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Error Probability of a Multilevel Digital System With Intersymbol Interference and Gaussian Noise

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In a previous paper a series expansion method for calculating the error probability of a binary digital AM system in the presence of intersymbol interference and additive gaussian noise was derived.[1] *In this paper those results are extended to the multilevel case. In the examples calculated for a four-level system, this method is* 10^4 *times faster than the exhaustive method and is* 10^2 *times more accurate than the Chernoff bound. The actual computation time with an 11-sample approximation to the real system impulse response is only 1.3 seconds with the GE Mark II time-sharing system.*

I. INTRODUCTION

In a recent paper[1] we have developed a new method to calculate the error probability of a binary digital data system in the presence of intersymbol interference and additive gaussian noise. A similar method has also been reported by M. I. Celebiler and O. Shimbo.[2] The purpose of this paper is to extend the previous results to multilevel systems.

The existing methods for the estimation of the error probability are the Chernoff bound or the worst case bound,[3,4] the results of which are generally too loose. Another alternative is the time-consuming exhaustive method.[3] For example, it would require $4^{10}(\approx 10^6)$ calculations of the error function to find the error probability of a four-level digital system where intersymbol interference resulted from ten nonzero samples of the channel impulse response.

II. DERIVATION OF THE EXPRESSION FOR ERROR PROBABILITY

For a $2m$-level digital AM system, the corrupted received sequence at the input to the receiver detector is

$$y(t) = \sum_{l=-\infty}^{\infty} a_l r(t - lT) + n(t), \tag{1}$$

where

$n(t)$ is additive gaussian noise,

$a_l = \pm 1, \pm 3, \cdots, \pm(2m - 1)$ with equal probability,

and

$r(t)$ is the given noiseless system impulse response.

At the detector, $y(t)$ is sampled every T seconds to determine the amplitude of the transmitted signal. At sampling time t_0, the sampled signal is,

$$y(t_0) = a_0 r(t_0) + \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} a_l r(t_0 - lT) + n(t_0). \tag{2}$$

The first term is the desired signal while the second and the third terms represent the intersymbol interference and gaussian noise respectively.

The set of slicing levels are,[5]

$$0, \pm 2r(t_0), \pm 4r(t_0), \cdots, \pm(2m - 2)r(t_0). \tag{3}$$

Given a particular transmitted signal level, a_0, the conditional error probability is,

$$P_r(e/a_0) = \begin{cases} P\{y(t_0) \geqq -2(m-1)r(t_0)\}, & a_0 = -(2m-1), \\ P\{y(t_0) \leqq 2(m-1)r(t_0)\}, & a_0 = 2m-1, \\ P\{(y(t_0) \geqq (a_0+1)r(t_0))U(y(t_0) \leqq (a_0-1)r(t_0))\}, & \\ & a_0 \neq \pm(2m-1), \end{cases} \tag{4}$$

where AUB is the union of the events A and B.

Substituting equation (2) into (4), we obtain

$$P_r(e/a_0) = \begin{cases} P\{\sum_{l\neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}, & a_0 = -(2m-1), \\ P\{\sum_{l\neq 0} a_l r(t_0 - lT) + n(t_0) \leqq -r(t_0)\}, & a_0 = 2m-1, \\ P\{(\sum_{l\neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0))U, & \\ \quad (\sum_{l\neq 0} a_l r(t_0 - lT) + n(t_0) \leqq -r(t_0))\}, & a_0 \neq \pm(2m-1). \end{cases} \tag{5}$$

Since $\sum_{l\neq 0} a_l r(t_0 - lT)$ and $n(t_0)$ are equally likely to be positive or negative, equation (5) reduces to

$$P_r(e/a_0) = \begin{cases} P\{\sum_{l\neq 0} a_l r(t_0 - lT) + n(t_0) \geq r(t_0)\}, & a_0 = \pm(2m-1), \\ 2P\{\sum_{l\neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}, & a_0 \neq \pm(2m-1). \end{cases} \tag{6}$$

The error probability of the system is,

$$P_e = \sum_{\text{all } a_0} P_r(e/a_0)P_r(a_0),$$

$$= \frac{2m-1}{m} P\{\sum_{l\neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}. \tag{7}$$

We note that equation (7) is similar to equation (7) of Ref. 1, with the only exception that the a_l can now assume multiple values. According to equation (9) of Ref. 1, we obtain the following expansion for P_e,

$$P_e = \frac{2m-1}{2m} \operatorname{erfc}\left(-\frac{r(t_0)}{\sqrt{2}\,\sigma}\right) + \frac{2m-1}{m} \sum_{k=1}^{\infty} \frac{1}{(2k)!}\left(\frac{1}{2\sigma^2}\right)^k \frac{1}{\sqrt{\pi}} \exp\left[-\frac{r^2(t_0)}{2\sigma^2}\right] \cdot H_{2k-1}\left(\frac{r(t_0)}{\sqrt{2}\,\sigma}\right) M_{2k}, \tag{8}$$

where

σ^2 is the noise power,
H_{2k-1} is the Hermite polynomial,
erfc is the complementary error function,
M_{2k} is the $2k$th moment of the random variable X,

and

$$X = \sum_{l\neq 0} a_l r(t_0 - lT), \qquad a_l = \pm 1, \pm 3, \cdots, \pm(2m-1). \tag{9}$$

The moments can again be obtained through the characteristic function of X without the explicit evaluation of the distribution function. The characteristic function is,

$$\Phi(\omega) = \prod_{l\neq 0} \left\{ \sum_{k=-m+1}^{m} \exp\,[j\omega(2k-1)r(t_0 - lT)]/2m \right\},$$

$$= \prod_{l\neq 0} \left\{ \frac{1}{2m} \sin\,[2m\omega r(t_0 - lT)] \cdot \csc\,[\omega r(t_0 - lT)] \right\}. \tag{10}^*$$

* See Ref. 6, equation (1.342).

Therefore,

$$\Phi'(\omega) = \Phi(\omega)\{2m \sum_{l\neq 0} r(t_0 - lT) \cot [2m\omega(t_0 - lT)] - \sum_{l \neq 0} r(t_0 - lT) \cot [\omega r(t_0 - lT)]\}. \tag{11}$$

Since $M_{2k} = (-1)^k\Phi^{2k}(0)$ and $M_{2k-1} = 0$, we obtain a recurrence formula for M_{2k} by successive differentiation of equation (11),

$$M_{2k} = -\sum_{i=1}^{k} \binom{2k-1}{2i-1} M_{2(k-i)}(-1)^i \frac{2^{2i}[(2m)^{2i} - 1]}{2i} \mid B_{2i} \mid \cdot [\sum_{l\neq 0} r(t_0 - lT)^{2i}], \tag{12}$$

where B_{2i} is the Bernoulli number obtained by series expansion of cotangent function about the origin. Knowing that $M_0 = 1$, all the M_{2k}'s can be calculated successively through equation (12) for an N-sample approximation of the channel pulse response. The N-sample truncation is equivalent to the approximation of $\sum_{l\neq 0} r(t_0 - lT)^{2k}$ by $(N - 1)$ summation terms.

The error probability of a $2m$-level system can thus be obtained by equations (8) and (12). In the special case $m = 1$, equations (8) and (12) agree with the results of the binary system, i.e., equations (9) and (15) of Ref. 1.

III. TRUNCATION ERROR BOUND

The error incurred by truncating the series expansion of equation (8) at a finite term $n - 1$ is,

$$R_{2n} = \frac{2m-1}{m} \sum_{k=n}^{\infty} \frac{1}{(2k)!} \left[\frac{1}{2\sigma^2}\right]^k \frac{1}{\sqrt{\pi}} \exp\left[-\frac{r^2(t_0)}{2\sigma^2}\right] \cdot H_{2k-1}\left(\frac{r(t_0)}{\sqrt{2}\,\sigma}\right) M_{2k} . \tag{13}$$

Let

$$\lambda = \text{maximum} \{\mid \sum_{l\neq 0} a_l r(t_0 - lT) \mid\}, = (2m - 1) \sum_{l\neq 0} \mid r(t_0 - lT) \mid. \tag{14}$$

It can be shown that the moments satisfy the following inequality,

$$M_{2k+2p} \leqq M_{2k}\lambda^{2p}, \qquad p = 0, 1, 2, \cdots \tag{15}$$

For $(2k - 1) \gg x$, the Hermite polynomials are upper bounded by,

$$| H_{2k-1}(x) | \leqq 2^{k-\frac{1}{2}}[(2k-3)!!]\sqrt{2k-1} \exp [x^2/2]. \tag{16}$$

Substituting equations (16) and (15) into equation (13) and grouping the terms into geometric series, we obtain an upper bound for R_{2n}.

$$| R_{2n} | \leqq \epsilon_{2n} = \frac{2m-1}{m} \frac{1}{\sqrt{2\pi}} (\exp [-r^2(t_0)/4\sigma^2]) \cdot \frac{M_{2n}}{(2\sigma^2)^n} \frac{1}{n!\sqrt{2n-1}} \frac{1}{\left[1 - \frac{[(2m-1) \sum_{l \neq 0} | r(t_0 - lT) |]^2}{2n\sigma^2}\right]}. \tag{17}$$

IV. EXAMPLE

We have calculated the error probability of a four-level digital AM system with the received pulse given by

$$r(t) = \frac{T}{\pi t} \sin (\pi t/T).$$

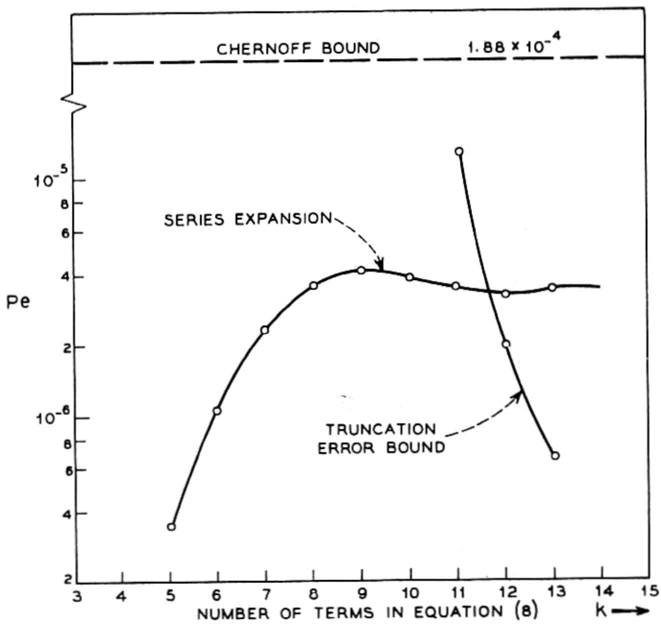

Fig. 1—Comparison of Chernoff Bound and series expansion method. Sin $[\pi t/T]/(\pi/T)t$ pulse, 11-pulse truncation approximation, $t_0 = 0.05T$, $(S/N) = 24$ dB.

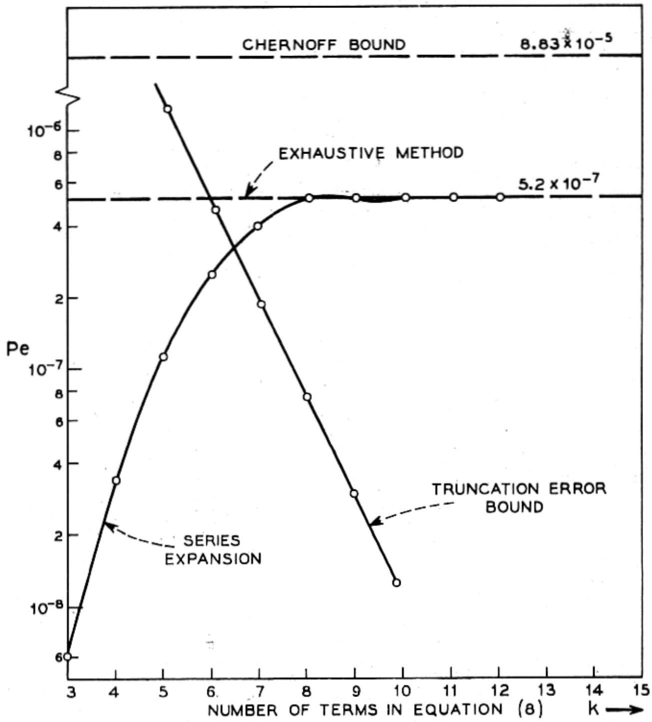

Fig. 2—Comparison of series expansion method with Chernoff Bound and exhaustive method. [SIN $\pi t/T$]/$(\pi/T)t$ pulse, 5-pulse truncation approximation, $t_0 = 0.05T$, $(S/N) = 24$ dB.

With an 11-sample approximation, a S/R of 24 dB and a sampling time of $t_0 = 0.05\ T$, the error probability obtained by the Chernoff[7] * bound is 2×10^{-4}. The result obtained by our method is 3.4×10^{-6} indicating an improvement of two orders of magnitude. The convergence of equation (8) is presented in Fig. 1. Reasonable accuracy is achieved after eight terms of the series are calculated. A check of the accuracy of our method by comparison with the exhaustive method is impossible in this case because the latter requires 10^4 times more computation time as compared to the series expansion method. Instead, we checked our method with the exhaustive method for the case of a five-sample approximation. The results agree well and are presented in

* 11-sample approximation to the channel response is used in calculating the Chernoff bound.

Fig. 2. The computation time spent on GE Mark II time-sharing system is approximately 1.3 seconds for the series expansion method in both the eleven-sample and the five-sample approximations. The truncation error bounds are also presented in both figures.

V. CONCLUSIONS

We have extended the series expansion evaluation of the error probability of a binary digital AM system in the presence of intersymbol interference and additive gaussian noise to the $2m$-level systems. The results are extremely encouraging. For the case examined the series expansion method was calculated several orders of magnitude faster than the exhaustive method, and it was more accurate than the Chernoff bound by two orders of magnitude.

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