) $1 > \mu \geqq 1/2$, and (*iii*) $\mu > 1$ with their physical fading environments will be discussed in the following sections (VI, VII, and VIII respectively).

VI. PREVAILING SQUARE LAW OF DEEP FADES

For the nondiversity fading signals of most radio links, $f(\alpha, \beta)$ is neither singular nor zero at ($\alpha = -1, \beta = 0$). Then $\mu = 1$ and $f(\alpha, \beta) = H(\alpha, \beta)$. Equations (31) to (37) under this situation become

$$P(V \leqq L) = \sum_{S=0}^{\infty} d_{2S+2} L^{2S+2}, \tag{38}$$

$$= \pi f(-1, 0) L^2 + d_4 L^4 + d_6 L^6 + \cdots, \tag{39}$$

$$f(\alpha, \beta) = H(\alpha, \beta), \tag{40}$$

$$d_2 = \pi f(-1, 0) = \pi H(-1, 0), \tag{41}$$

$$d_{2S+2} = \frac{\pi}{(S + 1)!\, 2^{2S}} \sum_{\nu=0}^{\nu=S} \frac{f_{2S-2\nu,2\nu}(-1, 0)}{(\nu!)(S - \nu)!}, \tag{42}$$

$$f_{2S-2\nu,2\nu}(-1, 0) = \frac{\partial^{2S}}{\partial\alpha^{2S-2\nu}\,\partial\beta^{2\nu}} f(\alpha, \beta) \Bigg|_{\substack{\alpha=-1\\ \beta=0}}, \tag{43}$$

$$p(L) = \sum_{S=0}^{\infty} (2S + 2)\, d_{2S+2} L^{2S+1}, \tag{44}$$

$$P(V \leqq L) \underset{L\to 0}{\longrightarrow} \pi f(-1, 0) L^2, \tag{45}$$

and

$$p(L) \underset{L\to 0}{\longrightarrow} 2\pi f(-1, 0) L. \tag{46}$$

Equation (45) means that as long as $f(\alpha, \beta)$ is neither singular nor zero at ($\alpha = -1, \beta = 0$), then the cumulative amplitude distribution in the deep-fade region always obeys the square law

$$P(V \leqq L) \propto L^2, \qquad L_{\text{up}} \geqq L \geqq 0. \tag{12}$$

Notice that this conclusion does not depend on any specific probability density function $f(\alpha, \beta)$ for the interfering vector as long as $f(\alpha, \beta)$ is smooth and $\infty > f(-1, 0) > 0$. The conclusion applies for α and β either normal or not, either dependent* or independent, either with zero mean or with nonzero means. The magnitude R and the phase θ of the interfering vector can be either dependent* or independent and θ can be either uniformly or nonuniformly distributed. Therefore, this conclusion covers a wide class of signal fading problems.

Apparently the simple condition, $\infty > f(-1, 0) > 0$, is appropriate to the nondiversity fading signals of most radio links because the square law of deep fades is representative of the experimental data.[1–6,11–14,15–16]†

Notice that the first terms of equations (39) and (45), πL^2, are the area of the two-dimensional region on (α, β) plane bounded by the circle, $L^2 = (1 + \alpha)^2 + \beta^2$, in which $V \leqq L$, as shown in Fig. 3.

The coefficient $f(-1, 0)$ in equation (45) has been observed to depend upon path length, operating frequency, path profile, and geographical factors. From the experimental data of a large number of radio links, it is possible to deduce an empirical formula of $f(-1, 0)$ as a function of these parameters.[22,23,24,25]

In equation (12), the upper bound, L_{up}, of signal level below which

* This conclusion does not hold if the correlation coefficient between α and β or between R and θ is unity because the joint probability density $f(\alpha, \beta)$ becomes singular.

† This theoretical result also explains an experimental fact that the observed amplitude distributions of atmospheric radio noise are also characterized by the square law (12) in the small amplitude region.[21]

the square law applies also depends upon fading environment. Our experimental data show that L_{up} of most line-of-sight microwave radio links is above 0.3 (i.e., above -10 dB).

If $f(-1, 0)$ is negligibly small, so that the first term in the power series (39) can be neglected, then the second term, d_4L^4, dominates, with the result that the amplitude distribution follows the power law $P(V \leqq L) \propto L^4$. We have observed this behavior on a short radio link in the signal range from -10 dB to -20 dB (i.e., $0.3 \geqq L \geqq 0.1$). For fade depths deeper than -20 dB (i.e., $0.1 > L \geqq 0$) the quadratic term of (39) again dominates; and the transition region between $P(V \leqq L) \propto L^2$ and $P(V \leqq L) \propto L^4$ occurs at about -20 dB for this short path.

An obvious reason that $f(-1, 0)$ is small for short radio links is that the multipath length differences are mostly less than a half-wavelength.

VII. A DOMINANT COMPONENT INTERFERING SIGNAL

For this case, μ is bounded by $1 > \mu \geqq 1/2$. By equation (36)

$$P(V \leqq L) \underset{L\to 0}{\longrightarrow} \frac{\pi H(-1, 0)}{\mu} L^{2\mu}, \qquad 1 > \mu \geqq \tfrac{1}{2}, \tag{47}$$

and the corresponding power law of deep fades is

$$P(V \leqq L) \propto L^{2\mu}, \qquad 1 > \mu \geqq \tfrac{1}{2}. \tag{48}$$

Since μ is less than unity for this case, then as $L \to 0$, the probability of deep fades decreases more slowly than square law (12). Physically, this means that the deep-fade problem for these links is more severe. The experimental data of two oversea paths (shown as curve 2 in Fig. 16.5a and as curve 1 in Fig. 16.5b of Reference 15) follow this power law (48) of severe fading.

It is true, however, that some overwater radio links still obey the square law (12) rather than (48).[15,17] The reason is that because of the geometry of the radio link, the singularity in the density function $q(R, \theta)$ may occur, if it exists at all, at a position far away from the infinite fade point ($R = 1, \theta = \pi$). Then in the neighborhood of the infinite fade point ($R = 1, \theta = \pi$), the density function $q(R, \theta)$ (or equivalently $f(\alpha, \beta)$) may still be a smooth function.

For overland paths, it is possible that an exceptionally calm and stratified atmosphere would also create a stable, dominant interfering signal over a sustained period. Then the joint probability density function $f(\alpha, \beta)$ may also be singular at the position of this stable, dominant interfering signal. Therefore, the results of this section on the power law of severe fading may also occur on an overland radio link.

VIII. DIVERSITY SYSTEMS

When $\mu > 1$, the power law of deep fades is

$$P(V \leqq L) \underset{L \to 0}{\longrightarrow} \frac{\pi H(-1, 0)}{\mu} L^{2\mu}, \qquad \mu > 1, \tag{49}$$

$$P(V \leqq L) \propto L^{2\mu}, \qquad \mu > 1. \tag{50}$$

As L decreases, the probability of deep fades decreases faster than those following the square law (12). Physically, this means the problem of fading for this case is less severe than those following the square law (12).

The experimental data[1,2,5] of composite signals of the outputs of diversity combining systems show that the amplitude distributions of composite signals in the deep-fade region obey the power-law equation (50) rather than the square-law equation (12).

Since $\mu > 1$ implies $f(\alpha, \beta)$ has a zero at the deep-fade point ($\alpha = -1$, $\beta = 0$), these results show that the artificial active combining devices of diversity combining systems serve to create a zero at ($\alpha = -1, \beta = 0$) of order $2(\mu - 1)$ of the density function $f(\alpha, \beta)$ of the equivalent interfering vector of the output composite signal. The value of μ depends on the order of diversity. By comparing the power law (50) to the experimental data[1,2,5] and the theoretical results on the diversity systems, we find that for most overland paths, the value of μ for the composite signal is equal to the order of diversity.

IX. ONE-ECHO MODEL

In the model described by equation (15) for the fading signal, if there is only one echo and if the magnitude of this echo is a constant, then $R \equiv A$ is a constant rather than a random variable. For this idealized case, the joint probability density function $q(R, \theta)$ of R and θ contains a delta function

$$q(R, \theta) = \delta(R - A)W(\theta), \tag{51}$$

where $W(\theta)$ is the probability density function of the random relative phase θ between the echo $Ae^{i\theta}$ and the constant vector.

For convenience, we shall call this specialized model the one-echo model. (In the literature, it is also known as the two-ray model.)

In the analysis, we shall assume that $A \leqq 1$. The case where $A > 1$

can be treated similarly simply by switching the roles of echo and the constant vector.

Since the joint probability density function of the interfering vector of the one-echo model does not belong to the class of $f(\alpha, \beta)$ discussed in Section IV of this part, the results of previous sections are not directly applicable. Nevertheless, substituting the density function (51) into the general formulation (21) for $P(V \leqq L)$ and integrating over R (see Fig. 4) yields

$$P(V \leqq L) = \int_{\pi-\theta_L}^{\pi+\theta_L} W(\theta)\, d\theta, \tag{52}$$

where

$$\theta_L = \cos^{-1}\left(\frac{1 + A^2 - L^2}{2A}\right), \qquad (1 + A) \geqq L \geqq (1 - A). \tag{53}$$

Since the behavior of $W(\theta)$ in the neighborhood of $\theta = \pi$ is important for the analysis of deep fades, we shall assume that $W(\theta)$ is smooth in this neighborhood so that the Taylor series expansion of $W(\theta)$ is applicable. Then

$$W(\theta) = W(\pi) + W_1(\pi)(\theta - \pi) + \frac{W_2(\pi)}{2!}(\theta - \pi)^2 + \cdots, \tag{54}$$

where

$$W_n(\pi) = \frac{d^n}{d\theta^n} W(\theta)\bigg|_{\theta=\pi}, \qquad n = 1, 2, 3, \cdots. \tag{55}$$

Substituting equation (54) into equation (52) and carrying out the integration yields

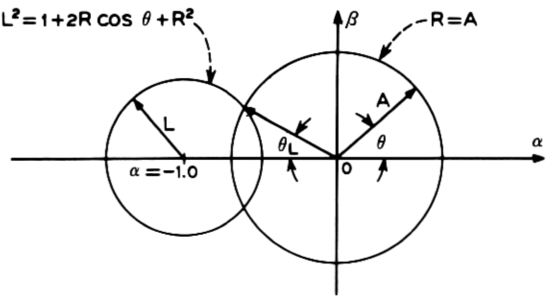

Fig. 4—The range of phase, $(\pi - \theta_L) \leqq \theta \leqq (\pi + \theta_L)$, in which $V \leqq L$.

$$P(V \leqq L) = 2 \sum_{S=0}^{\infty} \frac{W_{2S}(\pi)}{(2S+1)!} \left[\cos^{-1} \left(\frac{1 + A^2 - L^2}{2A} \right) \right]^{2S+1},$$

$$(1 + A) \geqq L \geqq (1 - A). \qquad (56)$$

If the magnitudes of the two vectors are equal, then $A = 1$, and equation (56) becomes

$$P(V \leqq L) = 2 \sum_{S=0}^{\infty} \frac{W_{2S}(\pi)}{(2S+1)!} \left[\cos^{-1} \left(1 - \frac{L^2}{2} \right) \right]^{2S+1}, \qquad 2 \geqq L \geqq 0. \qquad (57)$$

Since

$$\cos^{-1} \left(1 - \frac{L^2}{2} \right) = \sin^{-1} \left[L \left(1 - \frac{L^2}{4} \right) \right]^{\frac{1}{2}}; \qquad (58)$$

$$\therefore \cos^{-1} \left(1 - \frac{L^2}{2} \right) \xrightarrow[L \to 0]{} L \qquad (59)$$

Then the behavior of $P(V \leqq L)$ given by equation (57) in the deep-fade region is

$$P(V \leqq L) \xrightarrow[L \to 0]{} 2W(\pi)L; \qquad (60)$$

$$\therefore p(L) \xrightarrow[L \to 0]{} 2W(\pi). \qquad (61)$$

This result shows that as long as the probability density function $W(\theta)$ of the random phase θ is neither singular nor zero at $\theta = \pi$, then the cumulative amplitude distribution of the one-echo model with $R \equiv 1$ always obeys the power law

$$P(V \leqq L) \propto L \qquad (62)$$

in the deep-fade region no matter whether $W(\theta)$ is uniform or not. Equation (57) shows that the nonuniform part of $W(\theta)$ contributes only to the high-order terms of $P(V \leqq L)$ and does not affect the behavior of $P(V \leqq L)$ in the deep fade region.

If the distribution of the random relative phase is uniform in $(0, 2\pi)$, then

$$W(\theta) = \frac{1}{2\pi}$$
$$W_n(\pi) = 0, \qquad n \geqq 1. \qquad (63)$$

Equation (56) specialized to this case is

$$P(V \leqq L) = \frac{1}{\pi} \cos^{-1} \left(\frac{1 + A^2 - L^2}{2A} \right). \qquad (64)$$

If $A = 1$, then equation (64) becomes

$$P(V \leqq L) = \frac{1}{\pi} \cos^{-1}\left(1 - \frac{L^2}{2}\right). \tag{65}$$

As far as the deep-fade region is concerned, equation (57) can also be written as

$$P(V \leqq L) = 2W(\pi)L + 0(L^{1+\eta}), \qquad \eta > 0; \tag{66}$$

$$\therefore p(L) = 2W(\pi) + 0(L^\eta), \tag{67}$$

where $0(L^\eta)$ is a symbol to denote the component which goes to zero at a rate equal to or faster than that of L^η as $L \rightarrow 0$.

Although the one-echo model of this section and the other two cases discussed in Appendix C do not exhaust all the situations where $H(\alpha, \beta)$ is not analytic, the main objective is to show that the assumption of Taylor series expansion of $H(\alpha, \beta)$ in Section V of this part is not strictly necessary for the derivation of the power law of deep fades.

To unify the representations for all the cases considered in this paper, we shall rewrite equations (66) and (67) as

$$P(V \leqq L) = 2W(\pi)L^{2\mu} + 0(L^{2\mu+\eta}) \tag{68}$$

and

$$p(L) = 2W(\pi)L^{2\mu-1} + 0(L^{2\mu-1+\eta}), \qquad \mu = \tfrac{1}{2}, \quad \eta > 0. \tag{69}$$

Then the amplitude distributions of deep fades of all the cases discussed in Part 1 can be summarized as

$$P(V \leqq L) = d_2L^{2\mu} + 0(L^{2\mu+\eta}) \tag{70}$$

and

$$p(L) = 2\mu\, d_2L^{2\mu-1} + 0(L^{2\mu-1+\eta}), \qquad \mu \geqq \tfrac{1}{2}, \quad \eta > 0. \tag{71}$$

If $H(\alpha, \beta)$ is continuous at $(\alpha = -1, \beta = 0)$, then $d_2 = \pi H(-1, 0)/\mu$. If $H(\alpha, \beta)$ is discontinuous at $(\alpha = -1, \beta = 0)$, then $d_2 = \pi \bar{H}(-1, 0)/\mu$. For a one-echo model, $d_2 = 2W(\pi)$ and $\mu = 1/2$.

Part 2. Expected Number of Fades and Average Fade Duration

Contents

I. INTRODUCTION AND SUMMARY

In Part 1 we investigated the amplitude distribution of a fading signal. In a long time period, the cumulative amplitude distribution $P(V \leqq L)$ tells us the expected fraction of this time period that the signal will fade below any specified signal level L. However, $P(V \leqq L)$ does not tell us anything about the dynamic aspects of the fading signal. For example, a large number of short fades and a small number of long fades may have the same amplitude distribution.

Some communication systems may tolerate the short fades but not the long fades. Furthermore, in the design of a diversity combining device, a distortion equalizer, or an automatic gain controlling device to combat the fading problem, one needs information on the dynamic behavior of the fading signal.

In Part 2 we present the results of our investigation on the expected number $N(L)$ per unit time that the signal $V(t)$ fades below a given signal level L; and the average duration $\bar{t}(L)$ of fades below L.

The analysis is based on the general integral formulation of $N(L)$ by Rice[26,27] and Vigants[4] and our results for $P(V \leqq L)$ in Part 1. Again, we do not impose the restrictive assumption of the complex Gaussian model so that the theoretical results may be applied to a wide class of fading problems.

In the study of $N(L)$ and $\bar{t}(L)$, the experimental data[1-6] for $N(L)$ or $\bar{t}(L)$ are often plotted on a log scale as shown in Figs. 5 and 6. It is an experimental fact that the data for $N(L)$ and $\bar{t}(L)$ can be well represented by straight lines on this kind of graph paper for fade depth deeper than -10 dB, as shown in Figs. 5 and 6. The slopes of these straight lines are directly related to the power laws of $N(L)$ and $\bar{t}(L)$ in the deep-fade region. The experimental observations of $N(L)$ and $\bar{t}(L)$ are summarized below:

(*i*) The experimental data show that $N(L)$ for most nondiversity fading signals obeys the power law, $N(L) \propto L$, in the deep-fade region.

(*ii*) For a short radio link from Villa Rica to Palmetto, Georgia, the $N(L)$ of a nondiversity signal follows the cubic power law, $N(L) \propto L^3$, in the deep-fade region.

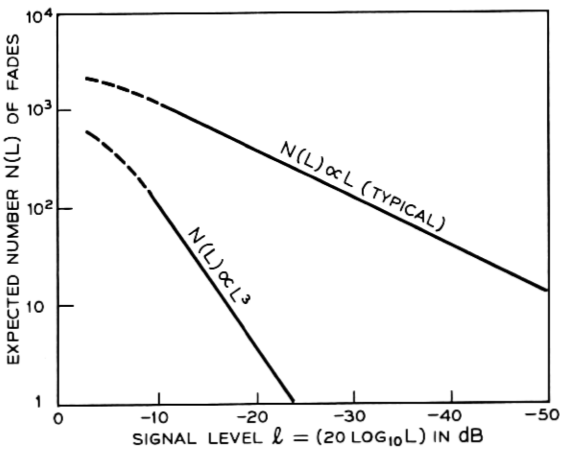

Fig. 5—Number of fades below signal level L.

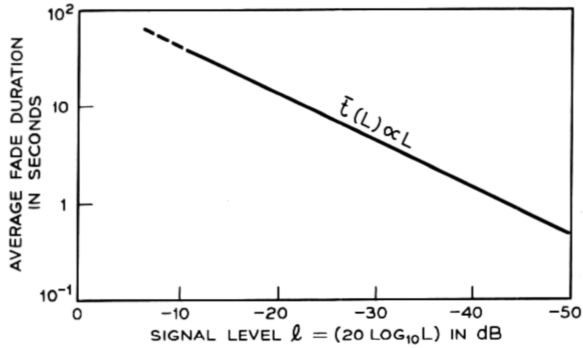

Fig. 6—Average duration of fades below signal level L.

(*iii*) The experimental data of $N(L)$ of composite fading signals of most dual diversity systems also follows the cubic power law, $N(L) \propto L^3$, in the deep-fade region.

(*iv*) The available experimental data on average fade duration $\bar{t}(L)$ all obey the universal power law, $\bar{t}(L) \propto L$, in the deep-fade region. This includes the fading signals of nondiversity systems, diversity systems, long radio links, and short radio links.

In summary:

(*i*) Our theory indicates that if the joint probability density function $f(\alpha, \beta)$ of the resultant interfering vector, $Re^{i\theta} = \alpha + j\beta$, is a smooth function which is neither singular nor zero at the infinite fade point ($\alpha = -1, \beta = 0$), then for small L

$$N(L) \simeq \pi a_o f(-1, 0)L \propto L,$$

where a_o is a constant approximately equal to the average positive derivative of the amplitude of the fading signal in the deep-fade region.

(*ii*) If $f(\alpha, \beta)$ is singular or zero at ($\alpha = -1, \beta = 0$) then

$$N(L) \simeq \pi H(-1, 0)a_o L^{2\mu-1} \propto L^{2\mu-1}$$

for $\mu \geqq 1/2$ and small L. The cubic power law, $N(L) \propto L^3$, of dual diversity systems and short radio links can be explained by this result when $\mu = 2$, which means $f(-1, 0)$ is zero or is negligibly small.

(*iii*) The theory predicts that the average fade duration always obeys the power law $\bar{t}(L) \propto L$ for small L no matter whether $f(\alpha, \beta)$ is smooth, singular or zero at ($\alpha = -1$, $\beta = 0$). This means the power law, $\bar{t}(L) \propto L$, is invariant with respect to variations of fading environment and diversity combinations of fading signals. This prediction agrees with the available experimental data.

(*iv*) The theoretical results on $N(L)$ and $\bar{t}(L)$ for the one-echo model are shown to be incompatible with the experimental data of most overland microwave radio links. Therefore, the one-echo model is not suitable for the study of the statistics of fading signals of these radio links.

(*v*) For line-of-sight radio links at 4 GHz and 6 GHz, the average positive derivative of the amplitude of the fading signal is estimated to range from 2×10^{-3} to 4×10^{-3} times V_{ref} per second, where V_{ref} is the signal level when there is no interference.

II. GENERAL FORMULATION FOR NUMBER OF FADES

The general expression for the expected number of fades per unit time of a random signal $V(t)$, below signal level L has been shown[26,27,4] to be

$$N(L) = \int_{\dot{V}=0}^{\dot{V}=\infty} \dot{V} p(\dot{V}, V) \,|_{V=L}\, d\dot{V}, \tag{72}$$

where $\dot{V} = dV/dt$, and $p(\dot{V}, V)$ is the joint probability density of $\dot{V}$ and V. For the sake of completeness, a brief derivation of (72) is included in Appendix E.

The joint probability density function $p(\dot{V}, V)$ can be written in terms of conditional probability[28] as

$$p(\dot{V}, V) = p_1(\dot{V} \mid V) p_2(V), \tag{73}$$

where $p_1(\dot{V} \mid V)$ is the conditional probability density of $\dot{V}$ under the condition that the signal level is V; and $p_2(V)$ is the probability density of V. Substituting (73) into (72) yields

$$N(L) = p_2(L) \int_{\dot{V}=0}^{\dot{V}=\infty} \dot{V} p_1(\dot{V} \mid L)\, d\dot{V}. \tag{74}$$

Let us define

$$\bar{\dot{V}}_+(L) = 2 \int_{\dot{V}=0}^{\dot{V}=\infty} \dot{V} p_1(\dot{V} \mid L)\, d\dot{V}. \tag{75}$$

The physical meaning of the definition (75) is that $\bar{\dot{V}}_+(L)$ is the conditional average positive derivative of V under the condition $V = L$.

The factor 2 in (75) is based upon the assumption that $p_1(\dot{V} \mid L)$ is symmetric about $\dot{V} = 0$.

Since $V(t)$ is a random fading signal, at a given signal level L, the value of its time derivative $\dot{V}$ is also random. In general, the conditional average positive derivative $\bar{\dot{V}}_+(L)$ is a function of signal level L.

By using definition (75), equation (74) becomes

$$N(L) = \tfrac{1}{2}\bar{\dot{V}}_+(L) p_2(L). \tag{76}$$

Or, equivalently,

$$N(L) = \tfrac{1}{2}\bar{\dot{V}}_+(L) \frac{\partial}{\partial L} P(V \leqq L), \tag{77}$$

which indicates $N(L)$ proportional to the conditional average positive derivative of the fading signal and to the probability density of fades at $V = L$.

III. GENERAL FORMULATION FOR AVERAGE FADE DURATION

In a long time interval, T, containing a large number of fades,* the expected total length of time that the random signal $V(t)$ spends below a specified signal level L is

$$t(L) = TP(V \leqq L). \tag{78}$$

The expected number of fades below L in this interval T is $TN(L)$. Therefore, the average duration of fades below L is

$$\bar{t}(L) = \frac{t(L)}{TN(L)} = \frac{P(V \leqq L)}{N(L)}. \tag{79}$$

Substituting (77) into (79) yields

$$\bar{t}(L) = \frac{1}{\bar{\dot{V}}_+(L)} \frac{2P(V \leqq L)}{\dfrac{\partial}{\partial L} P(V \leqq L)}. \tag{80}$$

Equation (80) shows that the average fade duration is inversely proportional to the conditional average positive derivative of the fading signal at $V = L$.

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* In our experiment on line-of-sight radio links, the typical time interval T is a whole summer of more than 100 days in which there are more than 500 fades below -10 dB relative to the nonfaded signal level.

IV. ASSUMPTION ON CONDITIONAL AVERAGE POSITIVE DERIVATIVE

Most existing theoretical work on $N(L)$ assumes that V and $\dot{V}$ are independent so that $\bar{\dot{V}}_+(L)$ becomes a constant which is independent of signal level L. In this paper we include the situation for which V and $\dot{V}$ are dependent and assume that $\bar{\dot{V}}_+(L)$ can be expanded into a Taylor series in the deep-fade region (i.e., small L); then

$$\bar{\dot{V}}_+(L) = a_o + a_1 L + a_2 L^2 + a_3 L^3 + \cdots, \tag{81}$$

where

$$a_0 = \lim_{L \to 0^+} \bar{\dot{V}}_+(L) = \bar{\dot{V}}_+(0^+), \tag{82}$$

$$a_1 = \lim_{L \to 0^+} \frac{\partial}{\partial L} \bar{\dot{V}}_+(L), \tag{83}$$

$$a_2 = \frac{1}{2!} \lim_{L \to 0^+} \frac{\partial^2}{\partial L^2} \bar{\dot{V}}_+(L), \quad \text{etc.} \tag{84}$$

The justification for this assumption is not trivial, and includes the following considerations:

(*i*) The theoretical results based on this assumption agree with the available experimental data.

(*ii*) For a complex Gaussian model, the conditional probability density function $p_1(\dot{V} \mid L)$ is known. Then with the help of the work of Rice[26,27] the integration indicated in (75) for $\bar{\dot{V}}_+(L)$ can be carried out in closed form. These explicit expressions are discussed in Section III of Part 3. The results of this model show that if the power spectrum of the Gaussian noise is symmetric with respect to the frequency of the sine wave (i.e., the constant unit vector), then $\bar{\dot{V}}_+(L) \equiv a_o$ is a constant independent of L. On the other hand, if the power spectrum of the Gaussian noise is asymmetric with respect to the signal frequency, then $\bar{\dot{V}}_+(L)$ is a function of L and the nonconstant terms in equation (81) cannot be omitted.

The theoretical work of Clarke,[29] Ossanna,[30] and Gans[31] on mobile radio indicate that the power spectrum of the fading signal is generally asymmetric with respect to the received carrier frequency unless the straight line joining the base station and the mobile antenna is perpendicular to the velocity of the mobile and the antenna pattern is symmetric with respect to this line. Therefore, the work on asymmetric power spectrum, and hence nonconstant $\bar{\dot{V}}_+(L)$, is not purely academic.

(*iii*) It is known[18] that the correlation between any real random variable $\epsilon(t)$ at instant t and its time derivative $\dot{\epsilon}(t + \zeta)$ at instant $(t + \zeta)$ vanishes if $\zeta = 0$. This is often used to support the assumption that V and $\dot{V}$ are independent and hence $\bar{\dot{V}}_+(L)$ is a constant. However, we know that the vanishing of correlation between $V(t)$ and $\dot{V}(t + \zeta)$ at $\zeta = 0$ does not imply the independency of V and $\dot{V}$ unless V is normally distributed. Including high-order terms in equation (81) removes the assumption of independency of V and $\dot{V}$ and enlarges the applicable scope of this theory.

(*iv*) In equations (82), (83), and (84) we define those coefficients of the Taylor series as the limits of $\bar{\dot{V}}_+(L)$ and its L-derivatives at $L = 0^+$ from the positive side. The reason is that $V(t)$ is the absolute value of a fluctuating complex signal; i.e.,

$$V(t) = | x + jy | = | Ve^{i\phi} | .$$

When the complex fluctuating signal $V(t)e^{i\phi(t)}$ crosses zero, its absolute value $V(t)$ may have a cusp at $V = 0$ as shown in Fig. 7. Therefore, the derivatives of $V(t)$ may not be well defined at $V = 0$. However, the limits of the derivatives at $V = 0^+$ from the positive side are well defined.

(*v*) Fig. 7 also shows that although $V = 0$ is a minimum of $V(t)$,

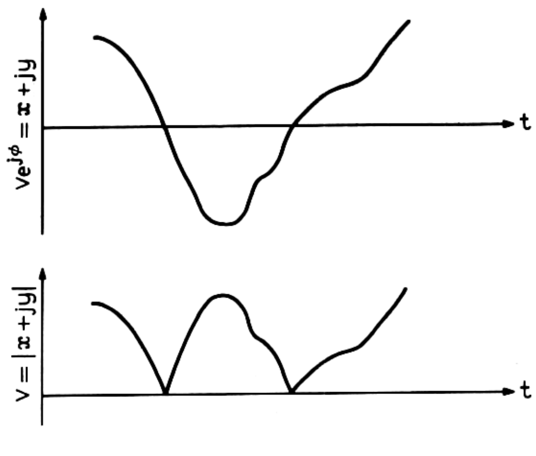

Fig. 7—The amplitude $V(t)$ may have a cusp at $V = 0$ even though the complex fading signal is a smooth time function. For convenience, the complex fading signal is plotted as a real function in this example.

$\dot{V}(t) = d/dt\, V(t)$ may not be zero at $V = 0^+$. Therefore, in equation (81), the constant term $a_o = \bar{\dot{V}}_+(0^+)$ does not vanish for most cases, and cannot be omitted from equation (81).

V. POWER SERIES FORM OF $N(L)$

Substituting the power series (35) for $(\partial/\partial L)\, P(V \leqq L)$ of Part 1 and the power series (81) for $\bar{\dot{V}}_+(L)$ into equation (77) yields

$$N(L) = \pi H(-1, 0) a_0 L^{2\mu-1} + \pi H(-1, 0) a_1 L^{2\mu} + \cdots \tag{85}$$

In the deep-fade region, the leading term dominates the power series (85). Therefore,

$$N(L) \cong \pi H(-1, 0) a_0 L^{2\mu-1}, \quad \text{for small} \quad L; \tag{86}$$

$$\therefore\ N(L) \propto L^{2\mu-1}, \quad \text{for small} \quad L. \tag{87}$$

Equations (85) and (86) show that if $a_o \neq 0$ and if $\mu < 1/2$, then $N(L) \rightarrow \infty$ as $L \rightarrow 0$. This seems to be nonphysical. Therefore, we require that $\mu \geqq 1/2$. This is the reason we impose this condition on the order of singularity of $f(\alpha, \beta)$ in Section IV of Part 1.*

5.1 *Prevailing Power Law of Number of Deep Fades*

For the nondiversity fading signals of most radio links, the probability density function $f(\alpha, \beta)$ of the resultant interfering signal is a smooth function which is neither singular nor zero at $(\alpha = -1, \beta = 0)$. Then $\mu = 1$ and $f(\alpha, \beta) = H(\alpha, \beta)$. Equations (85), (86), and (87) under this condition become

$$N(L) = \pi f(-1, 0) a_0 L + \pi f(-1, 0) a_1 L^2 + [2a_0\, d_4 + \pi f(-1, 0) a_2] L^2 + \cdots \tag{85'}$$

$$N(L) \cong \pi f(-1, 0) a_0 L, \quad \text{for small} \quad L. \tag{86'}$$

$$N(L) \propto L, \quad \text{for small } L. \tag{87'}$$

It is seen that as long as $\infty > f(-1, 0) > 0$, then the expected number $N(L)$ of deep fades always obeys the prevailing power law (86′). With reference to Fig. 5, the straight lines corresponding to the power law (87′) have inverse slopes of 20 dB per decade.

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* If we assume that $a_0 = 0$, then the only constraint on μ is $\mu > 0$ due to the unity total probability. However, assuming $a_0 = 0$ implies that the time derivative of V is always zero at $V = 0$. Such an assumption is unreasonable for multipath interference fading, but may be useful if the effects of random circuit interruptions, such as equipment failure, are included in the signal fading problem.

Apparently, the simple condition is easily satisfied by the nondiversity fading signals of most microwave radio links because the experimental[1–6] data of $N(L)$ are mostly characterized by the inverse slope of 20 dB per decade in the deep-fade region.

As L increases from zero towards unity, equation (85′) indicates that there may be a transition point beyond which high-order terms become significant and the slope begins to deviate.

For most microwave radio links, the transition points of $N(L)$ seem to be well above -10 dB. However, our latest experimental data show that the first transition point of $N(L)$ of a relatively short path (path length 15.87 miles, $f = 4$ GHz) is below -20 dB. The inverse slope of $N(L)$ in the region from -10 dB to -20 dB is approximately 20/3 dB per decade of number of fades. This indicates that the third term $[2a_o\, d_4 + \pi a_2 f(-1, 0)]L^3$ dominates in the region $0.3 > L > 0.1$ for this path.

VI. AVERAGE DURATION OF DEEP FADES

Substituting the power series (31) of $P(V \leqq L)$ of Part 1 and the power series (85) of $N(L)$ into equation (79) yields

$$\bar{t}(L) = \frac{\dfrac{\pi H(-1,0)}{\mu} L^{2\mu} + d_4 L^{2\mu+2} + d_6 L^{2\mu+4} + \cdots}{\pi H(-1,0) a_0 L^{2\mu-1} + \pi H(-1,0) a_1 L^{2\mu} +}. \tag{88}$$

In the deep-fade region, equation (88) becomes

$$\bar{t}(L) \cong \frac{1}{\mu a_0} L, \quad \text{for small} \quad L; \tag{89}$$

$$\therefore\ \bar{t}(L) \propto L, \quad \text{for small} \quad L. \tag{90}$$

On Fig. 6, the straight lines corresponding to the power law (90) have an inverse slope of 20 dB per decade of fade duration. The experimental data[1–6] agree with this conclusion on the slope of $\bar{t}(L)$ when plotted on Fig. 6.

VII. INVARIANCE OF POWER LAW OF AVERAGE FADE DURATION*

Equations (36), (87), and (90) show that in general $P(V \leqq L)$, $N(L)$, and $\bar{t}(L)$ obey the following set of power laws of deep fades:

* In equation (26) of Ref. 32, Rice has already predicted that the power law $\bar{t}(L) \propto L$ for small L may be applicable to cases more general than the complex Gaussian model even though most of his work in Ref. 32 is devoted to the statistics of a sine wave plus a narrowband Gaussian noise.

$$\left.\begin{aligned} P(V \leqq L) &\propto L^{2\mu} \\ N(L) &\propto L^{2\mu-1} \\ \bar{t}(L) &\propto L \end{aligned}\right\} \quad \text{for} \quad \mu \geqq \tfrac{1}{2} \quad \text{and small} \quad L. \qquad \begin{aligned}(91)\\(92)\\(93)\end{aligned}$$

It is seen that the power laws of $P(V \leqq L)$ and $N(L)$ depend on the value of μ which depends on whether $f(\alpha, \beta)$ is smooth, singular, or zero at $(\alpha = -1, \beta = 0)$. On the other hand, the power law (93) for the average fade duration is invariant with respect to μ. Since the behavior of $f(\alpha, \beta)$, and hence the value of μ, depends on fading environment, we conclude that the power law (93) for the average fade duration is insensitive to the fading environment in contrast to the power laws of $P(V \leqq L)$ and $N(L)$.

In equation (89), notice that μa_o, and hence $\bar{t}(L)$, does depend on the fading environment. However, it is the *power law*, $\bar{t}(L) \propto L$, which is insensitive to the fading environment.

Vigants,[4,5,6] Crawford, Hogg, and Kummer[12] have investigated the effects of diversity on $P(V \leqq L)$, $N(L)$ and $\bar{t}(L)$. The theoretical results and the experimental results of these authors show that in the deep-fade region, the diversity drastically changes the power laws of $P(V \leqq L)$ and $N(L)$ but does not affect the power law $\bar{t}(L) \propto L$. For example, the results of Vigants are shown in Table I. From this table it is seen that the power laws of $P(V \leqq L)$ and $N(L)$ depend on the diversity combination of fading signals, whereas the power law, $\bar{t}(L) \propto L$, of average fade duration is invariant.

VIII. INCOMPATIBILITY BETWEEN ONE-ECHO MODEL AND OVERLAND RADIO LINKS

Equations (60), (61), (91), (92), and (93) show that $p(L)$, $N(L)$, and $\bar{t}(L)$ of the one-echo model (with equal magnitudes, $A = 1$) in the

TABLE I—EFFECTS OF DIVERSITY ON POWER LAWS OF DEEP FADES

	Nondiversity	Diversity
$P(V \leqq L)$	L^2	$(1/q)\,L^4$
$N(L)$	cL	$(2c/q)\,L^3$
$\bar{t}(L)$	$(1/c)\,L$	$(1/2c)\,L$

Remark: In this table, the parameter, c, as defined by Vigants, is equal to a_0 of this paper; and the parameter, q, as defined by Vigants, is equal to $2/\pi H(-1, 0)$ of this paper.

deep-fade region are

$$\left.\begin{aligned} p(L) &\propto L^0 \\ N(L) &\propto L^0 \\ \bar{t}(L) &\propto L \end{aligned}\right\} \quad \text{for small} \quad L. \qquad \begin{aligned} &(94) \\ &(95) \\ &(96) \end{aligned}$$

On the other hand, the long-term experimental data of a nondiversity signal of most overland microwave radio links indicate that

$$\left.\begin{aligned} p(L) &\propto L \\ N(L) &\propto L \\ \bar{t}(L) &\propto L \end{aligned}\right\} \quad \text{for small} \quad L. \qquad \begin{aligned} &(97) \\ &(98) \\ &(99) \end{aligned}$$

The experimental results (97) and (98) disagree with (94) and (95) of the one-echo model.

In view of this disagreement, we may want to check the effect of the assumption (81) of $\bar{\dot{V}}_{+}(L)$ on the theoretical results of the one-echo model. Although we know that the constant term a_o of (78) generally does not vanish, yet we may deliberately set $a_o = 0$ and see what kind of theoretical results we get.

If we do so, the theoretical results of the one-echo model become

$$\left.\begin{aligned} p(L) &\propto L^0 \\ N(L) &\propto L \\ \bar{t}(L) &\propto L^0 \end{aligned}\right\} \quad \text{for small} \quad L. \qquad \begin{aligned} &(100) \\ &(101) \\ &(102) \end{aligned}$$

Under this modified assumption, $p(L)$ and $\bar{t}(L)$ of the one-echo model disagree with the experimental results (97) and (99). Similarly, forcing the coefficients of other higher order terms of (81) to zero also yields theoretical results which disagree with the experimental results. Therefore, we conclude that the one-echo model is not suitable for the study of the fading signals of most overland microwave radio links.

However, we emphasize that the experimental data mentioned in this section are restricted to the long-term data of overland microwave radio links. Therefore, the incompatibility of the one-echo model with these data does not necessarily exclude the use of this model for the study of other fading problems.

IX. APPROXIMATE AVERAGE POSITIVE DERIVATIVE $\bar{\dot{V}}_+$ OF LINE-OF-SIGHT MICROWAVE RADIO LINKS

For line-of-sight microwave radio links, let V_{ref} be the signal level when there is no interference, and $V_{\text{fad}}(t)$ be the random fading signal when the interference appears. In our analysis

$$V(t) = \frac{V_{\text{fad}}(t)}{V_{\text{ref}}} \tag{103}$$

is a normalized fading signal.

By comparing the experimental data of $\bar{t}(L)$ and the theoretical equation (89) for $\bar{t}(L)$, we can estimate the value of a_o of the radio link. The value of μ in equation (89) can be determined from the experimental data on the power laws (36) and (86) for $P(V \leqq L)$ and $N(L)$ of the same radio link.

Our experimental data of several line-of-sight microwave radio links in Ohio and Georgia indicate that the value of a_o ranges from 2×10^{-3} to 4×10^{-3}. In the deep-fade region where L is small, equation (81) shows that $\bar{\dot{V}}_+(L) \cong a_o$.

$$\left| \frac{d}{dt} V(t) \right|_{\text{average}} \cong a_0 = 2 \times 10^{-3} \sim 4 \times 10^{-3}. \tag{104}$$

Substituting (103) into (104) yields

$$\left| \frac{d}{dt} V_{\text{fad}}(t) \right|_{\text{average}} \cong (2 \times 10^{-3} \sim 4 \times 10^{-3}) \cdot V_{\text{ref}} \ . \tag{105}$$

Thus, the average positive derivative of the unnormalized fading signal, $V_{\text{fad}}(t)$, of these microwave radio links ranges from 2×10^{-3} to 4×10^{-3} times V_{ref} per second.

These approximate values of average positive derivative are valid only in the deep-fade region because they are deduced from the experimental data of deep fades. The path length of these radio links ranges from 15 miles to 36 miles. The operating frequencies are in 4-GHz and 6-GHz bands.

X. GENERALIZED ASSUMPTION ON $\bar{\dot{V}}_+(L)$

In Part 1, we indicated that the assumption of the Taylor series expansion of $H(\alpha, \beta)$ is not strictly necessary for the validity of the power law of $P(V \leqq L)$ deduced from the experimental data. In this section, we point out that the assumption of the Taylor series expansion of $\bar{\dot{V}}_{+}(L)$ in Section IV of this part is also not strictly necessary for the validity of the power laws of $N(L)$ and $\bar{t}(L)$. From a theoretical viewpoint, the assumption of $\bar{\dot{V}}_{+}(L)$ can be generalized to the following form:

$$\bar{\dot{V}}_{+}(L) = a_o + 0(L^{\eta}), \qquad \eta > 0 \tag{106}$$

where $0(L^{\eta})$ is a symbol to denote the component which goes to zero at a rate equal to or faster than that of L^{η} as $L \rightarrow 0$.

In assumption (106), we do not require the existence of the limits in equations (83), (84), etc. Therefore, the assumption (106) is less restrictive than the assumption (81). It can be shown that the power laws of $N(L)$ and $\bar{t}(L)$ of deep fades based on (106) are the same as those based on (81). However, at the present time, we do not have any practical evidence to necessitate the use of (106). Therefore, we merely point out the possibility but do not explicitly carry out this generalized analysis.

Part 3. Special Topics on Statistics of a Fading Signal

Contents

I. INTRODUCTION AND SUMMARY

In Parts 1 and 2, the analysis was oriented towards an explanation of the experimentally observed common behavior of a fading signal $Ve^{i\phi}$. The basic assumptions of the theoretical model are kept to a minimum in order to include the widest possible variation in practical fading environments. During the development of this general analysis, we have gained a new insight into several topics related to fading signals as investigated by previous authors.

Part 3 of this paper is a collection of theoretical treatments of several special topics relating our generalized analysis to the work of previous authors. These topics include the sum of n unit vectors with random phases; a sine wave plus Gaussian noise, m-distributions, chi-distribution, Rayleigh distribution, and log normal distribution.

In summary:

(*i*) In Section II of this part, the results of Part 1 are applied to find the amplitude distribution of the sum of n unit vectors with uniformly distributed random phases. For $n \geqq 3$, the analysis shows that the amplitude distribution always follows the square law $P(V \leqq L) \propto L^2$ for small L. On the other hand, when $n = 2$, the amplitude distribution follows the power law $P(V \leqq L) \propto L$ for small L.

(*ii*) In Section III, we investigate the model of a sine wave plus a narrowband Gaussian noise for the fading signal. By using the closed-form solutions of Rice, it is shown that if the power spectrum of the Gaussian noise is symmetric with respect to the frequency of the sine wave, then the amplitude V and its time derivative $\dot{V}$ are independent; and the conditional average positive derivative $\bar{\dot{V}}_+(L)$ is a constant. On the other hand, if the power spectrum is not symmetric, then V and $\dot{V}$ are dependent; and the conditional average positive derivative $\bar{\dot{V}}_+(L)$ is a function of signal level $V = L$.

As an example, the fading signal spectral density of a mobile radio is generally not symmetric with respect to the received carrier frequency. Therefore, in the analysis of $N(L)$ and $\bar{t}(L)$, it is not safe to assume that V and $\dot{V}$ are always independent.

(*iii*) In Section IV, we investigate the theoretical condition (147) on the joint probability density function $f(\alpha, \beta)$ of the interfering vector such that the amplitude distribution of the fading signal belongs to the family of m-distributions which includes normal distribution, Rayleigh distribution, Maxwell distribution, and all of chi-distributions as special cases.

It is also shown that the set of m-distributions behave like a log normal distribution within a small range (148) of signal level near its rms value. This result shows that in the interpretation of the experimental data, one must be cautious in attempting to estimate the tails of the distribution by an extension from the middle section of the distribution.

(*iv*) We find that in general, the integral transformation (20) from

$f(\alpha, \beta)$ into $P(V \leqq L)$ is not unique. Physically this means the signals of fading environments with different $f(\alpha, \beta)$ can have the same amplitude distribution $P(V \leqq L)$. As an example, this nonuniqueness shows that specifying a Rayleigh distribution for $P(V \leqq L)$ does not necessarily imply that there are a large number of interfering signals; nor does it necessarily imply that the real and imaginary parts of the fading signal are normally distributed with zero mean.

II. SUM OF n UNIT VECTORS WITH UNIFORM RANDOM PHASES

The amplitude distribution of the sum of two unit vectors with uniformly distributed random relative phase has been shown in Section IX of Part 1 to be, for $0 \leqq L \leqq 2$,

$$P(V \leqq L) = \frac{1}{\pi} \cos^{-1}\left(1 - \frac{L^2}{2}\right), \tag{107}$$

and

$$p(L) = \frac{\partial}{\partial L} P(V \leqq L) = \frac{1}{\pi} \frac{1}{\left[1 - \left(1 - \frac{L^2}{2}\right)^2\right]^{\frac{1}{2}}}. \tag{108}$$

In the deep-fade region where L is small, this amplitude distribution obeys the power law

$$P(V \leqq L) \propto L. \tag{109}$$

The sum of n unit vectors with uniformly distributed random phases has been investigated previously by many authors.[33,34–37] The mathematics involved in obtaining the amplitude distribution for any arbitrary $n \geqq 3$ is fairly complicated. Computer numerical integration is needed to show the distribution explicitly. In this section, we shall avoid the complicated mathematics and shall apply the results of Part 1 to show that the amplitude distribution for any arbitrary $n \geqq 3$ in the deep-fade region always follows the square law:

$$P(V \leqq L) \propto L^2, \quad \text{for small } L. \tag{110}$$

The sum of n unit vectors with random phases can be written as

$$Ve^{i\phi} = \sum_{i=1}^{i=n} e^{i\theta_i} \tag{111}$$

$$= \left[1 + \sum_{i=2}^{i=n} e^{i(\theta_i - \theta_1)}\right] e^{i\theta_1} \tag{112}$$

$$= [1 + Re^{i\theta}]e^{i\theta_1}, \tag{113}$$

where

$$Re^{i\theta} = \sum_{i=2}^{i=n} e^{i(\theta_i - \theta_1)}. \tag{114}$$

Rosenbaum[38] has indicated that if all the phases $\{\theta_i\}_{i=1}^{i=n}$ of the unit vectors are independently and uniformly distributed in $(0, 2\pi)$, then $Ve^{i\phi}$ and $Re^{i\theta}$ have circular symmetric probability density functions; i.e., the amplitude and the phase are independent and the phase is uniformly distributed in $(0, 2\pi)$.

It then follows that the random signal represented by equation (113) for any arbitrary $n \geqq 3$ is a special case of Appendix B. The case for $n = 2$ is an exception because the joint probability density function $q(R, \theta)$ contains a delta function whereas the $q(R, \theta)$ in Appendix B is assumed to be a smooth function.

The sum of three unit vectors can be considered as a unit vector suffering interference by a random vector $R(t)e^{i\theta(t)}$ which is the sum of the other two unit vectors. The amplitude distribution $g(R)$ of $R(t)$ is given by equation (108) except for the replacement of the notation L by R. Equation (108) implies

$$g(1) = \frac{2}{\pi\sqrt{3}}. \tag{115}$$

Therefore, $g(R)$ for this case is a smooth function which is neither singular nor zero at $R = 1$. Then equation (165) shows that

$$P(V \leqq L) \cong \tfrac{1}{2}g(1)L^2 = \frac{1}{\pi\sqrt{3}} L^2 \propto L^2. \tag{116}$$

The sum of n unit vectors, $n \geqq 3$, can be considered as a unit vector suffering interference for a random vector $R(t)e^{i\theta(t)}$ which is the sum of the other $(n - 1)$ unit vectors. It is obvious that $g(1) \neq 0$ simply because each of the $(n - 1)$ unit vectors has unity amplitude. Then, the results (see Appendix B) imply

$$P(V \leqq L) \propto L^2, \quad \text{for small } L \text{ and } n \geqq 3. \tag{117}$$

On a log-versus-dB graph paper, as shown in Fig. 1, the power law (109) implies a straight line with an inverse slope of 20 dB per decade of probability whereas the square law (117) implies a straight line with the same inverse slope of 10 dB per decade of probability as that of Rayleigh distribution.

Therefore, we conclude that for $n = 2$, the distribution of deep fades is characterized by the inverse slope of 20 dB per decade of probability whereas for any $n \geqq 3$, the distribution of deep fades is always characterized by the inverse slope of 10 dB per decade of probability. This conclusion agrees with the numerical results of Norton, et al., in Fig. 2 of Reference 33.

III. A SINE WAVE PLUS A GAUSSIAN NOISE

The statistical behavior of a sine wave plus narrowband Gaussian random noise has been investigated in great detail by Rice.[26–27,32] In this section we shall apply our general analysis to this case to show the consistency of our results with the work of Rice. Furthermore, we shall also use the closed-form solution of $N(L)$ and $\bar{t}(L)$ obtained by Rice to show that the conditional average positive derivative $\bar{\dot{V}}_+(L)$ can be either a constant or a function of signal level L, depending on whether the power spectrum of the noise is symmetric or asymmetric with respect to the frequency of the sine wave.

In this model, the interfering vector, $Re^{i\theta} = \alpha + j\beta$, represents the envelope of a narrowband Gaussian noise; the constant vector represents the sine wave with a constant amplitude and frequency f_a. The joint probability density function $f(\alpha, \beta)$ is a two-dimensional normal density function; i.e.,

$$f(\alpha, \beta) = \frac{1}{2\pi b_0} \exp\left[-(\alpha^2 + \beta^2)/2b_0\right], \tag{118}$$

where α and β are assumed to be independent normal random variables with the same variance b_o and zero mean.

The well known Rice distribution for the amplitude of this model is

$$p(L) = \frac{\partial}{\partial L} P(V \leqq L) = \frac{L}{b_0} I_0\left(\frac{QL}{b_0}\right) \exp\left(\frac{-L^2 - Q^2}{2b_0}\right), \tag{119}$$

where $I_o(\sim)$ is the modified Bessel function of zero$^{\text{th}}$ order, and Q is the magnitude of the sine wave. In our analysis, $Q = 1$ because all the signals are normalized to the magnitude of the constant vector.

Rice[27] has also shown that the joint probability density function $p(\dot{V}, V)$ for this model is

$$p(\dot{V}, V) = \frac{V}{(2\pi)^{\frac{3}{2}}\sqrt{Bb_0}} \int_{-\pi}^{\pi} \exp\left\{\frac{-1}{2Bb_0}\left[B(V^2 - 2VQ\cos\phi + Q^2) + (b_0\dot{V} + b_1 Q\sin\phi)^2\right]\right\} d\phi, \tag{120}$$

where ϕ is the phase of the resultant fading signal $Ve^{i\phi}$; f_q is the frequency of the sine wave, $w(f)$ is the power spectrum of the Gaussian noise, and

$$b_n = (2\pi)^n \int_0^\infty w(f)(f - f_q)^n \, df, \qquad n = 0, 1, 2, \tag{121}$$

$$B = b_0 b_2 - b_1^2 . \tag{122}$$

3.1 *Amplitude Distribution of Deep Fades*

The normal density function (118) is obviously a smooth function which is neither singular nor zero at the infinite fade point ($\alpha = -1$, $\beta = 0$). Then the results of Section VI of Part 1 predict that the amplitude distribution of the fading signal in the deep-fade region is

$$P(V \leqq L) \cong \pi f(-1, 0) L^2 \tag{123}$$

$$\cong \frac{1}{2b_0} \exp\left(\frac{-1}{2b_0}\right) \cdot L^2 . \tag{124}$$

On the other hand, the limiting form of the Rice distribution for small L is

$$p(L) \cong \frac{L}{b_0} \exp\left(\frac{-Q^2}{2b_0}\right) ; \tag{125}$$

$$\therefore P(V \leqq L) \cong \frac{1}{2b_0} \exp\left(\frac{-Q^2}{2b_0}\right) \cdot L^2, \quad \text{for} \quad L \ll Q = 1. \tag{126}$$

It is seen that our result (124) agrees with the Rice distribution in the deep-fade region.

The square law (124) implies that on a log-versus-dB graph paper, the Rice distribution in the deep-fade region is always characterized by the prevailing inverse slope of 10 dB per decade of probability. The numerical results of Norton, et al., in Fig. 5 of Reference 33 agree with this prediction.

3.2 *Symmetric Power Spectrum and Constant* $\bar{\dot{V}}_+(L)$

If the power spectrum $w(f)$ of the Gaussian noise is symmetric about f_q , then $b_1 = 0$ and the integration of (120) under this condition yields

$$p(\dot{V}, V) = \left\{ \frac{1}{\sqrt{2\pi b_2}} \exp\left[\frac{-\dot{V}^2}{2b_2}\right] \right\} \left\{ \frac{V}{b_0} I_0\left(\frac{QV}{b_0}\right) \exp\left[\frac{-V^2 - Q^2}{2b_0}\right] \right\} \tag{127}$$

$$= p_1(\dot{V} \mid V) p_2(V) \tag{128}$$

$$= p_3(\dot{V}) p_2(V), \tag{129}$$

where

$$p_2(V) = \frac{V}{b_0} I_0\left(\frac{QV}{b_0}\right) \exp\left[\frac{-V^2 - Q^2}{2b_0}\right] \tag{130}$$

is the Rice distribution for V;

$$p_3(\dot{V}) = \frac{1}{\sqrt{2\pi b_2}} \exp\left[\frac{-\dot{V}^2}{2b_2}\right] \tag{131}$$

is a normal density function for $\dot{V}$; and

$$b_2 = \frac{B}{b_0} = \dot{V}^2_{rms} \quad \text{when} \quad b_1 = 0. \tag{132}$$

Equations (127) and (129) show that if the power spectrum is symmetric about f_a, then the envelope V and its time derivative $\dot{V}$ are independent, and $\dot{V}$ is normally distributed.

Substituting (131) and (132) into the definition (75) for $\bar{\dot{V}}_+(L)$ yields

$$\bar{\dot{V}}_+(L) = \frac{2\dot{V}_{rms}}{\sqrt{2\pi}} = a_0 = \text{constant}. \tag{133}$$

Therefore, the conditional average positive derivative $\bar{\dot{V}}_+(L)$ for this model is a constant if the power spectrum is symmetric about f_a.

Substituting (130) and (133) into the general expression (76) for $N(L)$ yields

$$N(L) = \frac{\dot{V}_{rms}}{\sqrt{2\pi}} \frac{L}{b_0} I_0\left(\frac{QL}{b_0}\right) \exp\left[\frac{-L^2 - Q^2}{2b_0}\right] \tag{134}$$

$$= \tfrac{1}{2} a_0 p_2(L). \tag{135}$$

Then

$$\bar{t}(L) = \frac{P(V \leqq L)}{N(L)} = \frac{2\int_0^L p_2(V)\,dV}{a_0 p_2(L)}. \tag{136}$$

In the deep-fade region where $L \ll Q = 1$,

$$N(L) \cong \frac{\dot{V}_{rms}}{\sqrt{2\pi}} \frac{1}{b_0} \exp\left[\frac{-Q^2}{2b_0}\right] \cdot L \tag{137}$$

$$\cong \pi a_0 f(-1, 0) L, \tag{138}$$

$$\bar{t}(L) \cong \frac{1}{a_0} L. \tag{139}$$

It is seen that equations (138) and (139) agree with equations (86′) and (89) of Part 2.

3.3 *Asymmetric Power Spectrum and Nonconstant* $\bar{\dot{V}}_+(L)$

If the power spectrum $w(f)$ is not symmetric about f_q, then V and $\dot{V}$ are dependent and $b_1 \neq 0$. The joint probability density function $p(V, \dot{V})$ for this case cannot be written as the product of the individual probability density functions of V and $\dot{V}$. For this case, Rice[27] has obtained $N(L)$ by substituting (120) into the general expression (72) and carrying out the integration. This gives

$$N(L) = \frac{p_2(L)\sqrt{\dfrac{B}{b_0}}}{\sqrt{2\pi}\, I_0\left(\dfrac{QL}{b_0}\right)} \sum_{n=0}^{n=\infty} \frac{1}{2^n \cdot n!} \left(\frac{-b_0\gamma^2}{QL}\right)^n \cdot \left[I_n\left(\frac{QL}{b_0}\right) + \frac{b_0\gamma^2}{QL} I_{n+1}\left(\frac{QL}{b_0}\right)\right], \tag{140}$$

where $I_n(\sim)$ is the modified Bessel function of order n, and

$$\gamma^2 = \frac{b_1 Q^2}{Bb_0}. \tag{141}$$

Comparing equation (140) and the general expression (76) for $N(L)$ shows that

$$\bar{\dot{V}}_+(L) = \frac{2\left(\dfrac{B}{2\pi b_0}\right)}{I_0\left(\dfrac{QL}{b_0}\right)} \sum_{n=0}^{n=\infty} \frac{1}{2^n \cdot n!} \left(\frac{-b_0\gamma^2}{QL}\right)^n \cdot \left[I_n\left(\frac{QL}{b_0}\right) + \frac{b_0\gamma^2}{QL} I_{n+1}\left(\frac{QL}{b_0}\right)\right]. \tag{142}$$

It is seen that when V and $\dot{V}$ are dependent, then the conditional average positive derivative $\bar{\dot{V}}_+(L)$ is a function of signal level $V = L$.

The expected number of fades $N(L)$ and the average fade duration $\bar{t}(L)$ for this case in the deep-fade region are

$$N(L) \cong \pi a_0 f(-1, 0) L$$

$$\cong \frac{a_0}{2b_0} \exp\left(\frac{-Q^2}{2b_0}\right) \cdot L$$

$$\bar{t}(L) \cong \frac{1}{a_0} L,$$

where

$$a_0 = \lim_{L \to 0^+} \bar{\dot{V}}_+(L) = \left(\frac{2B}{\pi b_0}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-\gamma^2)^n}{(2^n \cdot n!)^2} \left[1 + \frac{\gamma^2}{2(n+1)}\right].$$

The work of Clarke,[29] Ossanna,[30] and Gans[31] on mobile radio indicates that the power spectrum of the fading signal is generally not symmetric with respect to the received carrier frequency unless the straight line joining the base station and the mobile antenna is perpendicular to the velocity of the mobile and the antenna pattern is symmetric with respect to this straight line. Therefore, in the theoretical work of $N(L)$ and $\bar{t}(L)$, it is not safe to assume that V and $\dot{V}$ are always independent.

IV. m-DISTRIBUTIONS, CHI-DISTRIBUTIONS, AND RAYLEIGH DISTRIBUTION

In the study of the experimental data of amplitude distributions of short-term high-frequency long-distance propagations, Nakagami[39] found that the set of experimental data can well be described by a family of m-distributions:*

$$p(L) = \frac{2 \cdot m^m}{\Gamma(m)\,\Omega^m} L^{2m-1} \exp\left[\frac{-mL^2}{\Omega}\right], \tag{143}$$

where Ω is the mean square value of the fading signal. The operating frequency ranged from 10 MHz to 20 MHz and the path length from 1500 kilometers to 9000 kilometers. Nakagami indicated that these results were obtained from short records of data from three to seven minutes in length in order to avoid the effects of slow fading on the distribution of rapid fading.

The various properties of the m-distributions have been investigated in detail by Nakagami.[39] It is easily shown that the set of chi-distributions[28] is a subset of m-distributions by setting $2m$ = any positive integer in (143). This means the normal distribution, Rayleigh distribution, and Maxwell distribution are also special cases of m-distributions when $m = 1/2$, 1, and 3/2 respectively. On Rayleigh paper, all the m-distributions appear to be straight lines passing through the common point of 50 percent at 0 dB, with different slopes which depend on the value of m. The graphical representation of m-distributions can be seen in Reference 39.

However, in Reference 39, one does not know the theoretical condition under which the amplitude distribution of a fading signal will

* To avoid possible confusion, we emphasize that the Nakagami distribution mentioned in Section I of Part 1 is not the m-distribution discussed in this section.

belong to this family of m-distributions. In this section we shall find the condition on the joint probability density function $f(\alpha, \beta)$ of the interfering vector such that the amplitude distribution $P(V \leqq L)$ will belong to m-distributions.

Expanding the exponential function in equation (143) into a power series gives

$$p(L) = \frac{2 \cdot m^m}{\Gamma(m)\Omega^m} \sum_{S=0}^{S=\infty} \frac{1}{S!} \left(\frac{-m}{\Omega}\right)^S \cdot L^{2S+2m-1}. \tag{144}$$

Comparing equation (144) and the general power series (35) for $p(L)$ shows that

$$\mu = m, \tag{145}$$

$$(2S + 2\mu)\, d_{2S+2} = \frac{2(-1)^S}{\left(\frac{\Omega}{\mu}\right)^{S+\mu} \Gamma(\mu) \cdot S!}. \tag{146}$$

Substituting (34) into (146) gives

$$\frac{\pi}{2^{2S}} \sum_{\nu=0}^{\nu=S} \frac{H_{2S-2\nu,2\nu}(-1, 0)}{(\nu!)(S-\nu)!} = \frac{(-1)^S}{\left(\frac{\Omega}{\mu}\right)^{S+\mu} \cdot \Gamma(\mu)}, \qquad S = 0, 1, 2, \cdots \tag{147}$$

Thus, equations (145) and (147) are the general conditions on the interfering vector such that $P(V \leqq L)$ is an m-distribution.

In Section V of Part 2 we showed that $\mu \geqq 1/2$ which implies $m \geqq 1/2$. Nakagami[39] has also found this condition on the parameter m by a different approach. Since m can be any value $\geqq 1/2$, equation (143) represents an infinitely large family of distributions.

4.1 *Log Normal Behavior of m-Distribution Near the RMS Value*

The experimental data of optical propagation[40–42] and line-of-sight radio links show that the distributions of the signal scintillation near its average value are approximately log normal. Usually the accuracy of experimental data is best in the middle section of the distribution and deteriorates towards the tails. It is quite tempting to estimate the tails of the distribution by an extension from the middle section. The deviations of the experimental data at the tails are often attributed to the experimental error.

However, de Wolf[40] and Deltz and Wright[42] have pointed out that the use of the middle section of a log normal paper may not be a reliable test of the log normal distribution. The differentiation between the log normal distribution and certain other distributions may be significant only at the tails rather than the middle section of these distributions.

Nakagami[39] has pointed out that all the m-distributions behave like a log normal distribution for the fading signal $V(t)$ in the neighborhood of its rms value $\sigma_R = \sqrt{\Omega}$. The explicit bounds on the signal level within which this approximation holds are:

$$20 \left| \log_{10} \frac{L}{\sqrt{\Omega}} \right| \ll 4.3\ dB. \tag{148}$$

It is shown in Appendix F that the m-distribution within the signal range (148) is approximately equal to

$$p(L) = \left[\frac{2m^m}{L \cdot \Gamma(m)} e^{-m} \right] \exp\left[-2m(\ln L - \ln \sqrt{\Omega})^2\right], \tag{149}$$

which is a log normal distribution for the signal level L.

Therefore, the m-distributions, including the normal distribution, Rayleigh distribution, Maxwell distribution, and chi-distributions, all behave like a log normal distribution within the signal range (148). This result points out that in the interpretation of the experimental data, one must examine the behavior of the data not only inside but also outside of the range (148) in order to assert their distribution.

V. NONUNIQUE RELATION BETWEEN AMPLITUDE DISTRIBUTION AND $f(\alpha, \beta)$

From the general integral relation (20) between $P(V \leqq L)$ and $f(\alpha, \beta)$, it is seen that any component of $f(\alpha, \beta)$ that is antisymmetric with respect to $(1 + \alpha)$ and/or β will cancel out in the integration (20), and contributes nothing to $P(V \leqq L)$. This means that there are many different $f(\alpha, \beta)$'s, with the same symmetric* part and different antisymmetric* part, which correspond to the same amplitude distribution $P(V \leqq L)$.

Mathematically, this means the integral transformation (20) from $f(\alpha, \beta)$ into $P(V \leqq L)$ is not unique. Physically, this means the fading signals in fading environments with different $f(\alpha, \beta)$ can have the same amplitude distribution.

Furthermore, even if we restrict $f(\alpha, \beta)$ to functions symmetric with respect to $(1 + \alpha)$ and β, the relation between $f(\alpha, \beta)$ and $P(V \leqq L)$ is still not unique. We shall demonstrate this nonunique relation specifically by using the results for the m-distributions previously discussed. We notice that for each S, equation (147) is an algebraic equation

* With respect to $(1 + \alpha)$ and β.

for $(S + 1)$ unknowns $\{H_{2S-2\nu,2\nu}(-1, 0)\}_{\nu=0}^{\nu=S}$. It is then obvious that there are infinitely many different sets of $\{H_{2S-2\nu,2\nu}(-1, 0)\}_{\nu=0}^{\nu=S}$ which will satisfy equation (147) because there is only one equation for $(S + 1)$ unknowns. This nonuniqueness gives a great freedom for the wide variations of the individual term, $H_{2S-2\nu,2\nu}(-1, 0)$, which is the even-order partial derivative of $H(\alpha, \beta)$ at $(\alpha = -1, \beta = 0)$. From the Taylor series (27), it is seen that $H_{2S-2\nu,2\nu}(-1, 0)$ is the coefficient of the even-order term $(1 + \alpha)^{2S-2\nu}\beta^{2\nu}$ which is symmetric with respect to $(1 + \alpha)$ and β. Therefore, the relation between $f(\alpha, \beta)$ and $P(V \leqq L)$ is not unique even if $f(\alpha, \beta)$ is symmetric with respect to $(1 + \alpha)$ and β.

A more detailed discussion of this nonunique relation in polar coordinates is given in Appendix G.

VI. PHYSICAL MODEL AND RAYLEIGH DISTRIBUTION

In this section, we shall show that specifying a Rayleigh distribution for the amplitude of a complex fading signal $Ve^{i\phi}$ does not necessarily imply that there is a large number of interfering signals; nor does it necessarily imply that the real part and the imaginary part of the fading signal are normal with zero mean.

Let x and y be the real part and the imaginary part respectively of the complex fading signal $Ve^{i\phi}$, and let $F(x, y)$ be the joint probability density function of x and y. Since $V^2 = x^2 + y^2$, then the probability of $V \leqq L$ is the probability of x and y falling within the circular region

$$x^2 + y^2 \leqq L^2. \tag{150}$$

Therefore, $P(V \leqq L)$ is the integration of $F(x, y)$ over the circular region (150); i.e.,

$$P(V \leqq L) = \int_{y=-L}^{y=L} \int_{x=-\sqrt{L^2-y^2}}^{x=\sqrt{L^2-y^2}} F(x, y)\, dx\, dy. \tag{151}$$

6.1 *Number of Interfering Signals*

In the most common derivation of the Rayleigh distribution, the fading signal is assumed to consist of a large number of random, independent interfering signals,

$$Ve^{i\phi} = \sum_{i=1}^{n} E_i e^{i\theta_i} = x + jy. \tag{152}$$

Furthermore, it is assumed that none of the components $\{E_i\}_{i=1}^{i=n}$ predominates in the summation (152). Then by the central limit theorem, one argues that as the number, n, of interfering signals approaches infinity, the real part x and the imaginary part y of the fading signal $Ve^{i\phi}$ become independent normal random variables with the same variance and zero mean. This implies that V and ϕ are independent and ϕ is uniformly distributed in $(0, 2\pi)$. Under this condition, the distribution of the random amplitude V is Rayleigh.

However, by an observation similar to those in Section V of Part 3 and Appendix G, we realize that the transformation (151) from $F(x, y)$ into $P(V \leqq L)$ is not unique. Given an amplitude distribution $P(V \leqq L)$, there correspond infinitely many different $F(x, y)$'s. In other words, the independent normal distribution for x and y with the same variance and zero mean is only a sufficient condition but is not a necessary condition for the amplitude V to have a Rayleigh distribution. Since x and y do not have to be normal, then the number n of the interfering signals does not have to be large.

In Appendix G, we have shown that the relation between $P(V \leqq L)$ and $F(x, y)$ becomes unique if the following two additional conditions are imposed;

(*i*) V and ϕ are independent, and
(*ii*) ϕ is uniformly distributed in $(0, 2\pi)$.

For long radio links such as beyond-the-horizon radio links, the conditions *i* and *ii* seem to be applicable. However, for line-of-sight radio links, our experience indicate that the phase ϕ has much higher tendency of wide variation during the deep fade where V is small. This means for short radio links, V and ϕ may not be independent and ϕ may not be uniform. Therefore, in our general analysis we do not impose the conditions *i* and *ii*.

6.2 *Mean Values of x and y*

In the integral relation (151) the antisymmetric part of $F(x, y)$ contributes nothing to the amplitude distribution $P(V \leqq L)$, but does affect the mean value of x and y. Then by adding a suitable* antisymmetric function to $F(x, y)$, the mean values of x and y can be changed arbitrarily without affecting the amplitude distribution $P(V \leqq L)$ of the fading signal $Ve^{i\phi}$.

In other words, given an amplitude distribution $P(V \leqq L)$ of a complex fading signal $Ve^{i\phi}$, the mean values and the higher moments

* The probability density function $F(x, y)$ must be $\geqq 0$ for any x and y; therefore, the symmetric part of $F(x, y)$ must be $\geqq$ |antisymmetric part of $F(x, y)$| when the antisymmetric part is negative.

of x and y are not unique. (However, the moments of the amplitude V are unique.)

Therefore, specifying a Rayleigh distribution for the amplitude V does not necessarily imply that the mean values of x and y are zero. Physically if there is a direct path between the transmitter and the receiver of a radio link, then the mean values of x and y may not be zero. However, the results of this section show that the mere nonzero means of x and y do not necessarily exclude the Rayleigh distribution for the amplitude of the fading signal.

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APPENDIX A

List of Symbols and Their Definitions

A	The constant amplitude of the echo in the one-echo model.
a_0	The zeroth order term of the Taylor Series expansion of $\bar{V}_+(L)$ defined in equation (81).
a_n	The coefficient of the nth order term of the Taylor series expansion $\bar{V}_+(L)$ defined in equation (81).
B	$= b_0b_2 - b_1^2$ as defined in equation (122).
b_n, $n = 0, 1, 2$	Defined by equation (121).
C_4	The coefficient of the fourth order term L^4 of $P(V \leqq L)$ in equations (161) and (163).
C_r^n	Defined by equation (30).
d_{2S+2}	The coefficient of the power series representation of $P(V \leqq L)$ defined in equations (31) and (34).
$f(\alpha, \beta)$	Joint probability density function of α and β.
$F(x, y)$	Joint probability density function of the real part x and the imaginary part y of the fading signal.
f_q	The frequency of the sine wave.
$g(R)$	The probability density function of the amplitude R of the interfering vector.
$g_n(R)$	The nth order derivative of $g(R)$.
$H(\alpha, \beta)$	The smooth part of $f(\alpha, \beta)$ as defined in equation (23).
$H_{n,m}(\alpha, \beta)$	The partial derivative of $H(\alpha, \beta)$ as defined in equation (29).
$H(-1, 0)$	The value of $H(\alpha, \beta)$ at $(\alpha = -1, \beta = 0)$.
$\bar{H}(-1, 0)$	The average value of $H(\alpha, \beta)$ at $(\alpha = -1, \beta = 0)$ if $H(\alpha, \beta)$ is discontinuous at this point.
$I_n(\sim)$	Modified Bessel function of order n.
L	An arbitrarily specified signal level in the study of the statistics $P(V \leqq L)$, $N(L)$ and $\bar{t}(L)$.
m	An integer.
n	An integer.
$N(L)$	Expected number of fades per unit time below the specified signal level L.
$0(L^\eta)$	A symbol to denote a function which goes to zero at a rate equal to or faster than L^η as $L \to 0$ where $\eta > 0$.
$P(V \leqq L)$	Probability that the amplitude V of a fading signal fades below a specified signal level L.

$p(L)$	Probability density of the amplitude V at the specified signal level L.
$p_2(V)$	Probability density function of V. $p_2(V) = p(V)$.
$p(\dot{V}, V)$	Joint probability density function of $\dot{V}$ and V.
$p_1(\dot{V} \mid L)$	Conditional probability density of $\dot{V}$ under the condition $V = L$.
Q	The constant magnitude of the sine wave in the specialized model of a sine wave plus a Gaussian noise. In this paper, $Q = 1$ because of the normalization of the signal level.
$q(R, \theta)$	Joint probability density function of R and θ.
R	The amplitude of the resultant interfering vector $Re^{i\theta}$ of the fading signal model.
$Re^{i\theta}$	The resultant complex interfering vector.
r	An integer.
S	An integer.
t	A variable representing time.
$\bar{t}(L)$	Average duration of fades below L.
V	The amplitude of the envelope of a complex fading signal normalized to the nonfaded value V_{ref}.
$\dot{V}$	The time derivative of the normalized amplitude V of the fading signal.
$Ve^{i\phi}$	The envelope of the fading signal.
$\dot{V}_{\text{rms}}$	The rms value of the time derivative of V.
$V_{\text{fad}}(t)$	The unnormalized amplitude of the random fading signal.
V_{ref}	The nonfaded signal level when there is no interference.
$\bar{\dot{V}}_+(L)$	Conditional average positive derivative of V as defined in equation (75).
$W(\theta)$	Probability density function of the random relative phase θ of one-echo model.
$W_n(\theta)$	The nth order derivative of the probability density function $W(\theta)$.
x	The real part of the fading signal $Ve^{i\phi}$.
y	The imaginary part of the fading signal $Ve^{i\phi}$.
α	The real part of the interfering vector $Re^{i\theta}$.
β	The imaginary part of the interfering vector $Re^{i\theta}$.
$\Gamma(\sim)$	Gamma function.
γ^2	Defined by equation (141).
η	An arbitrary constant > 0.
θ	The phase of the resultant interfering vector $Re^{i\theta}$.

θ_L	Defined by equation (53).
μ	$(1 - \mu)/2$ is the order of singularity of the joint probability density function $f(\alpha, \beta)$ at $(\alpha = -1, \beta = 0)$ as defined in equation (23).
ν	An integer.
$\rho(V, \phi)$	Joint probability density function of the amplitude V and the phase ϕ of the fading signal.
$\sigma(V, \phi)$	A function of V and ϕ satisfying the homogeneous integral equation (207).
ϕ	The phase of the envelope of the complex fading signal.
Ω	The mean square value of an m-distributed random variable.

APPENDIX B

Amplitude Distribution In Polar Coordinates

Since the results of $P(V \leqq L)$ for $\mu = 1$ cover a large class of fading problems and since the statistical behavior of the interfering vector is sometimes described by the joint probability density function $q(R, \theta)$ of the interfering vector, we shall also obtain the power series representation of $P(V \leqq L)$ in terms of $q(R, \theta)$ when $\mu = 1$. By using the relations:

$$\alpha = R \cos \theta, \tag{153}$$

$$\beta = R \sin \theta, \tag{154}$$

and the Jacobian relation [28] between $f(\alpha, \beta)$ and $q(R, \theta)$ one can represent the coefficients $\{d_{2S+2}\}$ of (42) in terms of $q(R, \theta)$. This gives

$$P(V \leqq L) = \pi q(1, \pi)L^2 + d_4 L^4 + d_6 L^6 + \cdots \tag{155}$$

where

$$d_2 = \pi q(1, \pi) = \pi f(-1, 0), \tag{156}$$

$$d_4 = \frac{\pi}{8} [q(1, \pi) - q_{1,0}(1, \pi) + q_{2,0}(1, \pi) + q_{0,2}(1, \pi)] \quad \text{etc.}, \tag{157}$$

$$q_{n,m}(R, \theta) = \frac{\partial^{n+m}}{\partial R^n \, \partial \theta^m} q(R, \theta). \tag{158}$$

In the deep-fade region,

$$P(V \leqq L) \xrightarrow[L \to 0]{} \pi q(1, \pi)L^2, \tag{159}$$

and

$$p(L) \xrightarrow[L \to 0]{} 2\pi q(1, \pi) L. \tag{160}$$

B.1 *Circular Symmetric Probability Density Function*

In this subsection, we consider a special case where the interfering random vector, $Re^{i\theta}$, has a circular symmetric probability density function $q(R, \theta)$; i.e., R and θ are independent and θ is uniformly distributed in $(0, 2\pi)$. For this case, let $g(R)$ be the probability density function of the magnitude R of the interfering vector, then

$$P(V \leqq L) = \tfrac{1}{2} g(1) L^2 + C_4 L^4 + C_6 L^6 + \cdots , \tag{161}$$

$$q(R, \theta) = g(R) \frac{1}{2\pi} , \tag{162}$$

$$C_4 = \frac{1}{16} [g(1) - g_1(1) + g_2(1)] \quad \text{etc.}, \tag{163}$$

$$g_n(1) = \frac{d^n}{dR^n} g(R) \mid_{R=1} . \tag{164}$$

In the deep-fade region

$$P(V \leqq L) \xrightarrow[L \to 0]{} \tfrac{1}{2} g(1) L^2. \tag{165}$$

These results are used in Part 3 where we discuss the relation between our generalized analysis and the existing theoretical work.

APPENDIX C

Nonanalytic $H(\alpha, \beta)$

In Section V of Part 1, the analysis is carried out based on the assumption that $H(\alpha, \beta)$ can be expanded into a two-dimensional Taylor series. In this appendix, we shall investigate two cases where $H(\alpha, \beta)$ cannot be expanded into the two-dimensional Taylor series. The objective is to show that from a theoretical viewpoint, the assumption of the Taylor series expansion of $H(\alpha, \beta)$ is not strictly necessary for the validity of the power laws of deep fades discussed in this paper.

C.1 *Continuous $H(\alpha, \beta)$ With Unbounded Derivatives*

If $H(\alpha, \beta)$ is continuous at $(\alpha = -1, \beta = 0)$ but its first-order partial derivatives and/or its higher order partial derivatives are unbounded at $(\alpha = -1, \beta = 0)$, then $H(\alpha, \beta)$ in the neighborhood of $\alpha = -1$ and

$\beta = 0$ can be written as

$$H(\alpha, \beta) = H(-1, 0) + 0\{[(1 + \alpha)^2 + \beta^2]^{\eta/2}\}, \tag{166}$$

where $1 > \eta > 0$, and

$$0\{[(1 + \alpha)^2 + \beta^2]^{\eta/2}\}$$

is a symbol to denote the component which goes to zero at a rate equal to or faster than that of

$$[(1 + \alpha)^2 + \beta^2]^{\eta/2} \quad \text{as} \quad [(1 + \alpha)^2 + \beta^2] \to 0.$$

It is obvious that $H(\alpha, \beta)$ given by (166) cannot be expanded into Taylor series because the derivatives of $H(\alpha, \beta)$ are unbounded (i.e., singular) at ($\alpha = -1, \beta = 0$).

Substituting equations (166) and (23) into the general formulation (20) for $P(V \leqq L)$, and carrying out the integration yields

$$P(V \leqq L) = \frac{\pi H(-1, 0)}{\mu} L^{2\mu} + 0(L^{2\mu+\eta}), \tag{167}$$

$$p(L) = 2\pi H(-1, 0) L^{2\mu-1} + 0(L^{2\mu-1+\eta}), \tag{168}$$

where $0(L^{2\mu+\eta})$ is a symbol to denote the high-order terms which go to zero at a rate equal to or faster than that of $L^{2\mu+\eta}$ as $L \to 0$. Since $\eta > 0$, then in the deep-fade region

$$P(V \leqq L) \underset{L \to 0}{\longrightarrow} \frac{\pi H(-1, 0)}{\mu} L^{2\mu}, \tag{169}$$

which is the same as equation (36). Then the discussions and conclusions in Sections VI, VII, and VIII of Part 1 on the power laws of deep fades for $\mu = 1$, $1 > \mu \geqq 1/2$ and $\mu > 1$ are readily applicable to the present case even though the derivatives of $H(\alpha, \beta)$ are unbounded at ($\alpha = -1$, $\beta = 0$).

C.2 *Discontinuous* $H(\alpha, \beta)$

Suppose $H(\alpha, \beta)$ and its derivatives are bounded but are discontinuous at $\beta = 0$ so that

$$\lim_{\beta \to 0^+} H_{n,m}(\alpha, \beta) \neq \lim_{\beta \to 0^-} H_{n,m}(\alpha, \beta); \tag{170}$$

i.e.,

$$H_{n,m}(\alpha, 0^+) \neq H_{n,m}(\alpha, 0^-)$$

$$n = 0, 1, 2, \cdots, \quad m = 0, 1, 2, \cdots . \tag{171}$$

Then on each side of $\beta = 0$, the one-sided Taylor series expansion of $H(\alpha, \beta)$ is applicable. One for $\beta > 0$ and another for $\beta < 0$. Substituting these two Taylor series and equation (23) into the general formulation (20) for $P(V \leqq L)$ and carrying out the integration, one can show that

$$P(V \leqq L) \underset{L \to 0}{\longrightarrow} \frac{\pi \bar{H}(-1, 0)}{\mu} L^{2\mu}, \tag{172}$$

$$p(L) \underset{L \to 0}{\longrightarrow} 2\pi \bar{H}(-1, 0) L^{2\mu - 1}, \tag{173}$$

where

$$\bar{H}(-1, 0) = \tfrac{1}{2}[H(-1, 0^+) + H(-1, 0^-)] \tag{174}$$

is the average value of the discontinuous $H(\alpha, \beta)$ at $(\alpha = -1, \beta = 0)$.

It is seen that equation (172) is also the same as equation (36) except for the proper interpretation of $\bar{H}(-1, 0)$ when $H(\alpha, \beta)$ is discontinuous at $(\alpha = -1, \beta = 0)$. Therefore, the discussions and conclusions of Sections VI, VII, and VIII of Part 1 are also applicable to the present case.

APPENDIX D

Integration for Power Series of Amplitude Distribution

Substituting (27) and (23) into (20) gives

$$P(V \leqq L) = \sum_{n=0}^{n=\infty} \frac{1}{n!} \sum_{r=0}^{r=n} C_r^n H_{n-r,r}(-1, 0) I_{n-r,r} , \tag{175}$$

where

$$I_{n-r,r} = \int_{\beta=-L}^{\beta=L} \int_{\alpha=-1-\sqrt{L^2-\beta^2}}^{\alpha=-1+\sqrt{L^2-\beta^2}} [(1+\alpha)^2 + \beta^2]^{\mu-1} (1+\alpha)^{n-r} \beta^r \, d\alpha \, d\beta. \tag{176}$$

From (176) it is seen that if either $(n - r)$ or r is an odd integer, then $I_{n-r,r}$ vanishes because the integrand is antisymmetric. Therefore, $I_{n-r,r}$ does not vanish only when both $(n - r)$ and r are even integers. Then let $n = 2S$ and $r = 2\nu$. Equation (176) becomes

$$\begin{aligned} I_{2S-2\nu,2\nu} &= \int_{\beta=-L}^{\beta=L} \int_{\alpha=-1-\sqrt{L^2-\beta^2}}^{\alpha=-1+\sqrt{L^2-\beta^2}} [(1+\alpha)^2 + \beta^2]^{\mu-1} (1+\alpha)^{2S-2\nu} \beta^{2\nu} \, d\alpha \, d\beta \\ &= \frac{2\Gamma(S - \nu + \frac{1}{2})\Gamma(\nu + \frac{1}{2})}{(2S + 2\mu)\Gamma(S + 1)} L^{2S+2\mu}. \end{aligned} \tag{177}$$

Combining (175) and (177) gives

$$P(V \leqq L) = \sum_{S=0}^{S=\infty} L^{2S+2\mu} \cdot \left[\frac{2}{(2S)!} \sum_{\nu=0}^{\nu=S} \frac{C_{2\nu}^{2S} \Gamma(S - \nu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{(2S + 2\mu)\Gamma(S + 1)} H_{2S-2\nu,2\nu}(-1, 0) \right] \tag{178}$$

$$= \sum_{S=0}^{S=\infty} d_{2S+2} L^{2S+2\mu}, \tag{179}$$

where

$$d_{2S+2} = \frac{2}{(2S)!} \sum_{\nu=0}^{\nu=S} \frac{C_{2\nu}^{2S} \Gamma(S - \nu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2})}{(2S + 2\mu)\Gamma(S + 1)} H_{2S-2\nu,2\nu}(-1, 0). \tag{180}$$

APPENDIX E

Derivation for Expected Numbers of Fades

Suppose $T = t_2 - t_1$ is the time interval in which we want to find the expected number of times that the random fluctuating signal $V(t)$ crosses the signal level $V = L$. This interval is divided into a large number of smaller intervals of width Δt so short that each contains no more than one level crossing. We first consider the expected number of upward level crossings. The downward level crossings can be treated similarly.

In an infinitesimal interval Δt, the conditions for an upward level crossing of $V(t)$ are

$$\dot{V}(t) = \frac{\partial V(t)}{\partial t} > 0 \tag{181}$$

$$\dot{V}(t)\,\Delta t > [L - V(t)] > 0 \tag{182}$$

These two conditions are shown graphically in Fig. 8. On a $\dot{V}$ versus V plane, the region in which $\dot{V}$ and V satisfy conditions (181) and (182) are shown as the shaded area in Fig. 9. The integration of the joint probability density $p(\dot{V}, V)$ over this range will give the probability that $V(t)$ will have an upward level crossing in Δt,

$$P_{\Delta t}(L) = \int_{\dot{V}=0}^{\dot{V}=\infty} \int_{V=L-\dot{V}\Delta t}^{L} p(\dot{V}, V)\, dV\, d\dot{V} \tag{183}$$

$$\cong \Delta t \int_{\dot{V}=0}^{\dot{V}=\infty} \dot{V} p(\dot{V}, V) \Big|_{V=L} d\dot{V}. \tag{184}$$

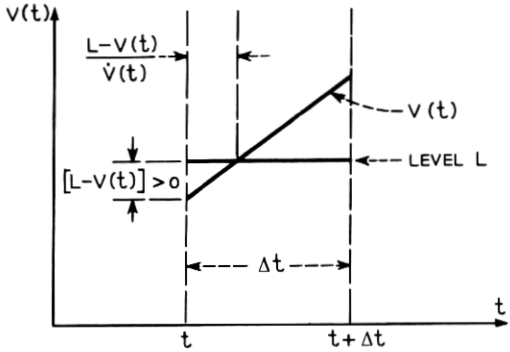

Fig. 8—The conditions for an upward level crossing of $V(t)$ in an interval Δt.

The expected number $N_{up}(L)$ of upward level crossings per unit time is

$$N_{up}(L) = \frac{P_{\Delta t}(L)}{\Delta t} = \int_{\dot{V}=0}^{\dot{V}=\infty} \dot{V} p(\dot{V}, V) \bigg|_{V=L} d\dot{V}. \tag{185}$$

Similarly, the expected number of downward level crossings per unit time is

$$N_{down}(L) = \int_{\dot{V}=-\infty}^{\dot{V}=0} | \dot{V} | p(\dot{V}, V) \bigg|_{V=L} d\dot{V}. \tag{186}$$

The total expected number of level crossings per unit time is

$$\begin{aligned} N_c(L) &= N_{up}(L) + N_{down}(L) \\ &= \int_{\dot{V}=-\infty}^{\dot{V}=\infty} | \dot{V} | p(\dot{V}, V) \bigg|_{V=L} d\dot{V}. \end{aligned} \tag{187}$$

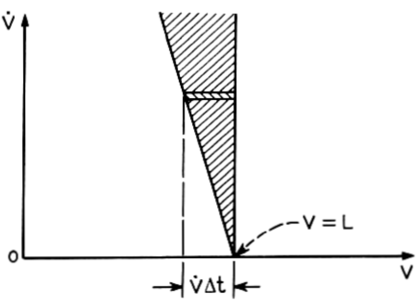

Fig. 9—The region in which $\dot{V}$ and V satisfy the conditions for an upward level crossing.

The expected number of fades per unit time below $V = L$ is

$$N(L) = \tfrac{1}{2}N_c(L) = N_{\text{up}}(L) = N_{\text{down}}(L)$$

$$= \int_{\dot{V}=0}^{\dot{V}=\infty} \dot{V}p(\dot{V}, V)\Big|_{V=L} d\dot{V}. \tag{188}$$

APPENDIX F

Log Normal Behavior of M-Distributions Near the RMS Value

Let

$$Z = 20 \log_{10}\left(\frac{L}{\sqrt{\Omega}}\right) = 20[\log_{10} L - \log_{10} \sqrt{\Omega}] \tag{189}$$

be the signal level in dB with respect to its rms value $\sqrt{\Omega}$. Equation (189) implies

$$\frac{Z}{M} = \ln\left(\frac{L}{\sqrt{\Omega}}\right), \tag{190}$$

$$\frac{L}{\sqrt{\Omega}} = \exp\left(\frac{Z}{M}\right), \tag{191}$$

$$\frac{\partial L}{\partial Z} = \frac{\sqrt{\Omega}}{M}\exp\left(\frac{Z}{M}\right) = \frac{L}{M}, \tag{192}$$

where

$$M = 20 \log_{10}e = 8.686 \text{ dB}. \tag{193}$$

Then the probability density functions of L and Z are related by the Jacobian relation:[28,18]

$$h(Z) = p(L)\left|\frac{\partial L}{\partial Z}\right| = \frac{L}{M}p(L). \tag{194}$$

Substituting the m-distribution (143) and equation (191) into (194) yields the following probability density of Z:

$$h(Z) = \frac{2m_m}{M\Gamma(m)}\left(\frac{L}{\sqrt{\Omega}}\right)^{2m} \exp\left[\frac{-mL^2}{\Omega}\right]$$

$$= \frac{2m^m}{M\Gamma(m)} \exp\left\{m\left[\frac{2Z}{M} - \exp\left(\frac{2Z}{M}\right)\right]\right\}. \tag{195}$$

Substituting the following power series

$$\exp\left(\frac{2Z}{M}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2Z}{M}\right)^n \tag{196}$$

into (195) gives

$$h(Z) = \left[\frac{2m^m}{M\Gamma(m)} e^{-m}\right] \exp\left\{-m\left[2\left(\frac{Z}{M}\right)^2 + \sum_{n=3}^{\infty} \frac{1}{n!}\left(\frac{2Z}{M}\right)^n\right]\right\}. \tag{197}$$

If the signal level L is very close to its rms value $\sqrt{\Omega}$, then

$$\left| \frac{L}{\sqrt{\Omega}} - 1 \right| \ll 1, \tag{198}$$

$$\left| \frac{2Z}{M} \right| = 2 \left| \ln \frac{L}{\sqrt{\Omega}} \right| \ll 1, \tag{199}$$

for

$$| Z | \ll M/2$$
$$= 4.3\ dB$$

Under this condition, we have

$$\sum_{n=3}^{\infty} \frac{1}{n!}\left(\frac{2Z}{M}\right)^n \ll 2\left(\frac{Z}{M}\right)^2. \tag{200}$$

Then $h(Z)$ in equation (197) becomes

$$h(Z) \cong \left[\frac{2m^m}{M\Gamma(m)} e^{-m}\right] \exp\left[-2m\left(\frac{Z}{M}\right)^2\right], \tag{201}$$

which is a normal distribution for Z. Substituting (190) and (201) into (194) yields

$$p(L) \cong \left[\frac{2m^m}{L\Gamma(m)} e^{-m}\right] \exp\left[-2m(\ln L - \ln \sqrt{\Omega})^2\right], \tag{202}$$

which is a log normal distribution for L. Therefore, within the range $| z | \ll M/2 = 4.3$ dB, all the m-distributions for any $m \geqq 1/2$ behave like a log normal distribution given by (202).

APPENDIX G

Nonunique Relation Between $P(V \leqq L)$ and $f(\alpha, \beta)$

In the definition of the fading signal model discussed in Section II of Part 1, the four random variables V, ϕ, α, and β are related by

$$\left.\begin{aligned} 1 + \alpha &= V \cos \phi \\ \beta &= V \sin \phi \end{aligned}\right\}. \tag{203}$$

Let $\rho(V, \phi)$ be the joint probability density function of the amplitude V and the phase ϕ of the fading signal. By the Jacobian[28,18] of the transformation (203), it is easily shown that

$$\rho(V, \phi) = \sqrt{(1 + \alpha)^2 + \beta^2}\, f(\alpha, \beta) = Vf(V \cos \phi,\ V \sin \phi). \tag{204}$$

In this appendix we shall show that the relation between $P(V \leqq L)$ and $\rho(V, \phi)$ is not unique. This then implies that the relation between $P(V \leqq L)$ and $f(\alpha, \beta)$ is also nonunique because of the simple algebraic relation (204) between $\rho(V, \phi)$ and $f(\alpha, \beta)$.

In the study of the amplitude distribution $P(V \leqq L)$ of a complex fading signal $Ve^{j\phi}$, it is often assumed that V and ϕ are independent with ϕ uniformly distributed in $(0, 2\pi)$. However, in the study of interference, distortion, FM radio system, radio navigation system, etc., many authors[27,43,44,19] have investigated the distribution of the random phase, $\phi(t)$. These results show that the distribution of phase is not always uniform. Furthermore, when the signal is weak (i.e., V is small), the phase is more likely to vary over wider range. This means the random variables V and ϕ are somewhat correlated. Therefore, V and ϕ generally can be either dependent or independent and ϕ can be either uniformly or nonuniformly distributed.

By definition,[28,18] the probability density $p(V)$ of V is the integration of $\rho(V, \phi)$ over the entire range of ϕ; i.e.,

$$p(V) = \int_{\phi=0}^{\phi=2\pi} \rho(V, \phi)\, d\phi. \tag{205}$$

Furthermore, the cumulative amplitude distribution $P(V \leqq L)$ is the integration of $p(V)$ from $V = 0$ to $V = L$. Therefore,

$$P(V \leqq L) = \int_{V=0}^{V=L} \int_{\phi=0}^{\phi=2\pi} \rho(V, \phi)\, d\phi\, dV. \tag{206}$$

Given a joint probability density function $\rho(V, \phi)$, then $P(V \leqq L)$ can be calculated by (206).

On the other hand, given an amplitude distribution $P(V \leqq L)$, equation (206) is an integral equation to solve for $\rho(V, \phi)$. An immediate question arising in solving the integral equation (206) is the uniqueness of the solution. A procedure to test the uniqueness of the solution is to consider the following homogeneous equation

$$0 = \int_{V=0}^{V=L} \int_{\phi=0}^{\phi=2\pi} \sigma(V, \phi)\, d\phi\, dV. \tag{207}$$

If the homogeneous equation (207) has a nontrivial solution, then the solution of (206) is not unique because, given any particular solution $\rho_P(V, \phi)$ of (206), then

$$\rho(V, \phi) = \rho_P(V, \phi) + c\sigma(V, \phi) \tag{208}$$

is also a solution of (206) where c is an arbitrary constant.*

It is obvious that all of the following functions

$$\sigma_n(V, \phi) = \xi_n(V) \sin (n\phi) + \zeta_n(V) \cos (n\phi); \qquad n = \pm 1, \pm 2, \pm 3, \cdots \tag{209}$$

are nontrivial solutions of the homogeneous equation (207) where $\xi_n(V)$ and $\zeta_n(V)$ are arbitrary functions of V. Notice that the nontrivial solutions (209) contain both symmetric and antisymmetric functions of ϕ. Furthermore, any arbitrary linear combination of $\{\sigma_n(V, \phi)\}$ is also a solution of the homogeneous equation (207). By the experience of Fourier series synthesis technique, we know that the linear combination of the set $\{\sigma_n(V, \phi)\}$ is able to represent a very large class of either simple or complicated functions of V and ϕ.

Therefore, given an amplitude distribution $P(V \leqq L)$, the integral equation (206) has infinitely many different solutions.

On the other hand, in equation (206), if one imposes the following two additional conditions:

(*i*) V and ϕ are independent, and
(*ii*) ϕ is uniformly distributed in $(0, 2\pi)$,

then

$$\rho(V, \phi) = \frac{1}{2\pi} p(V) = \frac{1}{2\pi} \frac{\partial}{\partial L} P(V \leqq L) \bigg|_{L=V} \tag{210}$$

is the only possible solution of (206).

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* Since the probability density function $\rho(V, \phi)$ must be $\geqq 0$ for any V and ϕ, then $\rho_p(V, \phi)$ must be $\geqq |c\sigma(V, \phi)|$ when $c\sigma(V, \phi)$ is negative.

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