A Telephone Traffic Model Based on Randomly Closing Crosspoints, and Its Relationships to Other Models

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I. INTRODUCTION AND SUMMARY

In the theory of traffic in telephone connecting networks it is on one hand a virtual necessity, for practical purposes, to compromise the true complexity of the system under study and to introduce drastic simplifying assumptions that allow some calculation to be done, and on the other, it is perfectly feasible to pursue basic theoretical studies without such compromise and simplification. For this reason, a spectrum of several mathematical models for describing traffic in networks has been developed in recent years.

These models range from "simple" ones that furnish an incomplete description based on strong stochastic independence assumptions, to "complicated" ones that exactly mirror network structure and routing. Each grade of model has its uses: "simple" ones for easy computation and involved ones for general understanding.

An example of a useful "simple" model is the probability linear graph[1] suggested by C. Y. Lee in 1955, an outgrowth of earlier work by L. E. Kittredge and E. C. Molina. At the other end of the scale, an example of a "complicated" model is the Markov process[2] proposed by the author in 1963 as an improvement of the "thermodynamic" model.[3]

We shall describe here another "simple" model, with a basic starting point similar to that of Lee, and then show how a certain natural restriction of this model yields in many cases precisely the thermodynamic model. Our presentation thus clarifies the known relationships between these models, and reveals some unsuspected ones; it strengthens our understanding of them by showing how the apparently realistic "complicated" models can arise through natural and relatively minor modfications of the "simple" ones.

Whereas Lee's model assumes that a *link* ℓ of the network is busy or idle with a probability p_ℓ , all links being independent of each other, we propose instead to assume that each (individual switch or) *crosspoint* c is closed with some probability p_c , again independently. This basis for calculating probabilities has the virtue of assigning probability to every possible way of closing switches, physically meaningful or not. We then modify the model by calculating all probabilities *conditional* on the system's being in a physically meaningful state. This procedure in effect rids our calculations of the irrelevant states by normalizing them out. The resulting new model we call the *crosspoint* model. If, as is usually the case, every call goes through the same number of switches, then conditioning in this way brings in the partition function[3] in a natural way, and the crosspoint model turns out to be formally equivalent to the thermodynamic one; when the latter is in turn modified so as to take realistic account of routing and calling rates, it becomes the Markov process model.

We stress that the suggestion made here of a new model does not really improve our capacity to calculate blocking, load-loss curves, or other practical items. Primarily, it provides a new derivation of the thermodynamic model from simple (and strong) first principles similar to those used for the probability linear graph model of Lee.

II. LEE'S MODEL AND ITS EXTENSION

The probability linear graph model has been extensively discussed[1,4] in the literature, so we include only a resume of the method: to calculate the congestion incurred by traffic between an inlet u and an outlet v, attention is focused on the graph G defined by the permitted paths through the network from u to v; G consists of all nodes and branches through which some path from u to v passes. Given any complete specification of which branches of G are busy and which are idle (at a particular juncture of network operation), it is possible to examine G to see if there is a path from u to v no branch of which is busy. The method now assigns a probability distribution to the possible occupancies by postulating that a link ℓ of G is busy with probability p_ℓ independently of all other links. The congestion for u and v is then calculated as the probability that this distribution assigns to the event "There is no path from u to v composed of idle branches."

We have described the probability linear graph model as assuming something only about certain events having to do with the graph G of paths for a particular call from u to v, and not as providing a probabilistic description of the busy or idle condition of all the links in the network. However, it is entirely possible to extend the probabilistic description, used in Lee's model for links of G, to all the links in the network. This extension is natural because if the description is believable for one inlet-outlet pair, it should be so for all such pairs, and so for all links. It will of course still give only an incomplete stochastic model, since it says rather little about what crosspoints are closed, so that in general it is not possible to tell what inlet is connected to what outlet.

However, the extension does shed some light on the character and accuracy of Lee's model, as we note in the next paragraph, and it also suggests the new model to be proposed.

The fashion in which this extended version of Lee's model works is clear: the states of the network, i.e., all the possible ways of closing crosspoints, whether physically meaningful or not, are partitioned according to the equivalence relation of "having the same links busy," and probabilities are assigned to these equivalence classes. In this situation, it is unfortunately true that physically meaningful and physically irrelevant (microscopic) states occur in the same equivalence class. Were this not so, one could try to remove the effect of the irrelevant states[5] by insisting that all probabilities be taken conditional on being in the set of relevant states. This set, however, does not have probability assigned to it.

III. THE CROSSPOINT MODEL

There is, nevertheless, a basic modification of Lee's approach in which the normalization device for eliminating the "irrelevant" states can be used. The change to be made is this: whereas Lee's model assigns a probability p_ℓ of being busy to each link ℓ, we propose to assign a probability p_c of being closed to each crosspoint c, all independently in both cases. The point is this: if it is known what crosspoints are closed, then it is known what links are busy, but not *vice versa*. This approach has the property of assigning probability to *every* state of the network, physically meaningful or not.

In particular, the set of meaningful states is assigned probability. Once this is true we can restrict attention to these states. We shall eliminate the effect of the irrelevant states by simply normalizing them out of the picture, i.e., by calculating all probabilities of interest *conditional* on being in the set of meaningful states. The distribution of probability (over the set S of physically meaningful states) obtained in this way we shall call the "crosspoint" model, because the basic events to which probability is assigned are closings or openings of crosspoints. In a similar vein, Lee's model might be called the "link" model, because the basic probabilities are assigned to the busy or idle condition of links.

IV. PROPERTIES OF THE CROSSPOINT MODEL

Let S be the set of physically meaningful ways of closing crosspoints, and for $x \in S$ let $c(x)$ be the number of switches or crosspoints closed in x. Let us suppose that all crosspoints have the same chance p of being closed. (This is likely to be true if the network traffic is uniform and if there is neither concentration nor expansion.) The basic *unconditional* probability assigned to x is then

$$p^{c(x)}(1-p)^{C-c(x)} \qquad x \in S$$

where C is the total number of crosspoints in the network. The probability of x conditional on being in the set S of physically meaningful states is then zero if $x \notin S$, and

$$\frac{p^{c(x)}(1-p)^{C-c(x)}}{\sum_{y \in S} p^{c(y)}(1-p)^{C-c(y)}}, \qquad \text{for} \quad x \in S,$$

or, with $\mu = p/(1-p)$,

$$\frac{\mu^{c(x)}}{\sum_{y \in S} \mu^{c(y)}}.$$

This is of the familiar Maxwell–Boltzmann form, with the function $c(\cdot)$ playing the role of the energy. The reader familiar with the thermodynamic model[3] will at once recognize the resemblance of the above expression to the basic (equilibrium) state probabilities in that model, which assigns a meaningful state $x \in S$ a probability

$$\frac{\lambda^{|x|}}{\sum_{y \in S} \lambda^{|y|}},$$

with $|y|$ = number of calls in progress in state y, and λ a positive constant. As we have pointed out, this distribution is obtained by maximizing the entropy functional subject to a fixed mean number of calls in progress. Exactly the same arguments[3] characterize the distribution over S in the crosspoint model, as follows:

Theorem: The distribution $\{q_x, x \in S\}$ *of probability over* S *which maximizes the entropy functional*

$$H(q) = -\sum_{x \in S} q_x \log q_x$$

subject to the constraint

$$\sum_{x \in S} q_x c(x) = c,$$

with c a fixed number >0, is given by

$$q_x = \frac{\mu^{c(x)}}{\sum_{y \in S} \mu^{c(y)}}$$

where $\mu > 0$ *is the constant determined uniquely by the equation*

$$c = \mu \frac{d}{d\mu} \log \sum_{x \in S} \mu^{c(x)}.$$

Thus the crosspoint model differs from the thermodynamic model only in that the average number of closed crosspoints, rather than that of calls in progress, is fixed while maximizing the entropy. In an important class of cases the two models formally coincide, even. This can be seen from the

Corollary: If every call goes through exactly s switches, then the state probabilities assigned by the crosspoint model with parameter p are exactly the same as those assigned by the thermodynamic model with parameter

$$\lambda = \mu^s = \left(\frac{p}{1-p}\right)^s.$$

For evidently in this case $c(x) = s \mid x \mid$. The property that every call goes through the same number of switches is possessed by virtually all the connecting networks used in practice.

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