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On the Design and Analysis of a Class of PCM Systems

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This paper considers the problem of transmitting bandlimited signals using binary signaling over a noise-free channel. An analytical framework is presented for the design and analysis of a class of PCM systems where peak error is of primary interest. For a specific class of input signals which includes deterministic amplitude-constrained bandlimited functions as well as bandlimited wide-sense stationary, second-order, random processes, results are obtained which provide trade-offs between the sampling rate, quantizer and reconstruction filter.

I. INTRODUCTION

This paper considers the problem of transmitting bandlimited signals using binary signaling over a noise-free channel. A functional block diagram of the type of PCM system under consideration is shown in Fig. 1.*

The two major differences between the problem considered here and previous work are the measure of system performance and signal classes considered. The measure of system performance usually considered is related to the integral mean-squared error.[1–3] This type of performance measure lends itself to a frequency-domain analysis. While this criterion is widely used, it doesn't provide direct information regarding the size of the error as a function of time. To investigate this time behavior, we use a time-domain approach and use the maximum error over time as a measure of system performance.

Because of the desire to consider input signals that engineers would normally call bandlimited [i.e., trigonometric polynomials, sinusoids as well as functions in $\mathfrak{L}_2(-\infty, \infty)$ with finite bandwidth] we treat a

* For special classes of signals, other types of PCM systems such as DPCM and Delta Modulation are sometimes used. These systems will not be considered here.

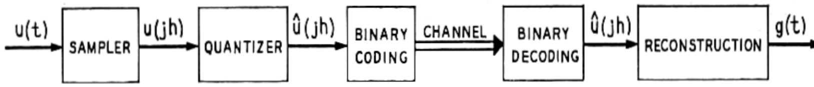

Fig. 1—UPCM system.

somewhat broader class of signals than normally considered in the literature. (The input class is usually considered to be a wide-sense stationary random process.)

In this paper, a technique is presented which simultaneously selects the sampling rate, quantizer and reconstruction filter in such a way as to minimize a bound on the peak error between the reconstructed and transmitted-signals. Since our interest centers on studying the trade-offs between the various system parameters, we desire precise mathematical results which are valid over the entire range of possible system parameters. By using an upper bound on system performance this aim was achieved for several classes of input signals. Using a criterion which evaluates a given encoding-decoding scheme in terms of its performance for the worst signal in the input class, a specific encoding-decoding algorithm, which will be called Uniform PCM or UPCM, is suggested and evaluated. The results are presented in the form of a set of normalized curves which plot an upper bound on the percentage error associated with the proposed UPCM system as a function of a normalized parameter, ρ, which represents the ratio of the bit rate to the bandwidth of the input class. The optimized values of the various parameters which define the system can also be determined from the plots (these include the sampling rate, the number of quantizing levels and the delay associated with the decoder).

II. AN ENCODING-DECODING ALGORITHM FOR THE TRANSMISSION OF BANDLIMITED FUNCTIONS

The class B_σ of functions consists of entire functions of exponential order one and type σ which are bounded on the real axis.[4] It includes all functions in $\mathfrak{L}_2(-\infty, \infty)$ having finite radian bandwidth, σ, and all trigonometric polynomials of degree $[\sigma]$.*

The performance of a given system is defined by

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* [ ] denotes integral part. A trigonometric polynomial of degree $[\sigma]$ has the form $\frac{1}{2}a_0 + \sum_{k=1}^{[\sigma]} (a_k \cos kt + b_k \sin kt)$.

$$\epsilon = \sup_{t \in [0,T]} \sup_{u \in B_\sigma(M)} \| u(t) - g(t; u) \| \tag{1}$$

where $u(t)$ is the value of the input signal at time t and $g(t; u)$ represents the value of the output signal at time t corresponding to the input function u. The input function u is an element of B_σ and the values of the input and output signals at any time $t \in [0, T]$ belong to a normed linear space, Ω. By proper choice of Ω and the norm on Ω, a variety of different input classes and performance measures can be treated. Specifically, define the set $B_\sigma(M)$ by

$$B_\sigma(M) = \{x : x \in B_\sigma\,,\ x(t) \in \Omega,\ \| x(t) \| \leqq M \ \forall\ t\}.$$

In arriving at the proposed algorithm, the following representation is used for elements of B_σ (and hence $B_\sigma(M)$:*

$$u(t) = \sum_{j=-\infty}^{\infty} u(jh)\theta_j(t) \tag{2}$$

where

$$\theta(t) = \frac{\sin \dfrac{\delta\sigma}{1-\delta} t}{\dfrac{\delta\sigma}{1-\delta} t} \; \frac{\sin \dfrac{\sigma}{1-\delta} t}{\dfrac{\sigma}{1-\delta} t}\,; \tag{3}$$

$$\theta_j(t) = \theta(t - jh), \qquad h = \frac{\pi(1-\delta)}{\sigma}. \tag{4}$$

This representation is valid for any $\delta \in (0, 1)$, hence δ may be chosen appropriately for each application. The parameter δ will be called the fractional guardband since the time between samples is $\pi(1 - \delta)/\sigma$ which is less than the time between samples π/σ which corresponds to the Nyquist rate.

In essence the proposed algorithm is a scheme for approximating any element of $B_\sigma(M)$† by one of a finite set of appropriately chosen functions. These functions are determined by first truncating the infinite series, (2). This truncation process produces an approximation to $u(t)$ (over the time interval $t \in [0, T]$) of the form

$$\bar{u}(t) = \sum_{j=-L}^{[T/h]+L} u(jh)\theta_j(t). \tag{5}$$

* This is a special case of a broader class of representations for elements of B_σ[5]. It was chosen because it provided the smallest bound on quantization error.

† The results presented in this paper are applicable to the class of signals having representation (2) with $\|u(jh)\| \leqq M$. This class is larger than $B_\sigma(M)$.

Thus we have replaced the requirement of transmitting an infinite number of sample values by the problem of transmitting a finite number. To achieve our ultimate objective, these sample values are then quantized and the quantized samples are used to reconstruct the signal

$$g(t) = \sum_{j=-L}^{[T/h]+L} \hat{u}(jh)\theta_j(t) \tag{6}$$

where $\hat{u}(jh)$ represents the quantized value of $u(jh)$.

The encoder will therefore consist of appropriate quantization of the input samples while the decoder simply represents the reconstruction of the truncated series using the quantized samples [i.e., the signal $g(t)$ given in equation (6)]. An analog interpretation of the decoding process is discussed in Section V. It should be noted that in order to construct $g(t)$ according to equation (6), a delay of $T + Lh$ seconds is required. The trade-offs between this delay and the accuracy of reconstruction will become clear as the results are presented.

Having constrained the general form of the proposed system, we now seek to determine the various parameters which define it (e.g., sampling rate and quantizer) in such a way as to minimize an upper bound on the value of ϵ. Because of the binary nature of the signaling which is being considered, the number of quantization levels is constrained to be 2^ν, where ν is an integer.* Thus ν represents the number of bits per sample. The bit rate is then given by

$$B = \frac{\nu}{h} \text{ b/s}. \tag{7}$$

It is convenient to define a normalized parameter ρ, which is given by

$$\rho = \frac{2\pi B}{\sigma} = \frac{B}{f} \tag{8}$$

where σ is the radian bandwidth of the input class and f is the bandwidth in Hz. In terms of these parameters, from equation (4), we have

$$\delta = 1 - \frac{2\nu}{\rho}. \tag{9}$$

Since $\delta \in (0, 1)$, the number of bits per sample must satisfy

$$1 \leqq \nu \leqq \left[\frac{\rho}{2}\right]. \tag{10}$$

* This is a practical constraint and not a theoretical one.

In terms of these parameters a simple bound on the reconstruction error results.

The analysis of the reconstruction error, given in the Appendix, shows that ϵ, defined by equation (1), satisfies

$$\epsilon \leqq \bar{\epsilon} = \frac{MK}{\pi^2 \delta L} + \frac{\sigma_q(\nu)}{\sqrt{\delta}} \tag{11}$$

where

$$K = \frac{\dfrac{T}{Lh} + 2}{\dfrac{T}{Lh} + 1}, \tag{12}$$

$$\sigma_q(\nu) = \sup_j \sup_{u \in B_\sigma(M)} \| u(jh) - \hat{u}(jh) \| \tag{13}$$

and δ is given by equation (9). Thus the actual value of ϵ which is achieved by the proposed algorithm may be less than the value $\bar{\epsilon}$ given by equation (11). In fact, since the reconstruction formula (6) is interpolatory, the error at sample points can never exceed the raw quantization error, $\sigma_q(\nu)$. The first term on the right-hand side of equation (11) may be viewed as the error due to truncating $u(t)$ as given by equation (2) while the second term represents the quantization error. The effect of the delay on the error can be easily seen from equations (11) and (12).

As a design procedure, one might first choose the quantizer which minimizes $\sigma_q(\nu)$ and then determine the value of ν [subject to (10)] which minimizes the error bound given by $\bar{\epsilon}$ [see equation (11)].

In the next sections, several important special cases are considered and explicit design curves are presented for these cases.

III. DETERMINISTIC, AMPLITUDE CONSTRAINED BANDLIMITED FUNCTIONS

In this case, $\Omega = R$ (the real line), and $\|x\| = |x|$. Thus

$$B_\sigma(M) = \{u : u \in B_\sigma,\ u(t) \in R,\ |u(t)| \leqq M \ \forall\ t\}^*.$$

The quantizer which minimizes $\sigma_q(\nu)$ for this case is shown in Fig. 2. The optimal value of $\sigma_q(\nu)$ is given by

$$\sigma_q(\nu) = \frac{M}{2^\nu}. \tag{14}$$

* Insofar as information content is concerned, if $M_1 \leqq u(t) \leqq M_2$, then it is equivalent to consider $|u(t)| \leqq M$, where $M = (M_2 - M_1)/2$.

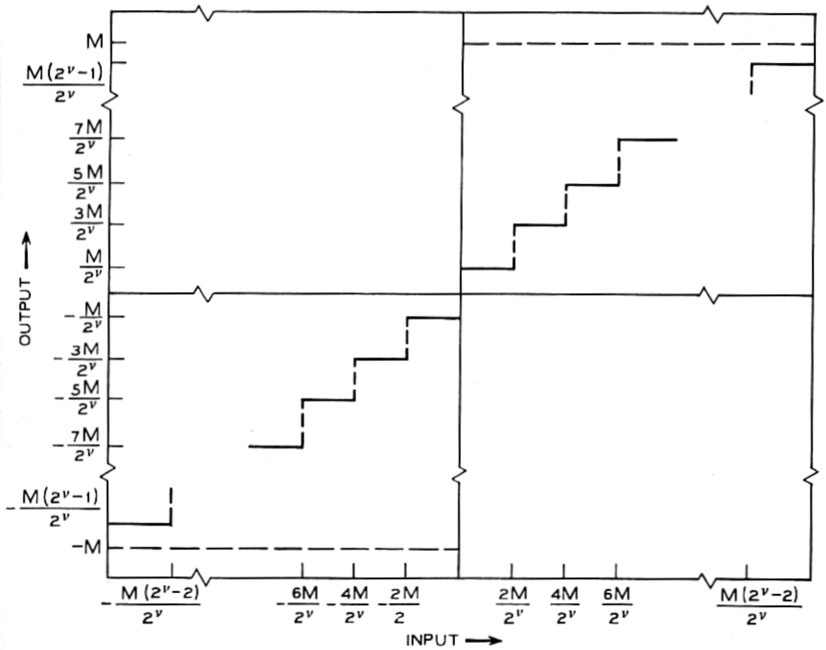

Fig. 2—Uniform quantizer.

Using equation (14), equation (11) becomes

$$\epsilon \leqq \bar{\epsilon} = \frac{MK}{\pi^2 \, \delta L} + \frac{M}{\sqrt{\delta} \, 2^{\nu}}. \tag{15}$$

For later use, we define

$$\bar{\epsilon} = \frac{MK}{\pi^2 \, \delta L} + \frac{K_0 M}{\sqrt{\delta} \, 2^{\nu}}. \qquad (16)^*$$

For the present case $K_0 = 1$ and equation (15) thus becomes $\epsilon \leqq \bar{\epsilon} = \bar{\epsilon}$.

The bound represented by equation (15) is valid for any allowable set of system parameters. For fixed values of ρ, T, and L, the value of ν which minimizes $\bar{\epsilon}$ can easily be found. If we define the normalized percentage error as $100(\bar{\epsilon}/2MK_0)$, then Fig. 3 plots this as a function of ρ, using the optimized values of ν. These values of ν (denoted ν^*) are indicated on the curves. Curves are plotted for $K_0L/K = 5, 10, 20, 40,$

* K_0 is introduced here for the purpose of unifying the results of this section and the next. For the class of signals in this section, K_0 can be replaced by unity.

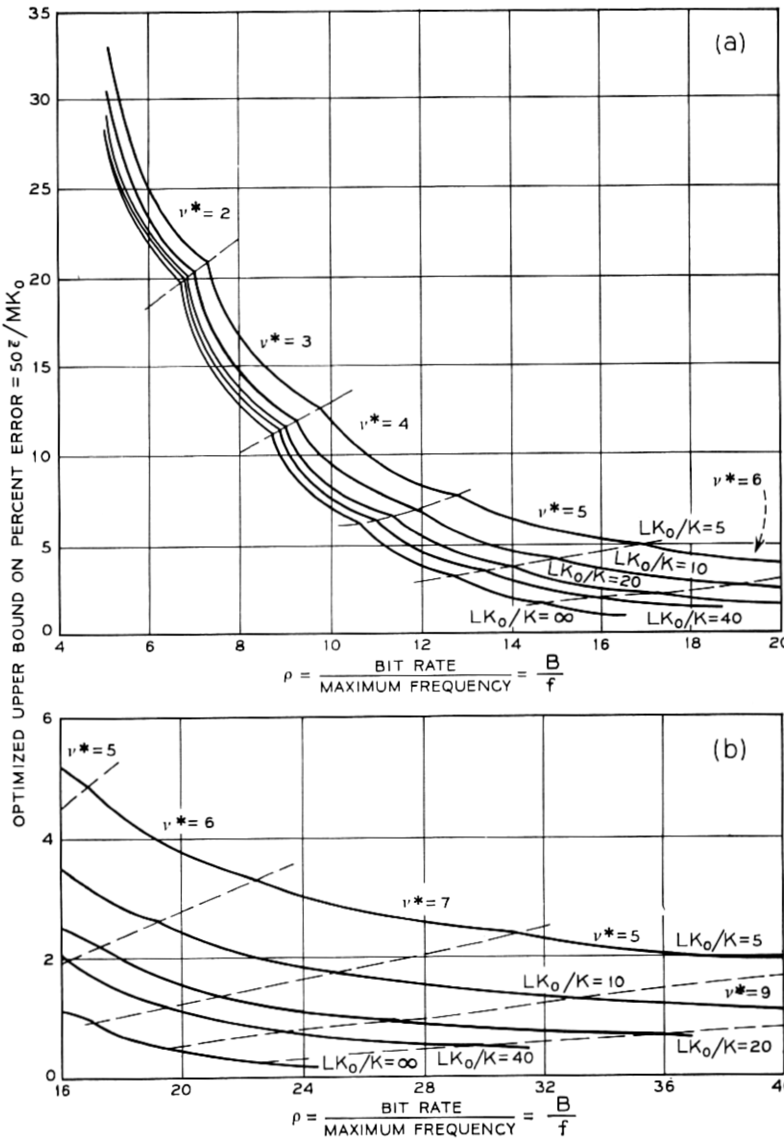

Fig. 3a and b—Performance curves. (Notes: 1. For analog implementation, use $K = 1$; delay $= (L + 1)\ h$. 2. For digital implementation, $K = (T/Lh + 2)/(T/Lh + 1)$; delay $= T + Lh$. 3. ν^* is the optimized number of bits per sample. 4. The optimum sampling rate is $1/h = B/\nu^*$.)

and ∞. Since $1 \leqq K \leqq 2$, for a given value of the ratio L/K, one can associate any value of L satisfying $L/K \leqq L \leqq 2L/K$. Using equation (12), the corresponding value of T can then be computed. The delay associated with reconstruction in accordance with equation (6) is then given by $T_d = T + Lh$. As discussed in Section V, when an analog implementation is used for the reconstruction, a value of $K = 1$ should be used and the delay associated with the filter which accomplishes the reconstruction is given by $T_d = (L + 1)h$.

To illustrate the use of these curves, consider the problem defined by the following parameter values:

$$\sigma = \pi \times 10^6,$$

$$B = 6.3 \times 10^6 \text{ b/s}.$$

Using these values, we compute $\rho = 12.6$. From the curves, for $L/K = 20$, we have $50\bar{\epsilon}/M = 4.67$ percent, $\nu^* = 5$. Using $M = \frac{3}{8}$, we have $\bar{\epsilon} = 0.035$. Since $B = \nu/h$, we have $h^* = 0.79 \times 10^{-6}$ seconds or $1/h^* = 1.26 \times 10^6$ samples/second. For $L = 40$, the delay associated with the error [using equation (6) to accomplish the reconstruction] is about 32 μs. For the corresponding analog implementation (See Section V) these results correspond to a value of $L = 20$ (and $K = 1$) with a corresponding delay of 16.6 μs.

It is clear that the normalized nature of the curves in Fig. 3 facilitates their use in a wide variety of ways. For example, one could easily answer questions such as:

(*i*) For a given class of signals, what is the smallest bit rate that can be used to guarantee an error not exceeding a prescribed level?

(*ii*) For a given bit rate, what is the largest class of signals that can be handled with a maximum error not exceeding a prescribed level?

In the next section, analogous results are developed for some important classes of random input signals.

IV. SOME IMPORTANT CLASSES OF RANDOM INPUTS

In this section we consider the case where Ω is the space of zero mean second-order random variables* with norm given by

$$\| x \| = (Ex^2)^{\frac{1}{2}} \equiv \sigma_x , \qquad x \in \Omega. \tag{17}$$

* There is no loss of generality in the zero-mean assumption.

Then

$$B_\sigma(M) = \{u : u \varepsilon B_\sigma , u(t) \varepsilon \Omega, \sigma_{u(t)} \leqq M\}. \tag{18*}$$

Thus $B_\sigma(M)$ consists of second-order random processes (with $\sigma_{u(t)} \leqq M$) with sample functions which are all in B_σ . For the important special case of wide-sense stationary random processes, $\sigma_{u(t)} \equiv \sigma_u$. For this case, the quantizer will be characterized by the property that it minimizes the mean-squared error at the sample times. The problem of designing such a quantizer for a given probability distribution of the input amplitude has been considered by B. Smith.[6] An approximate upper bound on the optimized quantizing error is given by

$$\sigma_q \leqq \bar{\sigma}_q \cong \bar{\sigma}_q = \frac{1}{2^\nu} \sqrt{\frac{2}{3}} \left[\int_0^\infty p^{\frac{1}{3}}(u)\, du \right]^{\frac{3}{2}} \tag{19}$$

where $p(u)$ is the probability density function of $u(t)$. The corresponding quantizer is described in Ref. 6.

For the important cases where $p(u)$ is uniform, gaussian, or exponential, $\bar{\sigma}_q$ can be written in the form

$$\bar{\sigma}_q = \frac{K_0 \sigma_u}{2^\nu} \tag{20}$$

where K_0 is a constant depending on the particular distribution function. Table I gives the value of K_0 for each of the input classes of interest. The bound on the system performance which is given by equation (11) can thus be written as

$$\epsilon \leqq \bar{\epsilon} \cong \bar{\epsilon} = \frac{MK}{\pi^2 \, \delta L} + \frac{K_0 M}{\sqrt{\delta}\, 2^\nu} \tag{21}$$

which is identical in form to the corresponding bound for the deterministic case [see equation (16)]. The use of the design-analysis curves of Fig. 3 for these cases is thus identical to the previously described deterministic case.

It is interesting to note that for the case of a uniform amplitude distribution as well as the amplitude constrained input class, $\sigma_q = \bar{\sigma}_q$ and the optimal quantizer is a uniform quantizer. In each of these cases $\epsilon \leqq \bar{\epsilon}$.

In the next section, the interpretation and implementation of the proposed algorithm will be discussed. It will be shown that the analog implementation takes the form of a PCM system where the sampling

* More precisely, $B_\sigma(M) = \{u(\cdot, \cdot): u(\cdot, \omega) \varepsilon B_\sigma, u(t, \cdot) \varepsilon \Omega, \|u(t, \cdot)\| \leqq M\}$.

TABLE I—INPUT SIGNAL DESCRIPTION*

Deterministic	Random: Uniform	Random: Gaussian	Random: Exponential
$\|u(t)\| \leqq M$	$p(u) = \begin{cases} \frac{1}{2a} & \|u\| \leqq a \\ 0 & \|u\| > a \end{cases}$	$p(u) = \frac{1}{\sqrt{2\pi}\,\sigma_u} \exp(-u^2/2\sigma_u^2)$	$p(u) = \frac{1}{\sqrt{2}\,\sigma_u} \exp(-\sqrt{2}u/\sigma_u)$
$K_0 = 1$	$K_0 = 1$	$K_0 = 1.65$	$K_0 = 2.12$
	$\sigma_u = \frac{a}{\sqrt{3}} \equiv M$	$M \equiv \sigma_u$	$M \equiv \sigma_u$

* All signals are in B_σ.

rate, quantizer and low-pass filter which is used to accomplish the decoding are carefully chosen.

V. ANALOG RECONSTRUCTION IN THE UPCM SYSTEM

In this section we discuss an analog reconstruction in the UPCM system. For the analog reconstruction, the system takes the form shown in Fig. 4. The identification of the low-pass filter results from the following analysis.

If we first consider the response, $\bar{g}(t)$, of a causal, stationary filter [with impulse response $h(t)$] to the input

$$u^*(t) = \sum_{j=-\infty}^{\infty} \hat{u}(jh)\,\delta(t - jh) \tag{22}$$

then

$$\bar{g}(t) = \sum_{-\infty}^{[t/h]} \hat{u}(jh)h(t - jh). \tag{23}$$

We now consider $h(t)$ to be a delayed, truncated (for negative times)

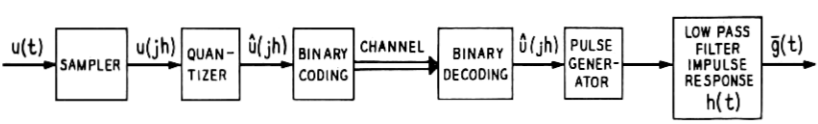

Fig. 4—Analog UPCM system.

version of $\theta(t)$ [see equation (3)]

$$h(t) = \theta(t - (L + 1)h)S(t) = \theta_{L+1}(t)S(t) \tag{24}$$

where $\theta(t)$ is given in equation (3) and $S(t)$ is the unit step. The response $\bar{g}(t)$ can thus be written

$$\bar{g}(t) = \sum_{-\infty}^{[t/h]} \hat{u}(jh)\theta_j(t - (L + 1)h). \tag{25}$$

If $\bar{u}(t)$, given in equation (5), were more accurately represented by

$$\tilde{u}(t) = \sum_{j=-\infty}^{[T/h]+L} u(jh)\theta_j(t), \qquad t \,\varepsilon\, [0, T] \tag{5$'$}$$

then $g(t)$, given by equation (6), would more accurately be represented (for $T = h$) by

$$\tilde{g}(t) = \sum_{j=-\infty}^{L+1} u(jh)\theta_j(t), \qquad t \,\varepsilon\, [0, T]. \tag{6$'$}$$

Equation (6′) can be made valid for all t with

$$\tilde{g}(t) = \sum_{j=-\infty}^{[t/h]+L+1} \hat{u}(jh)\theta_j(t). \tag{26}$$

Thus

$$\tilde{g}(t - (L + 1)h) = \sum_{-\infty}^{[t/h]} \hat{u}(jh)\theta_j(t - (L + 1)h). \tag{27}$$

We thus see that the output $\bar{g}(t)$ of the filter, with impulse response $h(t)$ given by equation (24), is a delayed version [delay of $(L + 1)h$ seconds] of the more accurate representation of $g(t)$ given by equation (6′). It should be noted that the more accurate representation of $g(t)$ was obtained by truncating only future values in the infinite sum as opposed to truncating both past and future values. The corresponding error bound is reduced to

$$\| u(t - (L + 1)h) - \bar{g}(t) \| \leq \frac{MK_0}{\sqrt{\delta}\, 2^{\nu}} + \frac{M}{\pi^2\, \delta L}. \tag{28)*$$

This corresponds to the case $K = 1$ for the curves shown in Fig. 3.

Figure 5 is a plot of $\theta_{L+1}(t)$ and its transform

$$\Theta_{L+1}(\omega) = \int_{-\infty}^{\infty} \theta_{L+1}(t) \exp(-j\omega t)\, dt.$$

* This can easily be obtained by eliminating the second term in equation (38) which corresponds to truncation of past values.

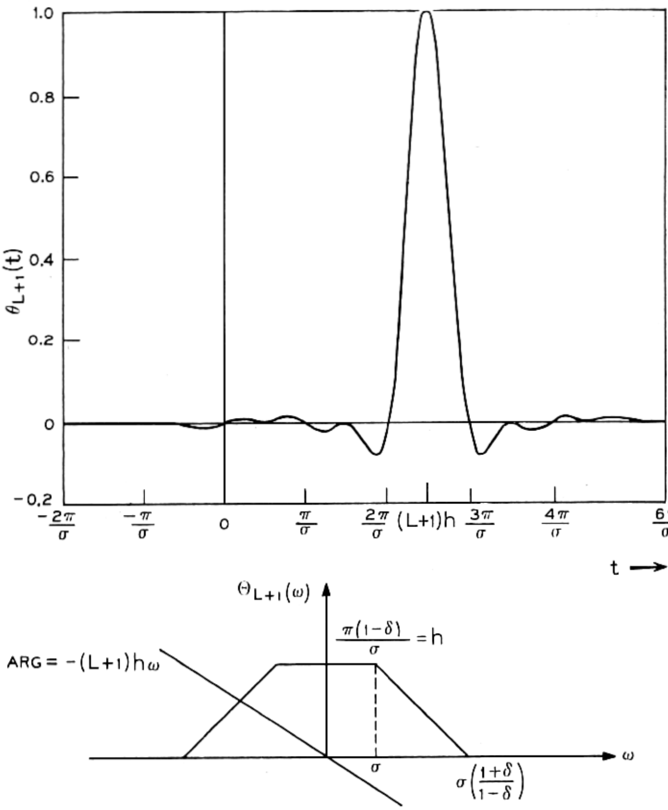

Fig. 5—Curve shown for $L = 4$, $\delta = 1/2$.

The causal impulse response $h(t)$ can be written as

$$h(t) = \theta_{L+1}(t) - \theta_{L+1}(t)S(-t)$$

where the second term represents the negative time tail of $\theta_{L+1}(t)$. The corresponding transform is

$$H(\omega) = \Theta_{L+1}(\omega) - \tilde{H}(\omega) \tag{29}$$

where $\tilde{H}(\omega)$ is the transform of $\theta_{L+1}(t)S(-t)$. The transform of the causal filter is thus seen to be a slight perturbation (for L sufficiently large) of the low-pass characteristic of $\Theta_{L+1}(\omega)$.

VI. DISCUSSION

In this paper, the design and analysis of a class of PCM systems has been considered. In arriving at these results, advantage has been taken of the fact that the sampling rate is higher than the Nyquist rate. The approach taken in this paper takes advantage of the alternate representations of the input class which are made possible by the higher sampling rate and chooses one for which a good bound on the effects of quantization errors can be derived.

In their present form the results which have been presented may be viewed as a step towards the study of peak error behavior of PCM systems. The extension of these results to input signals about which more information is available (e.g., power spectral information) is the next step towards providing a more generally applicable framework for the design and analysis of PCM systems where peak error is the natural criterion.

VII. ACKNOWLEDGMENTS

The authors would like to thank L. J. Forys for many helpful discussions. They are also indebted to Mrs. Sharon Miller for the excellent programming jobs she did in connection with this project.

APPENDIX

Derivation of Error Bound for UPCM System

The class B_σ of functions consists of entire functions of exponential order one and type σ which are bounded on the real axis.[4] It includes all functions in $\mathfrak{L}_2(-\infty, \infty)$ having finite radian bandwidth, σ, and all trigonometric polynomials of degree $[\sigma]$.*

Any element in B_σ has the representation[5]

$$u(t) = \sum_{j=-\infty}^{\infty} u(jh)\theta_j(t) \tag{30}$$

where

$$\theta(t) = \frac{\sin \dfrac{\delta\sigma}{1-\delta} t \sin \dfrac{\sigma}{1-\delta} t}{\dfrac{\delta\sigma}{1-\delta} t \quad \dfrac{\sigma}{1-\delta} t}, \tag{31}$$

$$\theta_j(t) = \theta(t - jh), \qquad h = \frac{\pi(1-\delta)}{\sigma}. \tag{32}$$

* [ ] denotes integral part. A trigonometric polynomial of degree $[\sigma]$ has the form $\frac{1}{2}a_0 + \sum_{k=1}^{[\sigma]} (a_k \cos kt + b_k \sin kt)$.

This representation is valid for any $\delta \in (0, 1)$, hence δ may be chosen appropriately for each application. The parameter δ will be called the fractional guardband since the time between samples is $\pi(1 - \delta)/\sigma$ which is less than the time between samples π/σ which corresponds to the Nyquist rate.

Let $\bar{u}(t)$ represent an approximation to $u(t)$ in the interval $[-T/2, T/2]$ obtained by truncating equation (30). Thus

$$\bar{u}(t) = \sum_{j=-N}^{N} u(jh)\theta_j(t) \tag{33}$$

and

$$u(t) - \bar{u}(t) = \sum_{|j|>N} u(jh)\theta_j(t). \tag{34}$$

Since $u \in B_\sigma(M)$, we have

$$\sup_{u \in B_\sigma(M)} \| u(t) - \bar{u}(t) \| \leqq M \sum_{|j|>N} | \theta_j(t) |. \tag{35}$$

Let $2N + 1 = T/h + 2L$, T/h odd, and $| j | > N$, then for $| t | \leqq T/2$,

$$| \theta_j(t) | \leqq \frac{1}{\pi^2 \delta} \frac{1}{\left(j - \frac{t}{h}\right)^2}. \tag{36}$$

Thus

$$\sup_{u \in B_\sigma(M)} \| u(t) - \bar{u}(t) \| \leqq \frac{M}{\pi^2 \delta} \sum_{j>N} \left\{ \left(j - \frac{t}{h}\right)^{-2} + \left(j + \frac{t}{h}\right)^{-2} \right\}. \tag{37}$$

Using the Sonin formula[7] to sum the remainder series in equation (37) yields

$$\sup_{u \in B_\sigma(M)} \| u(t) - \bar{u}(t) \|$$
$$\leqq \frac{M}{\pi^2 \delta} \left\{ \left(N + \frac{1}{2} - \frac{t}{h}\right)^{-1} + \left(N + \frac{1}{2} + \frac{t}{h}\right)^{-1} \right\}. \tag{38}$$

To obtain a uniform bound for $t \in [-T/2, T/2]$, we observe that the right side of equation (38) is maximized for $t = T/2$. Using this and $N + \frac{1}{2} = T/2h + L$ yields

$$\sup_{t \in [-T/2, T/2]} \sup_{u \in B_\sigma(M)} \| u(t) - \bar{u}(t) \| \leqq \frac{MK}{\pi^2 \delta L}, \tag{39}$$

where

$$K = \frac{\frac{T}{Lh} + 2}{\frac{T}{Lh} + 1}, \qquad 1 \leqq K \leqq 2.$$

Thus equation (39) represents a uniform bound on the truncation error associated with the finite sum approximation (33) to an arbitrary $u \in B_\sigma(M)$ over the time interval $-T/2 \leqq t \leqq T/2$. It is clear that this bound can be made arbitrarily small by appropriate choice of L, however the quantity $T + Lh$ represents the delay associated with the decoder and hence some compromise will generally be called for.

To evaluate the error due to quantizing the sample values, it is convenient to define the quantity $A(\delta)$ by

$$A(\delta) = \sup_{-\infty < t < \infty} \sum_{j=-\infty}^{\infty} |\theta_j(t)|. \tag{40}$$

$A(\delta)$ plays a crucial role in relating the effects of quantization error at sample times to the error between samples. In fact, one of the significant characteristics of the representation (30), which is not shared by the Cardinal series representation (for functions in $W_\sigma \subset B_\sigma$) is the ability to establish a useful bound on $A(\delta)$. To do this, we first apply the Cauchy-Schwarz inequality and obtain

$$A(\delta)^2 \leqq \sup_{-\infty < t < \infty} \sum_{j=-\infty}^{\infty} \left[\frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \right]^2 \cdot \sup_{-\infty < t < \infty} \sum_{j=-\infty} \left[\frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)} \right]^2. \tag{41}$$

The Parseval theorem for functions in B_σ which are also in $\mathfrak{L}_2(-\infty, \infty)$ provides the following explicit evaluations

$$\sum_{j=-\infty}^{\infty} \left[\frac{\sin \frac{\sigma}{1-\delta}(t - jh)}{\frac{\sigma}{1-\delta}(t - jh)} \right]^2 = 1, \tag{42}$$

$$\sum_{j=-\infty}^{\infty} \left[\frac{\sin \frac{\delta\sigma}{1-\delta}(t - jh)}{\frac{\delta\sigma}{1-\delta}(t - jh)} \right]^2 = \frac{1}{\delta}. \tag{43}$$

Hence

$$A(\delta) \leqq \frac{1}{\sqrt{\delta}}. \tag{44}$$

Consider a quantizer with 2^ν output levels. If we denote by $g(t)$ the signal

$$g(t) = \sum_{j=-N}^{N} \hat{u}(jh)\theta_j(t), \tag{45}$$

where $\hat{u}(jh)$ represents the quantized value of $u(jh)$, then

$$u(t) - g(t) = \sum_{j=-N}^{N} (u(jh) - \hat{u}(jh))\theta_j(t) + \sum_{|j|>N} u(jh)\theta_j(t). \tag{46}$$

Since the maximum quantization error *at sample times* is $\sigma_q(\nu)$, using equations (44) and (39) we obtain

$$\sup_{t\epsilon[-T/2,T/2]} \sup_{u\epsilon B_\sigma(M)} \| u(t) - g(t) \| \leqq \frac{\sigma_q(\nu)}{\sqrt{\delta}} + \frac{MK}{\pi^2\delta L} \triangleq \bar{\epsilon}. \tag{47}$$

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