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Time Dispersion in Dielectric Waveguides

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In dielectric waveguides operating at optical frequencies, the primary cause of time dispersion of narrow pulses can be mode conversion. In this paper we argue that under certain assumptions a dielectric waveguide acts as a linear system in intensity. That is, given the intensity input, the intensity output is equal to the input convolved with an intensity impulse response. We show that contrary to intuition, the width of the impulse response gets narrower when coupling between guided modes increases. Using the perturbation results of D. Marcuse, we obtain an interesting model of energy propagation down imperfect guides. We conclude that the intensity response width increases as the square root of the guide length for sufficiently long guides and approaches a gaussian shape for sufficiently long guides.

We conclude from the theory that the dispersion in dielectric waveguides may be orders of magnitude below that which was previously expected in guides of sufficiently long length having properly controlled large amounts of mode conversion. These theoretical results have not yet been verified experimentally.

I. INTRODUCTION

In multimode dielectric waveguides operating at optical frequencies, the primary cause of time dispersion of narrow pulses can be mode conversion. In a geometrically perfect guide with more than a single mode, energy initially launched in a given mode remains in that mode as it propagates down the guide. Physical guides have imperfections from perfect geometric shape (e.g., roughness at the core-cladding interface of a nominally right circular cylindrical guide) which allows energy to couple between modes during propagation down the guide. Since group velocities differ in general amongst the modes, a pulse of energy initially launched in a single mode or combination of modes will be broadened due to the spread of propagation times of different parts of the energy.

In this paper we argue that under certain assumptions, a dielectric waveguide acts as a linear system in intensity as well as in voltage. That is, we show that the relationship between the input intensity and output intensity of the guide is defined in terms of an intensity impulse response. We argue that this intensity impulse response, for sufficiently long guides, has a mean-square width about its mean which increases only linearly with length. Further, in the limit of very long guides, we argue that the response shape is gaussian. We also show that the greater the coupling between modes, the less the time dispersion—a result which at first contradicts intuition. Finally, we obtain quantitative results and an interesting model of an optical guide under certain assumptions.

We conclude from the theory that the dispersion in dielectric waveguides may be orders of magnitude below that which was previously expected in guides of sufficiently long length having properly controlled *large* amounts of mode conversion. These theoretical results have not yet been verified experimentally.

II. AN OUTLINE OF THE ARGUMENTS

We next outline the steps of the derivations to follow, so that the reader can follow the train of thought.

We start with the fact that the optical guide is a linear system in voltage. That is, if we expand the input signal in spatial modes and expand the output signal in the same modes, then the time varying coefficients of the modes at the output are related to the coefficients at the input by a set of voltage impulse responses. We then make an assumption about the associated set of transfer functions (Fourier transforms of the impulse responses) which allows us to argue that the set of average output intensities and the set of input intensities are also related by a set of impulse responses. Thus the guide is also linear in intensity under the assumptions.

We next argue that for sufficiently long guides, these intensity impulse responses coupling a chosen input mode coefficient and a chosen output mode coefficient are indifferent to the modes chosen except perhaps for a magnitude scale factor.

Finally, this allows us to show that for guides longer than the above scale, the intensity impulse response which is now in common for all input-output pairs has a mean-square deviation about its mean which grows linearly in length, and which approaches the gaussian shape in the limit of long guides.

III. THE OPTICAL GUIDE AS A LINEAR SYSTEM IN INTENSITY

3.1 *The Random Channel*

We now argue that under a simple assumption, the expected value of the intensity at the output of a random channel is related to the intensity of the input to the channel by a simple convolution with an intensity impulse response. We start with input baseband signal $a(t)$. We use $a(t)$ to linearly modulate a carrier $m(t)$ which may be coherent or a stationary (wide sense) random process centered at frequency f_0. We assume that the result $x(t) = a(t)m(t)$, has bandwidth B, i.e., its spectrum extends from $-B/2 + f_0$ to $B/2 + f_0$.

We pass $x(t)$ through a time invariant filter with a random impulse response $h(t)$ representing the channel, resulting in the final output $y(t)$. We define a Fourier transform relationship between the function $\Lambda(f)$ and the function $\lambda(t)$

$$\Lambda(f) = \int_{-\infty}^{\infty} \exp\,[i2\pi f(t)]\lambda(t)\,dt, \tag{1}$$

$$\Lambda(f) \Leftrightarrow \lambda(t).$$

By simple linear system theory if

$$\begin{aligned} X(f) &\Leftrightarrow x(t), \\ H(f) &\Leftrightarrow h(t), \\ Y(f) &\Leftrightarrow y(t), \end{aligned} \tag{2}$$

then

$$Y(f) = X(f)H(f).$$

Define the envelopes of $x(t)$ and $y(t)$ by

$$\begin{aligned} x(t) &= \sqrt{2}\ \mathrm{Re}\,\{x_e(t)\exp\,(i2\pi f_0 t)\}, \\ y(t) &= \sqrt{2}\ \mathrm{Re}\,\{y_e(t)\exp\,(i2\pi f_0 t)\}. \end{aligned} \tag{3}$$

The intensity of the input and output signals are defined as

$$\begin{aligned} I_{\mathrm{in}}(t) &= |x_e(t)|^2 = a^2(t)\,|m_e(t)|^2, \\ I_{\mathrm{out}}(t) &= |y_e(t)|^2, \end{aligned} \tag{4}$$

where $m_e(t)$ = carrier envelope.

Assumption: Stationarity of channel transfer function

$$\langle H^*(\alpha)H(f+\alpha)\rangle = \Gamma(f) \tag{5}$$

provided $f_0 - B/2 < \alpha, f + \alpha < f_0 + B/2$.

The assumption, while apparently arbitrary, is essential to the results which follow. Perturbation results of Marcuse,[1] to be discussed in Section 4.1, indicate that equation (5) may be satisfied for the input-output temporal transfer function of a given spatial eigenmode of an optical dielectric waveguide with mechanical imperfections, provided the mechanical imperfections satisfy constraints also to be discussed.

Define for any function $U(\alpha)$

$$\begin{aligned} U_+(\alpha) &= U(\alpha) \qquad \alpha \geqq 0, \\ &= 0 \qquad\quad\;\; \alpha < 0. \end{aligned} \tag{6}$$

Then it has been shown that (See Appendix C)

$$|y_e(t)|^2 \Leftrightarrow 2 \int_{-\infty}^{\infty} Y_+(f + \alpha) Y_+^*(\alpha)\, d\alpha. \tag{7}$$

Then clearly

$$|y_e(t)|^2 \Leftrightarrow 2 \int_{-\infty}^{\infty} X_+(f + \alpha) H_+(f + \alpha) H_+^*(\alpha) X_+^*(\alpha)\, d\alpha. \tag{8}$$

Using equation (5) we obtain

$$\begin{aligned} \langle |y_e(t)|^2 \rangle &\Leftrightarrow 2\Gamma(f) \int_{-\infty}^{\infty} \langle X_+(f + \alpha) X_+^*(\alpha) \rangle\, d\alpha, \\ \langle |y_e(t)|^2 \rangle &\Leftrightarrow \Gamma(f) I_{\text{in}}(f), \end{aligned} \tag{9}$$

where

$$I_{\text{in}}(f) \Leftrightarrow \langle I_{\text{in}}(t) \rangle.$$

Thus*

$$\langle I_{\text{out}}(t) \rangle = \langle I_{\text{in}}(t) \rangle * \gamma(t) \tag{10}$$

where

$$\gamma(t) \Leftrightarrow \Gamma(f).$$

Thus we have a linear system relationship between the channel input and output intensities.

3.2 *Extension to Vector Channels*

Suppose we have a vector channel (corresponding to multimode guide) consisting of a vector of L input functions

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* The notation $x(t) * y(t)$ signifies convolution:

$$x(t) * y(t) \triangleq \int_{-\infty}^{\infty} x(t - u) y(u) du.$$

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_L(t) \end{bmatrix}$$

and L output functions

$$\mathbf{Y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_L(t) \end{bmatrix}.$$

The input and output functions are related by an $L \times L$ matrix of impulse responses

$$\boldsymbol{\theta}(t) = [h_{ij}(t)] \tag{11}$$

where we have

$$y_l(t) = \sum_{j=1}^{L} h_{lj}(t) * x_j(t), \tag{12}$$

i.e.,

$$\mathbf{Y}(t) = \boldsymbol{\theta}(t) * \mathbf{X}(t).$$

Now consider a cascade of two vector channels having impulse response matrices ${}^1\boldsymbol{\theta}(t)$ and ${}^2\boldsymbol{\theta}(t)$. The input passes first through channel 1 and then through channel 2. The output of channel 2 is given by

$$\mathbf{Y}(t) = {}^2\boldsymbol{\theta}(t) * {}^1\boldsymbol{\theta}(t) * \mathbf{X}(t). \tag{13}$$

Define the envelope of $y_k(t)$, $y_{ke}(t)$: we know that

$$|y_{ke}(t)|^2 \Leftrightarrow 2 \int_{-\infty}^{\infty} Y_{k+}(\alpha + f) Y_{k+}^*(\alpha)\, d\alpha,$$

$$|y_{ke}(t)|^2 \Leftrightarrow 2 \int_{-\infty}^{\infty} \sum_j \sum_l \sum_m \sum_n {}^2H_{kl+}(f + \alpha){}^1H_{lj+}(f + \alpha)X_{j+}(f + \alpha) \tag{14}$$

$$\cdot {}^2H_{kn+}^*(\alpha){}^1H_{nm+}^*(\alpha)X_{m+}^*(\alpha)\, d\alpha.$$

Assumption a. Stationarity of Mode Transfer function.
b. Mode Transfer functions uncorrelated.

$$\langle {}^1H_{lj}(f + \alpha){}^2H_{kl}(f + \alpha){}^1H_{nm}^*(\alpha){}^2H_{rn}^*(\alpha)\rangle$$
$$= {}^1\Gamma_{lj}(f){}^2\Gamma_{kl}(f)\ \delta_{l,n}\ \delta_{j,m}\ \delta_{k,r} \tag{15}$$

where

$$\delta_{X,Y} \triangleq \text{Kronecka delta}$$

for $\{f + \alpha, \alpha\} \epsilon$ input signal bands.

We are implying that randomness, especially in phase, erases correlation between the transfer functions of different modes. The validity of this assumption for optical guides with more than one mode will be discussed in Section 4.1.

It then follows that

$$\langle |y_{ke}(t)|^2 \rangle \Leftrightarrow 2 \int_{-\infty}^{\infty} \sum_{j=1}^{L} \sum_{l=1}^{L} {}^1\Gamma_{lj}(f) {}^2\Gamma_{kl}(f) \langle X_{j+}(f+\alpha) X_{j+}^{*}(\alpha) \rangle \, d\alpha,$$
$$\langle |y_{ke}(t)|^2 \rangle \Leftrightarrow \sum_{j=1}^{L} \sum_{l=1}^{L} {}^2\Gamma_{kl}(f) {}^1\Gamma_{lj}(f) I_{\text{in}j}(f) \tag{16}$$

where

$$I_{\text{in}j}(f) \Leftrightarrow \langle | x_{je}(t) |^2 \rangle.$$

Thus under assumption (15) we have

$$\langle |y_{ke}(t)|^2 \rangle = \sum_{j=1}^{L} \sum_{l=1}^{L} {}^2\gamma_{kl}(t) * {}^1\gamma_{lj}(t) * \langle |x_{je}(t)|^2 \rangle \tag{17}$$

where

$$\gamma_{kl}(t) \Leftrightarrow \Gamma_{kl}(f).$$

Forming the matrix ${}^1G(t)$ with elements ${}^1\gamma_{kl}(t)$ and similarly ${}^2G(t)$; the vectors of input and output *intensities* are related by

$$| Y_e |^2 = {}^2G(t) * {}^1G(t) * |X_e|^2. \tag{18}$$

Thus the vector channel is a vector linear system in intensity as well as voltage [compare equation (18) to equation (13)].

3.3 *A Limit Theorem for a Cascade of Vector Channels*

Now consider a cascade of a large number M of vector channels, each behaving as described in Sections 3.1 and 3.2, i.e., if $|Y_e(t)|^2$ is the vector of the average intensity responses at the output, $|X_e(t)|^2$ the input intensity vector; we have

$$|Y_e(t)|^2 = \left(\mathop{*}_{j=1}^{M} \mathbf{G}_j(t) \right) * |\mathbf{X}_e(t)|^2 = \mathbf{G}_T * |\mathbf{X}_e(t)|^2$$

where

$$\mathbf{G}_T = \mathop{*}_{j=1}^{M} G_j(t) = \mathbf{G}_M * \mathbf{G}_{M-1} \cdots * \mathbf{G}_1 . \tag{20}$$

We would like to argue now that for sufficiently large M, all the elements of G_T are identical in waveform, differing at most by a constant. In other words, we would like to argue that the shape of the intensity response between any input mode and any output mode is indifferent to the choices of input and output modes, for sufficiently long guides, except perhaps for the magnitude of the responses.

We shall prove our results for a lossless two-mode guide. Define $\gamma_{ij}(t, b)$ as the $j \to i$ intensity impulse response for b sections of guide. We obtain (see Appendix A for derivation)

$$\gamma_{21}(t, b) = \sum_{R=1}^{b} \gamma_{11}(t, b - R) * \gamma_{21}(t) * \gamma_{22}^{*R-1}(t) \tag{21}$$

where

$$\gamma_{22}^{*R-1}(t) \triangleq \gamma_{22}(t) * \gamma_{22}(t) \cdots R - 1 \text{ times}$$

and

$$\gamma_{11}(t, 0) = \delta(t).$$

Now assume that the guide is lossless, i.e.,

$$\int_{-\infty}^{\infty} \{\gamma_{11}(t) + \gamma_{21}(t)\}\, dt = 1. \tag{22}$$

Further define

$$\gamma_{ij}(t) = a_{ij}\rho_{ij}(t),$$

$$\int_{-\infty}^{\infty} \rho_{ij}(t)\, dt = 1, \qquad 0 < a_{ij} < 1. \tag{23}$$

Thus

$$\gamma_{21}(t, b) = \sum_{R=1}^{b} \gamma_{11}(t, b - R) * \rho_{21}(t) * \rho_{22}^{*R-1}(t)(a_{21}a_{22}^{R-1}). \tag{24}$$

For the lossless guide, and b sufficiently large $\gamma_{11}(b - R) \approx \gamma_{11}(t, b)$ for $R \ll b$. Furthermore a convolution of $\gamma_{11}(t, b - R)$ with $\rho_{21}(t) * \rho_{22}^{*R-1}(t)$ is approximately equal to $\gamma_{11}(t, b - R)$ for $R \ll b$ since the response $\gamma_{11}(t, b - R)$, which is a convolution of $b - R$ terms, has a narrow spectrum compared to the other R term convolution for $b \gg R$. Furthermore $a_{21}a_{22}^{R-1} \to 0$ for R large. Thus for b sufficiently large

$$\gamma_{21}(t, b) \approx \gamma_{11}(t, b)(a_{21}/(1 - a_{22})). \tag{25}$$

Similarly we have

$$\gamma_{11}(t, b) = \sum_{R=1}^{b-1} \gamma_{12}(t, b - R) * \rho_{21}(t) * \rho_{11}^{*R-1}(t) a_{21} a_{11}^{R-1},$$

$$\approx \gamma_{12}(t, b) \frac{a_{21}}{1 - a_{11}} = \gamma_{12}(t, b), \tag{26}$$

(since $a_{21} + a_{11} = 1$). Thus

$$\mathbf{G}(t, b) \cong \rho(t, b) \begin{bmatrix} \dfrac{1}{1 + \eta} & \dfrac{1}{1 + \eta} \\ \dfrac{1}{1 + 1/\eta} & \dfrac{1}{1 + 1/\eta} \end{bmatrix} = \rho(t, b)\mathbf{A}, \tag{27}$$

where $\eta = a_{21}/a_{12}$, and $\rho(t, b) \triangleq \gamma_{11}(t, b) / \int_{-\infty}^{\infty} \gamma_{11}(t, b)\, dt$. Finally we obtain the response of a guide of kb sections

$$\mathbf{G}(t, kb) = \rho^{*k}(t, b)\mathbf{A}.$$

Note that $\mathbf{A}$ is idempotent, i.e., $\mathbf{A}^2 = \mathbf{A}$ and $\rho(t, b)$ is a positive unit area function.

3.4 *Application to Long Optical Guide*

For a multimode lossless of guide sufficiently long length, $l = kL$, with finite coupling between all modes, we can generalize equation (27) to conclude that the intensity impulse response between an input and output mode is a constant times some positive unit area function $\rho(t, L)$ convolved with itself l/L times when L is a scale on which equation (27) holds in the generalized case (more than two modes). Since the central limit theorem states that the convolution of a large number of unit area positive functions approaches a gaussian shape,* we conclude that the impulse response should approach a gaussian shape in the limit of long guides. Further, the impulse response's second moment about its mean increases linearly with increasing guide length for guides longer than L.† That is, the second moment about the mean of the response is

$$M_2(l) = M_2(L) l/L. \tag{28}$$

If τ_1 is the "differential delay" (time/meter) of propagation in the

* Provided that $\int_{-\infty}^{\infty} \rho(t, L) t^2 dt < \infty$, when we add similar independent random variables with finite second moments, the probability density of the sum, which is the *convolution* of the individual densities, approaches a gaussian shape.

† The second central moment of a convolution is the sum of the individual second central moments.

slowest modes and τ_2 is the differential delay in the fastest mode, then*

$$M_2(L) \leqq \frac{(\tau_1 - \tau_2)^2 L^2}{4} = \frac{(\Delta\tau L)^2}{4}. \tag{29}$$

Therefore,

$$M_2(l) \leqq \frac{(\Delta\tau)^2}{4} Ll$$

for any L where equation (27) holds [for an N mode case we have an $N \times N$ matrix multiplying $\rho(t, l)$].

IV. QUANTITATIVE RESULTS

4.1 *Perturbation Theory*

We shall now apply the above results to the case of a lossless slab dielectric waveguide previously studied by Marcuse.[1] We expand the input field to the guide as

$$\epsilon(t, x, 0) = \sum \epsilon_k(t, 0)\psi_k(x) \tag{30}$$

where x is the cross-sectional position parameter and the $\psi_k(x)$ are the eigenmodes of the guide. The field a distance l down the guide is written as

$$\epsilon(t, x, l) = \sum \epsilon_k(t, l)\psi_k(x). \tag{31}$$

We have a linear voltage impulse response relationship between the vector of input voltages $[\epsilon_k(t, 0)]$ and the vector of output voltages $[\epsilon_k(t, l)]$. Defining $E_k(\omega, l)$ as the Fourier transform of $\epsilon_k(t, l)$ we have

$$E_k(\omega, l) = \sum_j C_{kj}(\omega, l) E_j(\omega, 0). \tag{32}$$

Marcuse has shown that a *perturbation* theory solution for the $C_{kj}(\omega, l)$ is given by

$$C_{kj}(\omega, l) = \lambda_{kj} \exp\,[i\beta_j(\omega) l] \int_0^l g(z) \exp\,\{i[\beta_k(\omega) - \beta_j(\omega)]z\}\, dz \tag{33}$$

where λ_{kj} is a constant weakly dependent upon ω and $g(z)$ is the wall perturbation from straightness. It is assumed $k \neq j$ [For $k = j$, $C_{kj}(\omega, l) = 1$]. We have therefore

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* The right side of equation (29) is the mean-square intensity impulse response width if the response consists of an impulse of area ½ at the shortest delay and an impulse of area ½ at the longest delay.

$$C_{kj}(\omega, l)C_{np}^{*}(\omega + \sigma, l)$$
$$= \lambda_{kj}\lambda_{np}^{*} \exp \{i(\beta_j(\omega) - \beta_p(\omega + \sigma))l\} \int_0^l \int g(z)g(z')$$
$$\cdot \exp \{i[(\beta_k(\omega) - \beta_j(\omega))z - (\beta_n(\omega + \sigma) - \beta_p(\omega + \sigma))z']\}\, dz\, dz'. \tag{34}$$

Defining the correlation function

$$\langle g(z)g(z') \rangle = R_g(z - z') \tag{35}$$

(we assume $g(z)$ is a wide sense stationary process) we obtain

$$\langle C_{kj}(\omega, l)C_{np}^{*}(\omega + \sigma, l) \rangle = \lambda_{k1}\lambda_{np}^{*} \exp (i\, \Delta\beta_2 l) \int^l \int R_g(z - z')$$
$$\cdot \exp [i(\beta_k(\omega) - \beta_j \omega)(z - z')] \exp [i\, \Delta\beta(\omega, \sigma)z']\, dz\, dz' \tag{36}$$

where

$$\Delta\beta = (\beta_k(\omega) - \beta_n(\omega + \sigma)) - (\beta_j(\omega) - \beta_p(\omega + \sigma)),$$
$$\Delta\beta_2 = \beta_j(\omega) - \beta_p(\omega + \sigma).$$

If $R_g(z - z')$ drops off quickly for $(z - z')$ in an interval of length l, then we have the approximate result

$$\langle C_{kj}(\omega, l)C_{np}^{*}(\omega + \sigma, l) \rangle \simeq \lambda_{kj}\lambda_{np}^{*} l S_g(\beta_k(\omega) - \beta_j(\omega))$$
$$\cdot \exp (i\, \Delta\beta_2 l) \frac{1 - \exp [i\, \Delta\beta(\omega, \sigma)l]}{i\, \Delta\beta(\omega, \sigma)l} \tag{37}$$

where $S_g(\cdot)$ = Fourier Transform of $R_g(\cdot)$, and is assumed to be constant as a function of $\beta_k(\omega) - \beta_i(\omega)$ for ω within the excitation bandwidth. For $k = n$, $j = p$

$$\Delta\beta \simeq \left[\frac{\partial\beta_k(\omega)}{\partial\omega} - \frac{\partial\beta_j(\omega)}{\partial\omega}\right]\sigma \triangleq \Delta\tau_{kj}\sigma,$$
$$\Delta\beta_2 = \left(\frac{\partial\beta_j}{\partial\omega}\right)\sigma. \tag{38}$$

That is we assume σ is small enough so that there is negligible dispersion of energy travelling in a single mode. Thus, the intensity impulse response between input j and output k is (see Fig. 1)

$$\gamma_{kj}(t, l) \simeq \lambda_{mj}\lambda_{mj}^{*} S_g(\beta_m(\omega) - \beta_j(\omega))f(t - \tau_j l) \tag{39}$$

where

$$f(t) = 1, \qquad t \epsilon [0, \Delta\tau_{mj} l];$$
$$= 0, \qquad \text{otherwise};$$

for l small enough for the perturbation theory to hold.

From equations (36) through (38), it is clear that for cases where we do not have $k = n, j = p$ the correlation function $C_{kj}(\omega)C_{np}^*(\omega + \sigma)$ will be negligible provided $\Delta\beta(\omega, \sigma)$ is sufficiently large in the band of input frequencies. Thus we can use the perturbation length impulse response to find a long-guide response by means of equation (20).

From equation (37) we see that we increase coupling between modes by making the mechanical perturbation spectrum large at frequencies which correspond to the difference of the inverses of the phase velocities of the modes at excitation frequencies. It should be emphasized that while making the perturbation spectrum high at frequencies that couple guided modes, we will wish to avoid making it too high at frequencies that couple guided to unguided radiation modes since such coupling results in loss.

4.2 *A Hydraulic Model of Dispersion*

We shall now show that the perturbation results imply a model which is an interesting interpretation of the propagation process, and which allows easy computation of the response of a long guide.

Suppose energy traveled down the guide as follows. We start with a large number of indivisible bundles of energy at the guide input. Each bundle begins propagating down the guide randomly jumping from mode to mode. At any point down the guide, a bundle travels at the group velocity associated with the mode it is currently in. At any position, the probability that a bundle will jump to mode k, given that it is in mode j, in the next increment of distance dl is $\lambda_{kj}\lambda_{kj}^* S_g(\beta_k(\omega) - \beta_j(\omega))\, dl$.

Since we have a very large number of bundles, the output response of the guide in intensity should have the same shape as the probability distribution of the arrival time at the output of an individual bundle. For a short guide of length L, the probability that a bundle is in mode k given that it started in mode j is $|\lambda_{kj}|^2 S_g(\beta_k(\omega) - \beta_j(\omega))L$ and its arrival time distribution is given exactly by Fig. 1. Since this distribution is the perturbation solution for the intensity impulse response of a short guide, we see that the hydraulic model gives the same result as the perturbation theory. Further, a little thought will show (see Appendix B) that the extrapolation from a short guide to a long guide in the hydraulic model is analytically the same as equation (20). Thus any technique which can be used to determine the intensity impulse response characteristics using the hydraulic model will be valid for the solution of equation (20) using the perturbation results.

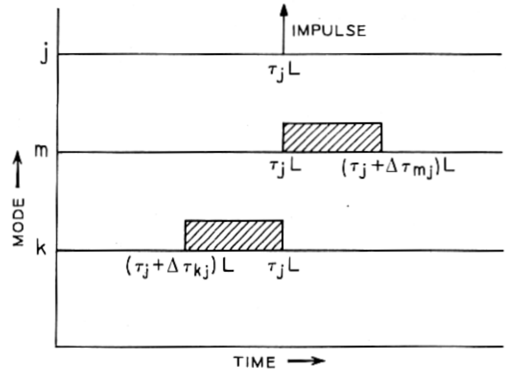

Fig. 1—Output intensity response for modes j, m and k, given mode j is excited.

4.3 *The Solution of Some Hydraulic Model Response*

4.3.1 *Characteristics*

We now wish to determine some probability density moments of the arrival time of a bundle of energy at the output of a long guide recalling that this has the shape of the guide impulse response.

Let $H(l')$ be the mode a bundle is in at distance l' down the guide. In that mode the bundle travels with differential delay (time/meter) $\tau_H = \tau(l')$. The total propagation time down the guide is

$$T = \int_0^l \tau(l')\, dl'. \tag{40}$$

The expected propagation time down the guide is

$$\langle T \rangle_{\text{av}} = \int_0^l \langle \tau(l')\, dl' \rangle_{\text{av}} . \tag{41}$$

The variance about the mean is

$$\begin{aligned} \langle (T - \langle T \rangle_{\text{av}})^2 \rangle_{\text{av}} &= \left\langle \int_0^l \int_0^l (\tau(l') - \langle \tau \rangle_{\text{av}}(l'))(\tau(l'') - \langle \tau \rangle_{\text{av}}(l''))\, dl'\, dl'' \right\rangle \\ &= \left[\int_0^l \int R_\tau(l', l'')\, dl'\, dl'' \right] - \langle T \rangle_{\text{av}}^2 \end{aligned} \tag{42}$$

where $R_\tau(l', l'') = E(\tau(l')\tau(l''))$. We need the correlation function $R_\tau(l, l')$ and the mean $\langle \tau \rangle_{\text{av}}(l')$.

4.4 *Calculation for a Lossless Two-Mode Guide*

For the lossless two-mode guide we have an energy bundle making a Poisson number of mode changes in any length L with mean $|\lambda_{12}|^2 S_g(\beta_1(\omega) - \beta_2(\omega))L$. The correlation function $R_\tau(l, l')$ (assuming we start off randomly in one of the modes) is that of a random telegraph wave[2] and is given by

$$R_\tau(l, l') = \tfrac{1}{4}\,|\Delta\tau_{12}|^2 \exp(-2\,|l - l'|/l_c) + \left(\frac{\tau_1 + \tau_2}{2}\right)^2 (l - l')^2 \qquad (43)$$

where

$$1/l_c = |\lambda_{12}|^2 S_g(\beta_1(\omega) - \beta_2(\omega)),$$

and

$$\Delta\tau_{12} = \left(\frac{\partial\beta_1}{\partial\omega} - \frac{\partial\beta_2}{\partial\omega}\right) = \tau_1 - \tau_2 .$$

We obtain the mean and second moment about the mean of the intensity response of a guide of length L.

$$\langle T\rangle_{\text{av}} = \left(\frac{\tau_1 + \tau_2}{2}\right)L,$$

$$\langle(T - \langle T\rangle_{\text{av}})^2\rangle_{\text{av}} = \frac{(\Delta\tau_{12})^2 l_c L}{4}\left[1 - \frac{l_c}{2L}(1 - \exp(-2L/l_c))\right], \qquad (44)$$

$$\lim_{[L/l_c]\to\infty} \langle(T - \langle T\rangle_{\text{av}})^2\rangle_{\text{av}} = \frac{(\Delta\tau_{12})^2 L l_c}{4},$$

where

$$l_c = [|\lambda_{12}|^2 S_g(\beta_1(\omega) - \beta_2(\omega))]^{-1},$$

$$\lim_{[L/l_c]\to 0} \langle(T - \langle T\rangle_{\text{av}})^2\rangle_{\text{av}} = \frac{(\Delta\tau_{12})^2 L^2}{4},$$

(compare equation (44) to equation (29)).

4.5 *Extension to the Two-Mode Guide With Loss*

We can use the hydraulic model to extend the above results to a two-mode guide with loss and differential loss.

Assume that when travelling at distance dl in mode j, a bundle of light has probability $\alpha_j dl$ of being absorbed.

If in travelling down a guide of length L, the bundle spends a distance L_1 in mode 1 and L_2 in mode 2, then the probability that it is not absorbed is

$$P = \exp\,[-(\alpha_1 L_1 + \alpha_2 L_2)] = \exp\,[-((\alpha_1 - \alpha_2)L_1 + \alpha_2 L)]. \quad (45)$$

The number of bundles entering the guide at time zero and arriving at the output end at time t in the presence of loss equals the number that would arrive at time t in the absence of loss times the probability that a bundle with total travel time t is not absorbed. But we have

$$\begin{aligned} L_1 + L_2 &= L, \\ t &= \tau_1 L_1 + \tau_2 L_2 = (\tau_1 - \tau_2)L_1 + \tau_2 L, \\ P &= \exp\,[-(\alpha_1 L_1 + \alpha_2 L_2)] = \exp\left\{-\left(\Delta\alpha\,\frac{(t - \tau_2 L)}{\Delta t_{12}} + \alpha_2 L\right)\right\} \end{aligned} \quad (46)$$

where

$$\Delta\tau_{12} = \tau_1 - \tau_2 \quad \Delta\alpha = \alpha_1 - \alpha_2.$$

With a little algebra we obtain

$$P = \exp\left[-\left\{\frac{\Delta\alpha}{\Delta\tau_{12}}\,(t - \langle\tau\rangle_{av}L) + \langle\alpha\rangle_{av}L\right\}\right] \quad (47)$$

where

$$\begin{aligned} \langle\alpha\rangle_{av} &= (\alpha_1 + \alpha_2)/2, \\ \langle\tau\rangle_{av} &= (\tau_1 + \tau_2)/2. \end{aligned}$$

Thus the intensity impulse response for a two-mode guide with loss is equal to the lossless response multiplied by P of equation (47).

Note that the gaussian shape for long guides still holds because the product of a gaussian and an exponential envelope is a shifted gaussian.

V. CONCLUSIONS

We can conclude at least one important result. Long optical fiber waveguides need not have large dispersion due to random imperfections if properly controlled mode coupling exists. From equation (44) we see that a mechanical perturbation spectrum which is peaked at frequencies that couple guided modes will lower dispersion. However, to avoid loss, we must not make the mechanical perturbation spectrum too high at frequencies that couple guided and radiating modes.

The above conclusions have been obtained by D. T. Young and H. E. Rowe,[3] for the two-mode guide by solving the coupled line equations directly under the assumption of white noise coupling.

APPENDIX A

We wish to derive equation (21) from the following relationship

$$G(t, b) = [\gamma_{ij}(t, b)] = [\gamma_{ij}(t)]^*[\gamma_{ij}(t)]^* \cdots (b \text{ Times}), \qquad i, j = 1, 2. \tag{48}$$

Equation (48) implies the following model shown in Fig. 2. The transfer function $\gamma_{21}(t, b)$ is the overall transmission response between input 1 and output 2. This can be obtained by adding up the transmission responses over all different paths between input 1 and output 2 using any desired bookkeeping scheme. Every path between input 1 and output 2 must pass through the $\gamma_{21}(t)$ function *for the last time* in some section. If a path passes through the $\gamma_{21}(t)$ function for the last time in the fifth section from the end, then it must pass through four $\gamma_{22}(t)$ functions on its way to output 2. The sum of the path transfer functions between input 1 and the input to the $\gamma_{21}(t)$ function in the fifth section from the end is $\gamma_{11}(t, b - 5)$. Thus the contribution to the overall transfer function between input 1 and output 2 due to all paths which pass through a $\gamma_{21}(t)$ function for the last time in the fifth section from the end is $\gamma_{11}(t, b - 5) * \gamma_{21}(t) * \gamma_{22}^{*4}(t)$. Equation (21) merely expresses the sum of the contributions over all positions of last passage through a γ_{21} (t) function.

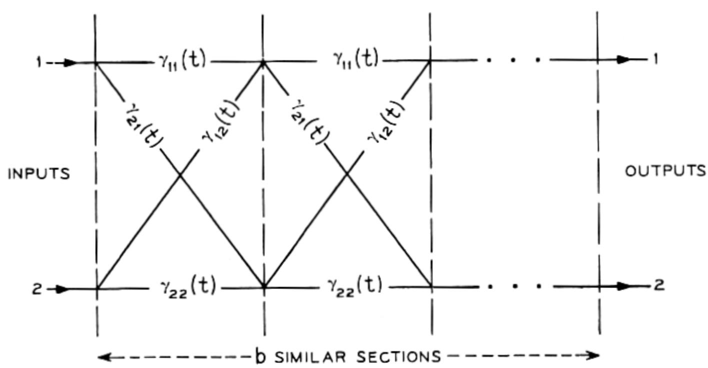

Fig. 2—*b*-section guide.

$$\gamma_{21}(t, b) = \sum_{R=1}^{b} \gamma_{11}(t, b - R) * \gamma_{21}(t) * \gamma_{22}^{*R-1}(t). \tag{21}$$

APPENDIX B

We wish to show that the equations for obtaining the intensity response of a long guide, given the intensity response of a short guide, for the hydraulic model are identical to the extrapolation equations for the intensity response given by equation (20).

Suppose we have the probability density, for a short guide, that a bundle of energy starting off in mode j of the guide (at time zero) arrives at the output of the guide at time t in mode i. Thus we have the matrix of densities $P(t, L)$ where L is the guide length and the elements $p_{ij}(t, L)$ are the previously described densities. Let $I_j(t, 0)$ be the probability density that a bundle of light arrives at input j at time t. Let $I_j(t, L)$ be the probability density that a bundle arrives at the output position L in mode j at time t. Let $I(t, \cdot)$ be the corresponding vectors. Using the laws of addition of random variables we obtain

$$I_i(t, L) = \sum_s p_{is}(t, L) * I_s(t, 0)$$

of in matrix notation

$$I(t, L) = P(t, L) * I(t, 0)$$

therefore

$$I(t, kL) = \left(\overset{k}{\underset{1}{*}} P(t, L)\right) * I(t, 0).$$

We see that the probability density of the output arrival mode and time of a bundle of energy for a long guide, which corresponds to the intensity response, is extrapolated from the short-guide response exactly as in equation (20). Thus since the perturbation results of Marcuse correspond to the hydraulic model in the limit of short guides and satisfy the conditions for extrapolation using equation (20), it follows that properties of the hydaulic model solution for long guides will correspond to the solution of equation (20) starting with these perturbation results. This is true no matter what techniques we use to find these hydaulic model properties.

APPENDIX C

We wish to establish that for a narrowband high frequency signal

$$y(t) = (y_e(t) \exp(i\omega_0 t) + y_e^*(t) \exp(-i\omega_0 t))/\sqrt{2}$$

of carrier frequency $f_0 = \omega_0/2\pi$ and envelope $y_e(t)$, the intensity is given by

$$|y_e(t)|^2 = 2 \int \left[\int_{-\infty}^{\infty} Y_+^*(f) Y_+(f + \alpha)\, df \right] \exp(-i2\pi\alpha t)\, d\alpha,$$

$$\Leftrightarrow 2 \int_{-\infty}^{\infty} Y_+^*(f) Y_+(f + \alpha)\, df.$$

Define

$$Y_e(f) = \int_{-\infty}^{\infty} y_e(t) \exp(i2\pi f t)\, dt, = \sqrt{2}\, Y_+(f + f_0),$$

[provided $y_e(t)$ is narrowband compared with f_0];

$$|y_e(t)|^2 = \int_{-\infty}^{\infty} Y_e(f) \exp(-2i\pi f t) Y_e^*(f') \exp(i2\pi f' t)\, df\, df',$$

$$= \int_{-\infty}^{\infty} \int \exp[-i2\pi(f - f')t][Y_e(f) Y_e^*(f')]\, df\, df',$$

$$= \int_{-\infty}^{\infty} \int^{\infty} \exp(-i2\pi\gamma t)[Y_e^*(f') Y_e(f' + \gamma)]\, d\gamma\, df',$$

where $\gamma = f - f'$;

$$= 2 \int_{-\infty}^{\infty} \exp(-i2\pi\gamma t)[Y_+^*(g) Y_+(g + \gamma)\, d\gamma\, dg]$$

where $g = (f' + f_0)$.

Q.E.D.

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